

State-Space Control Fundamentals (00860721)

lecture no. 12

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Outline

Discrete LQR

Discrete-time state observers and observer-based feedback

Case study 1

Case study 2

Outline

Discrete LQR

Discrete-time state observers and observer-based feedback

Case study 1

Case study 2

Discrete infinite-horizon LQR problem

Given

$$x[t+1] = Ax[t] + Bu[t], \quad x[0] = x_0$$

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, find u that

- stabilizes the systems ($\lim_{t \rightarrow \infty} x[t] = 0$),
- minimizes

$$\mathcal{J} = \sum_{t=0}^{\infty} \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix}$$

for given $Q = Q' \in \mathbb{R}^{n \times n}$, $R = R' \in \mathbb{R}^{m \times m}$, and $S \in \mathbb{R}^{n \times m}$.

Assumptions:

$\mathcal{A}_1$: (A, B) is stabilizable

$$\mathcal{A}_2: \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$$

no need for $R > 0$

$$\mathcal{A}_3: \begin{bmatrix} A - e^{-j\theta}I & B \\ Q & S \\ S' & R \end{bmatrix} \text{ has full column rank for all } \theta \in [-\pi, \pi]$$

DARE / Matlab: `idare(A,B,Q,R,S)` or `idare(A,B,Q,R)`

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $Q = Q' \in \mathbb{R}^{n \times n}$, $R = R' \in \mathbb{R}^{m \times m}$, and $S \in \mathbb{R}^{n \times m}$ be given. The matrix equation

$$\bar{X} = A' \bar{X} A + Q - (S + A' \bar{X} B)(R + B' \bar{X} B)^{-1}(S' + B' \bar{X} A)$$

is called the **discrete-time algebraic Riccati equation** (DARE). Its solution $\bar{X} \in \mathbb{R}^{n \times n}$ is said to be stabilizing if $\det(R + B' \bar{X} B) \neq 0$ and the matrix

$$A_K := A + BK, \quad \text{where } K = -(R + B' \bar{X} B)^{-1}(S' + B' \bar{X} A),$$

is Schur, i.e. having eigenvalues in $\mathbb{D}_1 := \{z \in \mathbb{C} \mid |z| < 1\}$. For LQR we are interested in DAREs with

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$$

which requires $Q \geq 0$ and $R \geq 0$, but not $R > 0$.

DARE: example

Let $A = a$, $B = 1$, $Q = q \geq 0$, $S = 0$, and $R = r \geq 0$. The DARE reads

$$\bar{X} = a^2 \bar{X} + q - \frac{a^2 \bar{X}^2}{r + \bar{X}} = \frac{ra^2 \bar{X}}{r + \bar{X}} + q$$

1. If $r = 0$, then the DARE becomes $\bar{X} = q$ and $r + \bar{X} \neq 0 \iff q \neq 0$.
In the latter case, $A_K = 0$ is Schur.
2. If $r > 0$, then the equation reads

$$\bar{X}^2 - (r(a^2 - 1) + q)\bar{X} - qr = 0 \quad \text{and} \quad \bar{X} \neq -r.$$

The first equation is solved by

$$\bar{X} = \frac{r(a^2 - 1) + q}{2} \pm \sqrt{\frac{(r(a^2 - 1) + q)^2}{4} + qr}$$

with $\bar{X} \geq 0$ for the "+" solution ($\bar{X} = 0$ iff $q = 0$ and $|a| \leq 1$ then).

DARE: example

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The first equation is solved by

$$\bar{X} = \frac{r(a^2 - 1) + q}{2} \pm \sqrt{\frac{(r(a^2 - 1) + q)^2}{4} + qr}$$

with $\bar{X} \geq 0$ for the “+” solution ($\bar{X} = 0$ iff $q = 0$ and $|a| \leq 1$ then).

DARE: example (contd)

2. Now,

$$r + \bar{X} = \frac{r(a^2 + 1) + q}{2} \pm \sqrt{\frac{(r(a^2 - 1) + q)^2}{4} + qr}$$

is zero only for its “−” part for $a = 0$. Furthermore,

$$A_K = a - \frac{a\bar{X}}{r + \bar{X}} = \frac{r(a^2 + 1) + q}{2ar} \mp \frac{\sqrt{(r(a^2 + 1) + q)^2 - (2ar)^2}}{2ar}$$

and it is Schur for the “+” solution of $\bar{X}$. Hence, the stabilizing

$$\bar{X} = \frac{r(a^2 - 1) + q}{2} + \sqrt{\frac{(r(a^2 - 1) + q)^2}{4} + qr}$$

and the resulting

$$A_K = \frac{a^2 + 1 + q/r}{2a} - \frac{\sqrt{(a^2 + 1 + q/r)^2 - 4a^2}}{2a} \xrightarrow{r \rightarrow 0} 0$$

DARE: solvability

Theorem 12.1

Let $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$. The DARE

$$\bar{X} = A' \bar{X} A + Q - (S + A' \bar{X} B)(R + B' \bar{X} B)^{-1}(S' + B' \bar{X} A)$$

has a stabilizing solution $\bar{X} \geq 0$ iff

1. (A, B) is stabilizable,

2. $\begin{bmatrix} A - \lambda I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank for all $\lambda \in \{z \in \mathbb{C} \mid |z| = 1\}$.

Moreover, $\bar{X} > 0$ iff the second condition is replaced with

2'. $\begin{bmatrix} A - \lambda I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank for all $\lambda \in \{z \in \mathbb{C} \mid |z| \leq 1\}$.

In any case, the stabilizing solution is unique.

Discrete LQR: solution / Matlab: `dlqr(A,B,Q,R,S)`

Theorem 12.2

If $\mathcal{A}_{1-3}$ hold, then the unique stabilizing controller that minimizes

$$\mathcal{J} = \sum_{t=0}^{\infty} \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix}$$

is

$$u[t] = -(R + B' \bar{X} B)^{-1} (S' + B' \bar{X} A) x[t],$$

where $\bar{X} = \bar{X}' \geq 0$ is the stabilizing solution to the DARE

$$\bar{X} = A' \bar{X} A + Q - (S + A' \bar{X} B)(R + B' \bar{X} B)^{-1} (S' + B' \bar{X} A)$$

The optimal cost is $\mathcal{J}_{\text{opt}} = x_0' \bar{X} x_0$ then.

Remark: Like in the continuous-time case, the optimal control law is the state feedback with $K = -(R + B' \bar{X} B)^{-1} (S' + B' \bar{X} A)$. The closed-loop "A" matrix $A + BK$ is Schur because $\bar{X}$ is the stabilizing solution.

Proof

As $K = -(R + B' \bar{X} B)^{-1} (S' + B' \bar{X} A) \implies S' + B' \bar{X} A = -(R + B' \bar{X} B)K$,
we have that

$$\bar{X} = A' \bar{X} A + Q - (S + A' \bar{X} B)(R + B' \bar{X} B)^{-1} (S' + B' \bar{X} A)$$

$$\Downarrow$$

$$Q = \bar{X} - A' \bar{X} A + K'(R + B' \bar{X} B)K$$

and, therefore, that

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} \bar{X} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \bar{X} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} -K' \\ I \end{bmatrix} (R + B' \bar{X} B) \begin{bmatrix} -K & I \end{bmatrix}$$

This is a discrete counterpart of a key relation in the continuous-time case and it plays a similar role in the discrete proof.

Proof (contd)

The quadratic form in the cost $\mathcal{J}$ is

$$\begin{aligned}
 & \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix} \\
 &= \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} -K' \\ I \end{bmatrix} (R + B' \bar{X} B) \begin{bmatrix} -K & I \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix} \\
 &\quad + \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \left(\begin{bmatrix} \bar{X} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \bar{X} \begin{bmatrix} A & B \end{bmatrix} \right) \begin{bmatrix} x[t] \\ u[t] \end{bmatrix} \\
 &= (u[t] - Kx[t])'(R + B' \bar{X} B)(u[t] - Kx[t]) \\
 &\quad + x'[t] \bar{X} x[t] - x'[t+1] \bar{X} x[t+1] \\
 &= \|(R + B' \bar{X} B)^{1/2}(u[t] - Kx[t])\|^2 + x'[t] \bar{X} x[t] - x'[t+1] \bar{X} x[t+1]
 \end{aligned}$$

where $M^{1/2} > 0$ is the matrix square root (Property 10, Slide 29, Lecture 1) of $M > 0$.

Proof (contd)

Thus,

$$\begin{aligned}
 \mathcal{J} &= \sum_{t=0}^{\infty} \left(\|(R + B' \bar{X} B)^{1/2} (u[t] - Kx[t])\|^2 \right. \\
 &\quad \left. + x'[t] \bar{X} x[t] - x'[t+1] \bar{X} x[t+1] \right) \\
 &= \sum_{t=0}^{\infty} \|(R + B' \bar{X} B)^{1/2} (u[t] - Kx[t])\|^2 + x_0' \bar{X} x_0 - \lim_{t \rightarrow \infty} x'[t] \bar{X} x[t]
 \end{aligned}$$

If u is stabilizing, then $\lim_{t \rightarrow \infty} x'[t] \bar{X} x[t] = 0$. Because $(R + B' \bar{X} B)^{1/2} > 0$,

$$\mathcal{J} = \sum_{t=0}^{\infty} \|(R + B' \bar{X} B)^{1/2} (u[t] - Kx[t])\|^2 + x_0' \bar{X} x_0 \geq x_0' \bar{X} x_0$$

whenever the system is asymptotically stable and the equality is attained iff $u = Kx$, which *is* stabilizing ($\bar{X}$ is the stabilizing solution). □

Suboptimal solutions

Corollary 12.3

Let $\mathcal{A}_{1-3}$ hold and the cost function

$$\mathcal{J} = \sum_{t=0}^{\infty} \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix}.$$

The control law

$$u[t] = -(R + B' \bar{X} B)^{-1} (S' + B' \bar{X} A) x[t] + (R + B' \bar{X} B)^{-1/2} v[t],$$

where $\bar{X} = \bar{X}' \geq 0$ is the stabilizing solution to the DARE

$$\bar{X} = A' \bar{X} A + Q - (S + A' \bar{X} B)(R + B' \bar{X} B)^{-1} (S' + B' \bar{X} A)$$

renders

$$\mathcal{J} = x_0' \bar{X} x_0 + \sum_{t=0}^{\infty} \|v[t]\|^2 = x_0' \bar{X} x_0 + \|v\|_2^2.$$

What if $R = 0$?

First, $\begin{bmatrix} Q & S \\ S' & 0 \end{bmatrix} \geq 0 \implies S = 0$. Otherwise, $\exists x_1 \neq 0$ such that $\|S'x_1\| \neq 0$, so

$$\begin{bmatrix} x_1' & -\alpha x_1' S \end{bmatrix} \begin{bmatrix} Q & S \\ S' & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ -\alpha S'x_1 \end{bmatrix} = x_1' Q x_1 - 2\alpha \|S'x_1\|^2 < 0$$

whenever α is large enough. Second, if $R = 0$ and $S = 0$, then

$$K = -(B' \bar{X} B)^{-1} B' \bar{X} A \implies A + BK = (I - B(B' \bar{X} B)^{-1} B' \bar{X}) A,$$

and

$$B' \bar{X} (A + BK) = B' \bar{X} A - B' \bar{X} B (B' \bar{X} B)^{-1} B' \bar{X} A = 0.$$

Because

$$\det(B' \bar{X} B) \neq 0 \implies \text{rank } B' \bar{X} = m$$

$$\text{otherwise, there is } 0 \neq \eta \in \mathbb{R}^n \text{ such that } \eta' B' \bar{X} = 0 \implies \eta' B' \bar{X} B = 0$$

there are m linearly independent $\eta_i \in \mathbb{R}^n$, viz. the columns of $\bar{X} B$, such that $\eta_i'(A + BK) = 0$, implying that

$$\implies A + BK \text{ has at least } m \text{ eigenvalues at the origin (deadbeat modes).}$$

What if $R = 0$?

First, $\begin{bmatrix} Q & S \\ S' & 0 \end{bmatrix} \geq 0 \implies S = 0$. Otherwise, $\exists x_1 \neq 0$ such that $\|S'x_1\| \neq 0$, so

$$\begin{bmatrix} x_1' & -\alpha x_1' S \end{bmatrix} \begin{bmatrix} Q & S \\ S' & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ -\alpha S'x_1 \end{bmatrix} = x_1' Q x_1 - 2\alpha \|S'x_1\|^2 < 0$$

whenever α is large enough. Second, if $R = 0$ and $S = 0$, then

$$K = -(B' \bar{X} B)^{-1} B' \bar{X} A \implies A + BK = (I - B(B' \bar{X} B)^{-1} B' \bar{X}) A.$$

and

$$B' \bar{X} (A + BK) = B' \bar{X} A - B' \bar{X} B (B' \bar{X} B)^{-1} B' \bar{X} A = 0.$$

Because

$$\text{— } \det(B' \bar{X} B) \neq 0 \implies \text{rank } B' \bar{X} = m$$

$$\text{otherwise, there is } 0 \neq \eta \in \mathbb{R}^m \text{ such that } \eta' B' \bar{X} = 0 \implies \eta' B' \bar{X} B = 0$$

there are m linearly independent $\eta_i \in \mathbb{R}^n$, viz. the columns of $\bar{X} B$, such that $\eta_i' (A + BK) = 0$, implying that

$$\text{— } A + BK \text{ has at least } m \text{ eigenvalues at the origin (deadbeat modes).}$$

Discrete-time return difference equality

Using the key relation, it can be shown that

$$(I + L'(z^{-1}))(R + B'\bar{X}B)(I + L(z)) = \begin{bmatrix} P'_x(z^{-1}) & I \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} P_x(z) \\ I \end{bmatrix}$$

where $L(z) = -K(zI - A)^{-1}B$ and $P_x(z) = (zI - A)^{-1}B$. If $m = 1$, $R = r$, and $S = 0$, then under $z = e^{j\theta}$

$$|1 + L(e^{j\theta})|^2 = \frac{r}{r + B'\bar{X}B} + \frac{[P_x(e^{j\theta})]'QP_x(e^{j\theta})}{r + B'\bar{X}B} \geq \overbrace{\frac{r}{r + B'\bar{X}B}}^{< 1 \text{ unless } \bar{X}B = 0}$$

and we can no longer guarantee that $|1 + L(e^{j\theta})| \geq 1$ for all θ (in fact, it is not). As a result, unless $\bar{X}B = 0$ (implying $K = 0$ if $S = 0$)

- there are frequencies θ where the sensitivity magnitude $|S(e^{j\theta})| > 1$
- continuous-time LQR margins ($\mu_g = \infty / \mu_{ph} \geq 60^\circ$) cannot be attained
perhaps expectably, discrete-time strictly proper systems cannot have $\mu_g = \infty$

Yet another LQR problem

Consider

$$x[t+1] = Ax[t] + Bu[t], \quad x[0] = x_0$$

for given $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Suppose it

- operates over the interval $\mathbb{Z}_{0..\eta}$ for a given $\eta \in \mathbb{N}$ (aka the horizon).

The goal is to find u that

- minimizes

$$\mathcal{J} = x'[\eta]Q_\eta x[\eta] + \sum_{t=0}^{\eta-1} \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix}.$$

for conventional Q , R , and S and some $Q_\eta = Q'_\eta \geq 0$.

Unusual:

- finite horizon $\implies$ no stability requirements
- terminal penalty $x'[\eta]Q_\eta x[\eta]$

Difference Riccati equation (DDRE)

For every $t \in \mathbb{Z}_{0..\eta-1}$ define the sequence of matrices, running back in time,

$$\begin{aligned}\bar{X}[t] = & A' \bar{X}[t+1] A + Q - (S + A' \bar{X}[t+1] B) \\ & \times (R + B' \bar{X}[t+1] B)^{-1} (S' + B' \bar{X}[t+1] A), \quad \bar{X}[\eta] = \bar{X}_\eta\end{aligned}$$

If $\mathcal{A}_{1-3}$ hold and $\bar{X}_\eta \geq 0$, then

$$\lim_{t \rightarrow -\infty} \bar{X}[t] = \bar{X}_\infty$$

where $\bar{X}_\infty$ is the stabilizing solution to the DARE

$$\bar{X}_\infty = A' \bar{X}_\infty A + Q - (S + A' \bar{X}_\infty B)(R + B' \bar{X}_\infty B)^{-1} (S' + B' \bar{X}_\infty A).$$

Moreover,

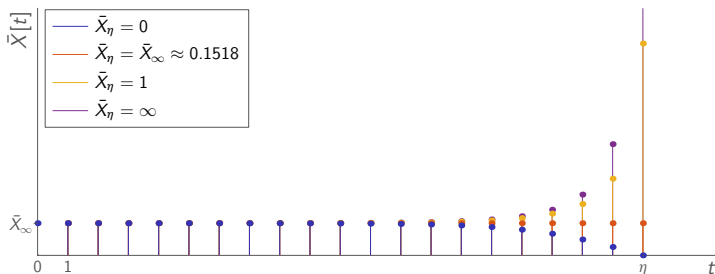
- if horizon is very long, then $\bar{X}[t] \approx \bar{X}_\infty$ for almost all t
- $\bar{X}_\eta = \bar{X}_\infty \implies \bar{X}[t] = \bar{X}_\infty$ for all t

DDRE: example

If $A = a$, $B = 1$, $Q = q \geq 0$, $S = 0$, and $R = r \geq 0$, then the DDRE reads

$$\bar{X}[t] = q + \frac{ra^2\bar{X}[t+1]}{r + \bar{X}[t+1]}.$$

For $a = r = 1$, $q = 0.02$, and the horizon $\eta = 20$, different $\bar{X}_\eta$ yield



Finite-horizon LQR: solution

Theorem 12.4

If $R + B'Q_\eta B > 0$, then the unique controller minimizing

$$\mathcal{J} = x'[\eta]Q_\eta x[\eta] + \sum_{t=0}^{\eta-1} \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix}$$

is

$$u[t] = -(R + B'\bar{X}[t+1]B)^{-1}(S' + B'\bar{X}[t+1]A)x[t],$$

where $\bar{X}[t] = \bar{X}'[t] \geq 0$ is the solution to the difference ARE

$$\begin{aligned} \bar{X}[t] &= A'\bar{X}[t+1]A + Q - (S + A'\bar{X}[t+1]B) \\ &\quad \times (R + B'\bar{X}[t+1]B)^{-1}(S' + B'\bar{X}[t+1]A), \quad \bar{X}[\eta] = Q_\eta \end{aligned}$$

The optimal cost is $\mathcal{J}_{opt} = x'_0\bar{X}[0]x_0$ then.

Proof

The relation

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} \bar{X} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \bar{X} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} -K' \\ I \end{bmatrix} (R + B' \bar{X} B) \begin{bmatrix} -K & I \end{bmatrix}$$

for the DARE solution $\bar{X}$ has a counterpart for DDRE, in the form

$$\begin{aligned} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} &= \begin{bmatrix} \bar{X}[t] & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \bar{X}[t+1] \begin{bmatrix} A & B \end{bmatrix} \\ &\quad + \begin{bmatrix} -K'[t] \\ I \end{bmatrix} (R + B' \bar{X}[t+1] B) \begin{bmatrix} -K[t] & I \end{bmatrix} \end{aligned}$$

where

$$K[t] := -(R + B' \bar{X}[t+1] B)^{-1} (S' + B' \bar{X}[t+1] A).$$

It holds for all $t \in \mathbb{Z}_{0..\eta-1}$.

Proof (contd)

The quadratic form in the cost $\mathcal{J}$ is

$$\begin{aligned}
 & \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix} \\
 &= \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} -K'[t] \\ I \end{bmatrix} (R + B' \bar{X}[t+1]B) \begin{bmatrix} -K[t] & I \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix} \\
 &\quad + \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \left(\begin{bmatrix} \bar{X}[t] & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \bar{X}[t+1] \begin{bmatrix} A & B \end{bmatrix} \right) \begin{bmatrix} x[t] \\ u[t] \end{bmatrix} \\
 &= (u[t] - K[t]x[t])'(R + B' \bar{X}[t+1]B)(u[t] - K[t]x[t]) \\
 &\quad + x'[t] \bar{X}[t] x[t] - x'[t+1] \bar{X}[t+1] x[t+1] \\
 &= \|(R + B' \bar{X}[t+1]B)^{1/2}(u[t] - K[t]x[t])\|^2 \\
 &\quad + x'[t] \bar{X}[t] x[t] - x'[t+1] \bar{X}[t+1] x[t+1]
 \end{aligned}$$

Proof (contd)

Thus,

$$\begin{aligned}\mathcal{J} &= x'[\eta]Q_\eta x[\eta] + \sum_{t=0}^{\eta-1} \left(\|(R + B'\bar{X}[t+1]B)^{1/2}(u[t] - K[t]x[t])\|^2 \right. \\ &\quad \left. + x'[t]\bar{X}[t]x[t] - x'[t+1]\bar{X}[t+1]x[t+1] \right) \\ &= x'[\eta]Q_\eta x[\eta] + x'_0\bar{X}[0]x_0 - x'[\eta]\bar{X}[\eta]x[\eta] \\ &\quad + \sum_{t=0}^{\eta-1} \|(R + B'\bar{X}[t+1]B)^{1/2}(u[t] - K[t]x[t])\|^2 \\ &= x'_0\bar{X}[0]x_0 + \sum_{t=0}^{\eta-1} \|(R + B'\bar{X}[t+1]B)^{1/2}(u[t] - K[t]x[t])\|^2 \geq x'_0\bar{X}[0]x_0\end{aligned}$$

Therefore, the unique control law minimizing $\mathcal{J}$ is $u[t] = K[t]x[t]$ and the optimal cost is the lower bound. □

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Discrete LQR

Discrete-time state observers and observer-based feedback

Case study 1

Case study 2

State reconstruction via full-order observers

Consider

$$\begin{cases} x[t+1] = Ax[t] + Bu[t], & x[0] = x_0 \\ y[t] = Cx[t] \end{cases}$$

To reconstruct x from the measurements y and u , construct the observer

$$\hat{x}[t+1] = A\hat{x}[t] + Bu[t] - L(y[t] - C\hat{x}[t]), \quad \hat{x}[0] = \hat{x}_0$$

The error $\epsilon = x - \hat{x}$ satisfies

$$\epsilon[t+1] = (A + LC)\epsilon[t], \quad \epsilon[0] = x_0 - \hat{x}_0$$

and decays asymptotically whenever $A + LC$ is Schur. Thus, there is a

- complete analogy with continuous-time observer results

and the design of L follows the same logic (just replace Hurwitz with Schur, Lyapunov inequality with discrete-time Lyapunov inequality, etc). Similar to the state-feedback case, we can have deadbeat response (finite convergence) via assigning $\text{spec}(A + LC) = \{0\}$.

State is a key

Like in the continuous-time case,

- estimating $z = C_z x$ goes via estimating x and generating $\hat{z} = C_z \hat{x}$.

Assume that discrete dynamics is derived via discretizing $\dot{x} = Ax + Bu$ with sampling period h , so that $\bar{x}[i+1] = \bar{A}\bar{x}[i] + \bar{B}\bar{u}[i]$. The sampled derivative of x , $\bar{\dot{x}}$ satisfies

$$\bar{\dot{x}}[i] = A\bar{x}[i] + B\bar{u}[i-1]$$

then (because we assume that $u(t) = \bar{u}[i]$ for $ih < t \leq (i+1)h$). Hence,

- estimating sampled $\bar{x}$ allows us to reconstruct $\bar{\dot{x}}$ as well.

Given a sampling instance s_i ,

$$x(s_i + \tau) = e^{A\tau} \bar{x}[i] + \int_0^\tau e^{A s} ds B \bar{u}[i] = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \exp \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix}$$

where $\bar{x}[i] = x(s_i)$. Hence,

- estimating sampled $\bar{x}$ allows us to reconstruct the analog x too.

State is a key

Like in the continuous-time case,

- estimating $z = C_z x$ goes via estimating x and generating $\hat{z} = C_z \hat{x}$.

Assume that discrete dynamics is derived via discretizing $\dot{x} = Ax + Bu$ with sampling period h , so that $\bar{x}[i+1] = \bar{A}\bar{x}[i] + \bar{B}\bar{u}[i]$. The sampled derivative of x , $\bar{\dot{x}}$ satisfies

$$\bar{\dot{x}}[i] = A\bar{x}[i] + B\bar{u}[i-1]$$

then (because we assume that $u(t) = \bar{u}[i]$ for $ih < t \leq (i+1)h$). Hence,

- estimating sampled $\bar{x}$ allows us to reconstruct $\bar{\dot{x}}$ as well.

Given a sampling instance s_i ,

$$x(s_i + \tau) = e^{A\tau} \bar{x}[i] + \int_0^\tau e^{A(\tau-s)} B \bar{u}[i] ds$$

where $\bar{x}[i] = x(s_i)$. Hence,

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Given a sampling instance s_i ,

$$x(s_i + \tau) = e^{A\tau} \bar{x}[i] + \int_0^\tau e^{As} ds B \bar{u}[i] = \begin{bmatrix} I & 0 \end{bmatrix} \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tau\right) \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix}$$

where $\bar{x}[i] = x(s_i)$. Hence,

- estimating sampled $\bar{x}$ allows us to reconstruct the analog x too.

Delay-free state observer

The full-order observer is strictly causal (at least one-step delay). Consider

$$\begin{cases} x_o[t+1] = Ax_o[t] + Bu[t] - L(y[t] - Cx_o[t]), & x_o[0] = x_{o,0} \\ \hat{x}[t] = x_o[t] - M(y[t] - Cx_o[t]) \end{cases}$$

for an arbitrary $M \in \mathbb{R}^{n \times p}$ offering extra degree of freedom. This system is delay free and the error dynamics, with $\epsilon_o := x - x_o$, are

$$\begin{cases} \epsilon_o[t+1] = (A + LC)\epsilon_o[t], & \epsilon_o[0] = x_0 - x_{o,0} \\ \epsilon[t] = (I + MC)\epsilon_o[t] \end{cases}$$

so that

$$\epsilon[t] = (I + MC)(A + LC)^t(x_0 - x_{o,0})$$

decays asymptotically whenever $A + LC$ is Schur. The extra parameter, M ,

- cannot affect the decay rate of the whole ϵ unless $\text{rank } C = n$
- can affect error in certain directions

Delay-free state observer (contd)

Let

$$M = -PC'(CPC')^{-1}$$

for any P such that $\det(CPC') \neq 0$. In this case $CM = -I$, so that

$$\hat{x} = x_0 - M(y - Cx_0) \implies C\hat{x} = y \quad \text{and} \quad C\epsilon = 0.$$

In other words, the measured part of x , Cx , is reconstructed perfectly, like in the reduced-order observer case.

Remark. In fact, we can always choose P to render p eigenvalues of $A + LC$ unobservable (and thus non-minimal) in the realization of the observer

$$G_{\hat{x}y}(z) = \left[\begin{array}{c|c} A + LC & -L \\ \hline I + MC & -M \end{array} \right].$$

offering another way to design a reduced-order observer. Indeed, in this case (cf. PBH)

$$\left[\begin{array}{c} A + LC - zI \\ I - PC'(CPC')^{-1}C \end{array} \right] PC'\eta = \left[\begin{array}{c} (A + LC - zI)PC'\eta \\ 0 \end{array} \right] = 0$$

if P is such that the columns of PC' span the spectral subspace of $A + LC$ of interest.

Linear-quadratic state estimator: formulation

Let

$$\begin{cases} x[t+1] = Ax[t] + B_w w[t] + B_u u[t], & x[0] = 0 \\ y[t] = Cx[t] + D_w w[t] \end{cases}$$

We need to reconstruct the state x from the measurements y and u so that the reconstructed state $\hat{x}$ is as “close” to x as possible. The linear-quadratic approach is to measure the closeness in terms of the cost function

$$\mathcal{J} = \sum_{i=1}^{m_w} \sum_{t=1}^{\infty} \|x[t] - \hat{x}_i[t]\|^2,$$

where $\hat{x}_i := \hat{x}$ under $w = e_i \delta$, where e_i is the i th standard basis of $\mathbb{R}^{m_w}$ and δ is the Kronecker delta (unit-intensity white signal). Assumptions:

$\mathcal{A}_4$: (C, A) is detectable

$\mathcal{A}_5$: $\begin{bmatrix} A - \lambda I & B_w \\ C & D_w \end{bmatrix}$ has full row rank for all $\lambda \in \{z \in \mathbb{C} \mid |z| = 1\}$

and we do not require $D_w D_w' > 0$, i.e. that measurement channels are noisy.

Linear-quadratic state estimator: causal solution

Theorem 12.5

If $\mathcal{A}_{4,5}$ hold, then the unique causal optimal estimator is

$$\begin{cases} x_o[t+1] = Ax_o[t] + B_u u[t] - L(y[t] - Cx_o[t]), & x_o[0] = 0 \\ \hat{x}[t] = x_o[t] - M(y[t] - C\hat{x}[t]) \end{cases}$$

where

$$\begin{bmatrix} L \\ M \end{bmatrix} = - \begin{bmatrix} B_w D'_w + A\bar{Y}C' \\ \bar{Y}C' \end{bmatrix} (D_w D'_w + C\bar{Y}C')^{-1}$$

and $\bar{Y} = \bar{Y}' \geq 0$ is the stabilizing solution to the DARE

$$\bar{Y} = A\bar{Y}A' + B_w B'_w - (B_w D'_w + A\bar{Y}C')(D_w D'_w + C\bar{Y}C')^{-1}(D_w B'_w + C\bar{Y}A')$$

The optimal cost $\mathcal{J}_{opt} = \text{tr}(\bar{Y} - \bar{Y}C'(D_w D'_w + C\bar{Y}C')^{-1}C\bar{Y})$.

This solution is known as the (steady-state) **Kalman filter** and remains the same if instead of x we optimally reconstruct $C_z x$.

Causal solution under $D_w = 0$ (no measurement noise)

In this case the transfer function of the Kalman filter is

$$G_{\hat{x}y}(z) = \left[\begin{array}{c|c} A(I - \bar{Y}C'(C\bar{Y}C')^{-1}C) & A\bar{Y}C'(C\bar{Y}C')^{-1} \\ \hline I - \bar{Y}C'(C\bar{Y}C')^{-1}C & \bar{Y}C'(C\bar{Y}C')^{-1} \end{array} \right].$$

Because $\text{rank } C = p$ (otherwise $C\bar{Y}C'$ cannot be nonsingular),

$$C(I - \bar{Y}C'(C\bar{Y}C')^{-1}C) = 0 \implies \text{rank}(I - \bar{Y}C'(C\bar{Y}C')^{-1}C) = n - p.$$

Hence, there exist full-rank matrices $U \in \mathbb{R}^{n \times (n-p)}$ and $V \in \mathbb{R}^{(n-p) \times n}$ such that $I - \bar{Y}C'(C\bar{Y}C')^{-1}C = UV$ (Lecture 1, Slide 9) and

$$G_{\hat{x}y}(z) = \left[\begin{array}{c|c} AUV & A\bar{Y}C'(C\bar{Y}C')^{-1} \\ \hline UV & \bar{Y}C'(C\bar{Y}C')^{-1} \end{array} \right] = \left[\begin{array}{c|c} VAU & VA\bar{Y}C'(C\bar{Y}C')^{-1} \\ \hline U & \bar{Y}C'(C\bar{Y}C')^{-1} \end{array} \right]$$

(because $V(sI - AUV)^{-1} = (sI - VAU)^{-1}V$, check it). Thus,

no measurement noise $\implies$ the order of Kalman filter is at most $n - p$ (the dimension of VAU), making it a reduced-order observer.

Causal solution under $D_w = 0$ (no measurement noise)

In this case the transfer function of the Kalman filter is

$$G_{\hat{x}y}(z) = \left[\frac{A(I - \bar{Y}C'(C\bar{Y}C')^{-1}C)}{I - \bar{Y}C'(C\bar{Y}C')^{-1}C} \mid \frac{A\bar{Y}C'(C\bar{Y}C')^{-1}}{\bar{Y}C'(C\bar{Y}C')^{-1}} \right].$$

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$$G_{\hat{x}y}(z) = \left[\frac{AUV}{UV} \mid \frac{A\bar{Y}C'(C\bar{Y}C')^{-1}}{\bar{Y}C'(C\bar{Y}C')^{-1}} \right] = \left[\frac{VAU}{U} \mid \frac{VA\bar{Y}C'(C\bar{Y}C')^{-1}}{\bar{Y}C'(C\bar{Y}C')^{-1}} \right]$$

(because $V(sl - AUV)^{-1} = (sl - VAU)^{-1}V$, check it). Thus,

no measurement noise $\implies$ the order of Kalman filter is at most $n - p$ (the dimension of VAU), making it a reduced-order system.

Causal solution under $D_w = 0$ (no measurement noise)

In this case the transfer function of the Kalman filter is

$$G_{\hat{x}y}(z) = \left[\frac{A(I - \bar{Y}C'(C\bar{Y}C')^{-1}C)}{I - \bar{Y}C'(C\bar{Y}C')^{-1}C} \mid \frac{A\bar{Y}C'(C\bar{Y}C')^{-1}}{\bar{Y}C'(C\bar{Y}C')^{-1}} \right].$$

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(because $V(sI - AUV)^{-1} = (sI - VAU)^{-1}V$, check it). Thus,

— no measurement noise $\implies$ the order of Kalman filter is at most $n - p$ (the dimension of VAU), making it a **reduced-order observer**.

Linear-quadratic state estimator: strictly causal solution

Theorem 12.6

If $\mathcal{A}_{4,5}$ hold, then the unique strictly causal optimal estimator is

$$\hat{x}[t+1] = A\hat{x}[t] + B_u u[t] - L(y[t] - C\hat{x}[t]), \quad \hat{x}[0] = 0$$

where

$$L = -(B_w D'_w + A\bar{Y}C')(D_w D'_w + C\bar{Y}C')^{-1}$$

and $\bar{Y} = \bar{Y}' \geq 0$ is the stabilizing solution to the DARE

$$\bar{Y} = A\bar{Y}A' + B_w B'_w - (B_w D'_w + A\bar{Y}C')(D_w D'_w + C\bar{Y}C')^{-1}(D_w B'_w + C\bar{Y}A')$$

The optimal cost $\mathcal{J}_{opt} = \text{tr}(\bar{Y}) \geq \text{tr}(\bar{Y} - \bar{Y}C'(D_w D'_w + C\bar{Y}C')^{-1}C\bar{Y})$.

This solution is known as the (steady-state) **Kalman predictor**, remains the same if instead of x we optimally reconstruct $C_z x$, and has deadbeat modes whenever D_w does not have full row rank.

Observer-based controller

With $u = K\hat{x} + v$, the closed-loop dynamics are

$$\begin{bmatrix} x[t+1] \\ x_o[t+1] \end{bmatrix} = \begin{bmatrix} A - BKMC & BK(I + MC) \\ -LC - BKMC & A_L + BK(I + MC) \end{bmatrix} \begin{bmatrix} x[t] \\ x_o[t] \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v[t]$$

The change of variables $x_o \rightarrow \epsilon_o = x - x_o$ results in

$$\begin{bmatrix} x[t+1] \\ \epsilon_o[t+1] \end{bmatrix} = \begin{bmatrix} A + BK & -BK(I + MC) \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x[t] \\ \epsilon_o[t] \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v[t]$$

and we return to the

- separation between state-feedback and observer error dynamics

in the closed-loop behavior. The gain M affects only $BK(I + MC)$, which is the effect of the estimation error ϵ_o on x .

Outline

Discrete LQR

Discrete-time state observers and observer-based feedback

Case study 1

Case study 2

Plant and control architecture

Consider an LTI plant P having the transfer function

$$P(s) = \frac{1}{s(s+1)},$$

(can be thought of as a voltage controlled DC motor). Its companion form

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

with the state $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$. Note that the state feedback

$$u = Kx = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1 y + k_2 \dot{y} =: k_p y + k_d \dot{y}$$

is a PD controller in this case.

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Requirements and LQR cost function

We would like to have

- fast convergence of $y \implies$ penalize $\|y\|_2$
- smooth convergence of $y \implies$ penalize $\|\dot{y}\|_2$
- small control signal $u \implies$ penalize $\|u\|_2$

Cost

$$J = \int_0^{\infty} (\lambda y^2(t) + (1 - \lambda) \dot{y}^2(t) + r u^2(t)) dt$$

for some weights $\lambda \in [0, 1]$ and $r > 0$ ($\lambda \uparrow 1$ encourages faster convergence, $\lambda \downarrow 0$ encourages smoother convergence, while the increase of r discourages large control energy).

Taking into account that $y = x_1$ and $\dot{y} = x_2$, we get

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & r \end{bmatrix}.$$

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Taking into account that $y = x_1$ and $\dot{y} = x_2$, we get

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \left[\begin{array}{cc|c} \lambda & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ \hline 0 & 0 & r \end{array} \right].$$

Some analytic formulae

The stabilizing

$$\bar{X} = \begin{bmatrix} \sqrt{\lambda} \alpha & \sqrt{\lambda r} \\ \sqrt{\lambda r} & \sqrt{r}(\alpha - \sqrt{r}) \end{bmatrix} > 0,$$

where $\alpha := \sqrt{1 - \lambda + 2\sqrt{\lambda r} + r} > \sqrt{r}$. The gain (PD):

$$K = -\sqrt{\frac{\lambda}{r}} \begin{bmatrix} 1 & (\alpha - \sqrt{r})/\sqrt{\lambda} \end{bmatrix} \implies \begin{cases} k_p = -\sqrt{\lambda/r} \\ k_d = -\alpha/\sqrt{r} + 1 \end{cases}$$

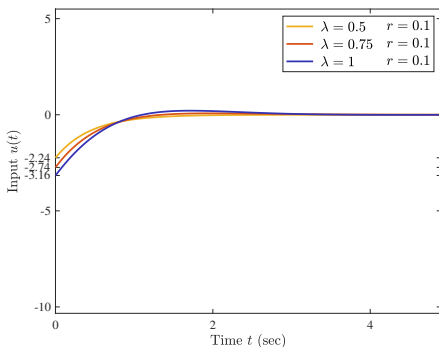
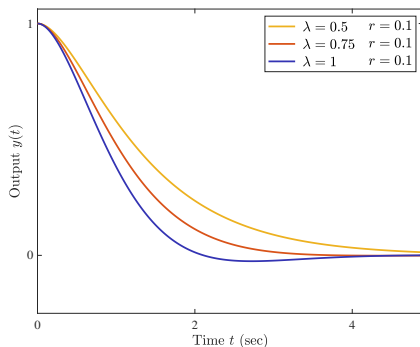
If $\dot{y}(0) = 0$, then $u(0) = -\sqrt{\lambda/r}$, increases as λ increases and r decreases.
Closed-loop eigenvalues:

$$\frac{-\alpha \pm \sqrt{1 - \lambda - 2\sqrt{\lambda r} + r}}{2\sqrt{r}}.$$

They are real iff $\lambda \leq 1 + 3r - 2\sqrt{2r^2 + r}$, which is always true if $r \geq 4$ or $\lambda \leq 0.5$. If $\lambda = 1$, then the damping factor $\zeta = \sqrt{2 + \sqrt{r}}/2 > 1/\sqrt{2}$.

Simulations of analog controller

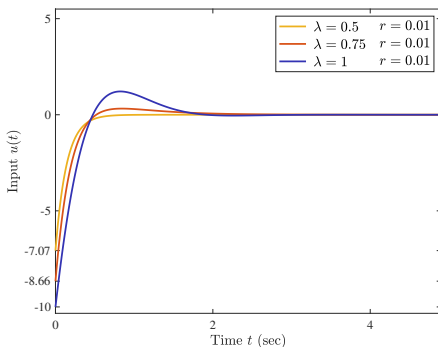
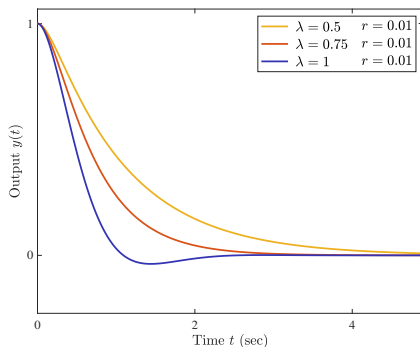
With (large) $r = 0.1$:



- under $\lambda = 1$ the response is faster, but overshoots (complex poles)
- as λ decreases, the response slows down, but smooths
- control effort decreases as λ decreases (always peaks at $-\sqrt{\lambda/r}$)

Simulations of analog controller (contd)

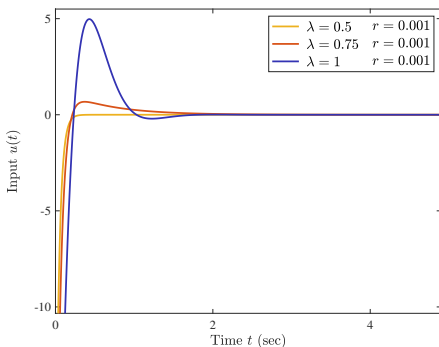
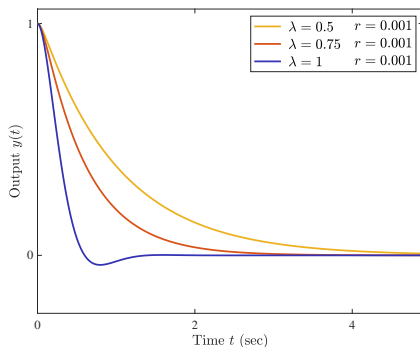
With (middle) $r = 0.01$:



- under $\lambda = 1$ the response is faster, but overshoots (complex poles)
- as λ decreases, the response slows down, but smooths
- control effort decreases as λ decreases (always peaks at $-\sqrt{\lambda/r}$)

Simulations of analog controller (contd)

With (small) $r = 0.001$:



- under $\lambda = 1$ the response is faster, but overshoots (complex poles)
- as λ decreases, the response slows down, but smooths
- control effort decreases as λ decreases (always peaks at $-\sqrt{\lambda/r}$)

Discretization

The discretized plant has the realization

$$\begin{cases} x[t+1] = \begin{bmatrix} 1 & 1 - e^{-h} \\ 0 & e^{-h} \end{bmatrix} x[t] + \begin{bmatrix} h - 1 + e^{-h} \\ 1 - e^{-h} \end{bmatrix} u[t] \\ y[t] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[t] \end{cases}$$

and its state has $x_1[i] = y(ih)$ and $x_2[i] = \dot{y}(ih)$. Hence, keeping

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & r \end{bmatrix}$$

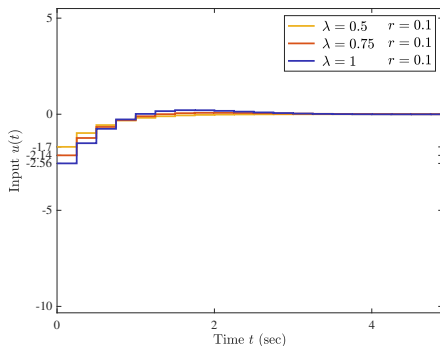
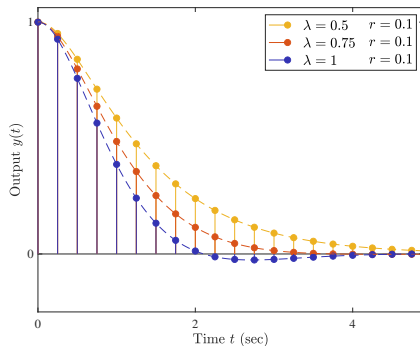
yields

$$\mathcal{J} = \sum_{i=0}^{\infty} (\lambda y^2(ih) + (1 - \lambda) \dot{y}^2(ih) + r u^2[i])$$

which has the advantage of being physical.

Simulations of digital controller

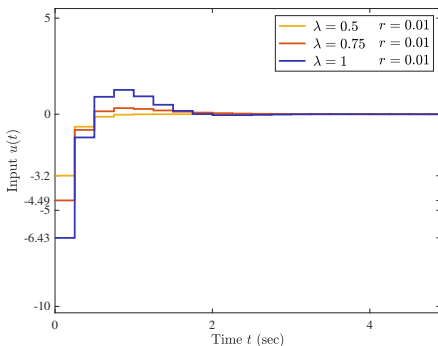
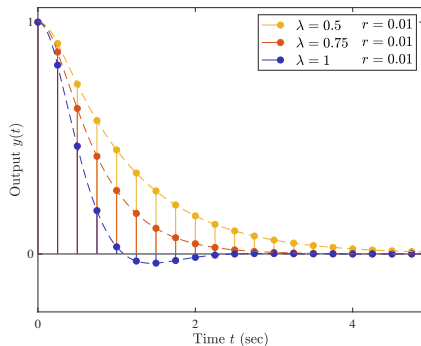
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- under $\lambda = 1$ the response is faster, but overshoots (complex poles)
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- intersample behavior is as expected

Simulations of digital controller (contd)

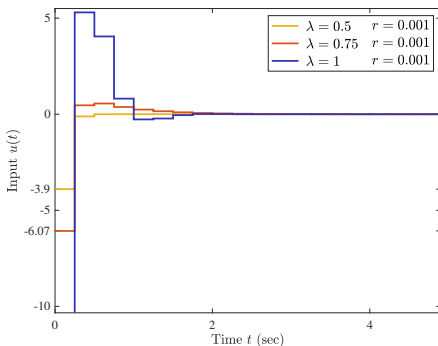
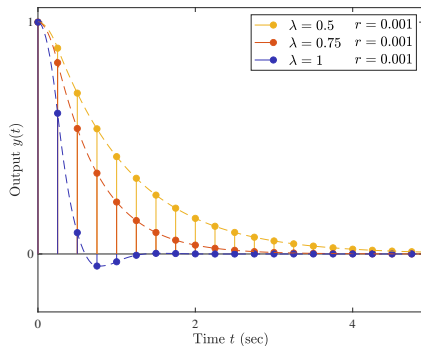
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Simulations of digital controller (contd)

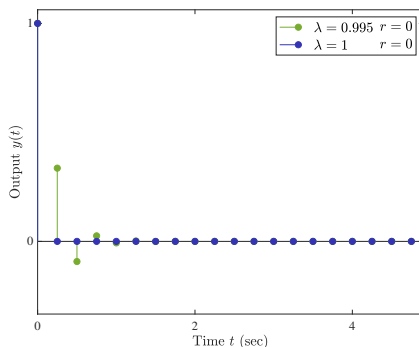
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- intersample behavior is as expected

Simulations of digital controller (contd)

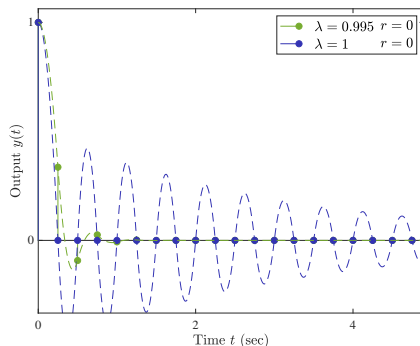
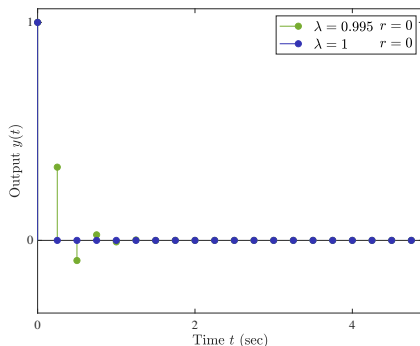
With $r = 0$:



- if $\dot{y}$ is not penalized, we have the deadbeat discrete response
 - the intersample behavior is poor (hidden oscillations)
 - oscillations are invisible in $\mathcal{J}$, so that it pays off to place pole at the sampling zero
 - even a tiny penalty of $\dot{y}$ changes that (the deadbeat $\hat{y}(ih)$ is huge)

Simulations of digital controller (contd)

With $r = 0$:



- if $\dot{y}$ is not penalized, we have the deadbeat discrete response
- the intersample behavior is poor (hidden oscillations)
oscillations are invisible in $\mathcal{J}$, so that it pays off to place pole at the sampling zero
- even a tiny penalty of $\dot{y}$ changes that (the deadbeat $\dot{y}(ih)$ is huge)

Finite-horizon formulation

The same discretized DC motor, under the same sampling period,

$$\begin{cases} x[t+1] = \begin{bmatrix} 1 & 1 - e^{-h} \\ 0 & e^{-h} \end{bmatrix} x[t] + \begin{bmatrix} h - 1 + e^{-h} \\ 1 - e^{-h} \end{bmatrix} u[t] \\ y[t] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[t] \end{cases}$$

and the same cost weights,

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \left[\begin{array}{cc|c} \lambda & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ \hline 0 & 0 & r \end{array} \right].$$

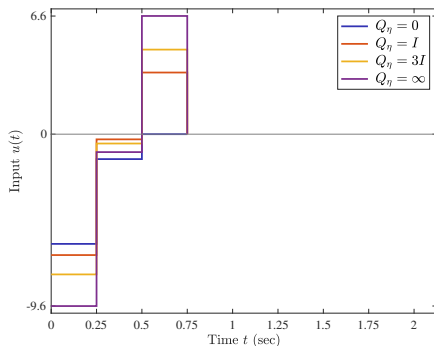
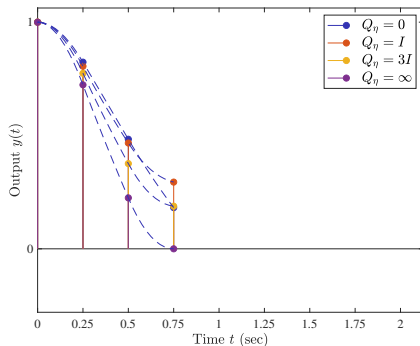
But now

$$\mathcal{J} = x'[\eta] Q_{\eta} x[\eta] + \sum_{t=0}^{\eta-1} (\lambda y^2(ih) + (1 - \lambda) \dot{y}^2(ih) + r u^2[i])$$

(remember, $x_1[i] = y(ih)$ and $x_2[i] = \dot{y}(ih)$).

Finite-horizon simulations

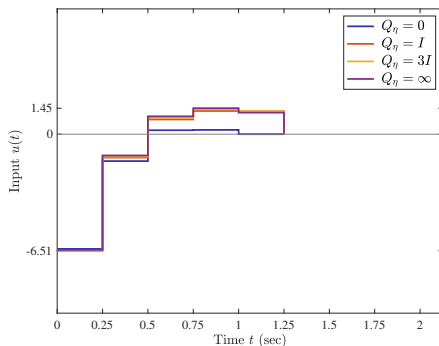
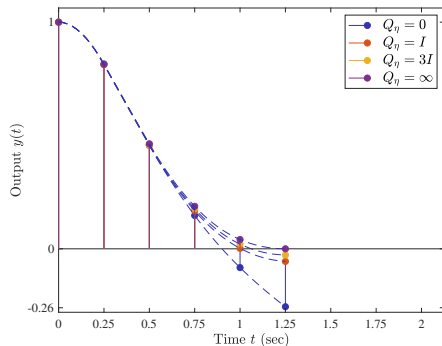
With $\lambda = 1$, $r = 0.01$, and $\eta = 3$:



- larger Q_η reduces $x[\eta]$
- with $Q_\eta = \infty$ end up with deadbeat behavior

Finite-horizon simulations (contd)

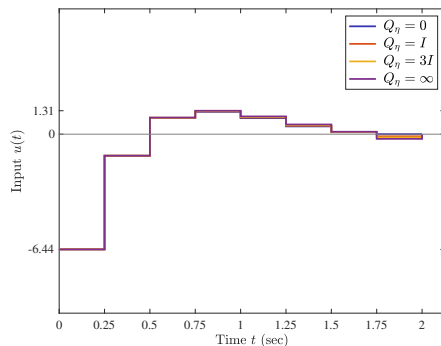
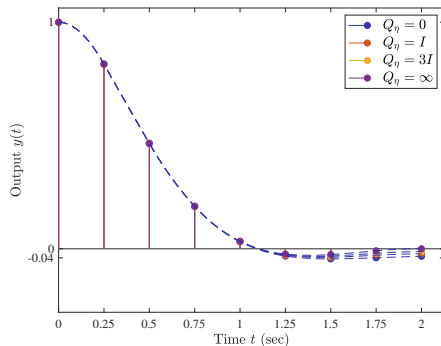
With $\lambda = 1$, $r = 0.01$, and $\eta = 5$:



- larger Q_η reduces $x[\eta]$
- with $Q_\eta = \infty$ end up with deadbeat behavior

Finite-horizon simulations (contd)

With $\lambda = 1$, $r = 0.01$, and $\eta = 8$:



- larger Q_η reduces $x[\eta]$
- with $Q_\eta = \infty$ end up with deadbeat behavior
- as η increases, approach infinite-horizon behavior

Outline

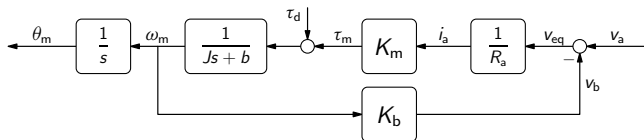
Discrete LQR

Discrete-time state observers and observer-based feedback

Case study 1

Case study 2

DC motor with constant torque disturbance



Model:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -(b + K_m^2/R_a)/J \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} 0 \\ 1/J \end{bmatrix}}_{B_d} \tau_d + \underbrace{\begin{bmatrix} 0 \\ K_m/(JR_a) \end{bmatrix}}_B v_a$$

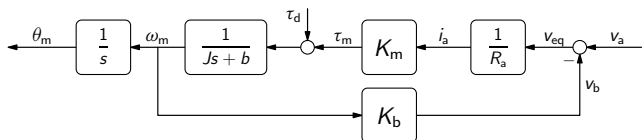
with measurable

$$x = \begin{bmatrix} \theta_m \\ \dot{\theta}_m \end{bmatrix} = \begin{bmatrix} \theta_m \\ \omega_m \end{bmatrix}.$$

Note, $B_d = BM_d$ for $M_d = R_a/K_m \implies$ matched disturbance. Disturbance is supposed to be constant, which suggest using the exosystem

$$\dot{\tau}_d = 0, \quad \tau_d(0) = \tau_0$$

DC motor with constant torque disturbance (contd)



Augmented system:

$$\begin{cases} \dot{x}_{\text{aug}} = \underbrace{\begin{bmatrix} A & B_d \\ 0 & 0 \end{bmatrix}}_{A_{\text{aug}}} x_{\text{aug}} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_{\text{aug}}} v_a, & x_{\text{aug}}(0) = \begin{bmatrix} x_0 \\ \tau_0 \end{bmatrix} \\ x = \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{C_{\text{aug}}} x_{\text{aug}} \end{cases}$$

with $x_{\text{aug}} = \begin{bmatrix} x \\ \tau_d \end{bmatrix}$. Choosing $e = x$, the regulator equation

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} + \begin{bmatrix} B_d \\ 0 \end{bmatrix} = 0 \implies \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} = \begin{bmatrix} 0 \\ -M_d \end{bmatrix}$$

(cf. Lecture 8, Slide 31) and the full-information $v_a = Kx - M_d \tau_d$, naturally.

Reduced-order observer

Because we measure x , we may opt for a reduced-order observer for τ_d . In the notation of Lecture 7, we need to reconstruct

$$z = \tau_d = \begin{bmatrix} 0 & 1 \end{bmatrix} x_{\text{aug}} \implies C_z = \begin{bmatrix} 0 & 1 \end{bmatrix} \implies \begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} 0 & I_2 \\ 1 & 0 \end{bmatrix}$$

The reduced-order observer

$$\begin{cases} \dot{x}_o = (C_z + LC_{\text{aug}})A_{\text{aug}}T_1x_o + (C_z + LC_{\text{aug}})(A_{\text{aug}}(T_2 - T_1L)x + B_{\text{aug}}u) \\ \hat{\tau}_d = x_o - Lx \end{cases}$$

requires

$$\left. \begin{matrix} C_z A_{\text{aug}} T_1 = 0 \\ C_{\text{aug}} A_{\text{aug}} T_1 = B_d \end{matrix} \right\} \implies \begin{cases} (C_z + LC_{\text{aug}})A_{\text{aug}}T_1 = LB_d \\ (C_z + LC_{\text{aug}})A_{\text{aug}}(T_2 - T_1L) = L(A - B_dL) \\ (C_z + LC_{\text{aug}})B_{\text{aug}} = LB \end{cases}$$

and Hurwitz LB_d .

Observer-based controller

Having the control law $v_a = Kx - M_d \hat{t}_d$ and the observer

$$\begin{cases} \dot{x}_o = LB_d x_o + L(A - B_d L)x + LBv_a \\ \hat{t}_d = x_o - Lx \end{cases}$$

the controller is (remember, $B_d = BM_d$)

$$\begin{cases} \dot{x}_o = LB_d x_o + L(A - B_d L)x + LB(Kx - M_d(x_o - Lx)) = L(A + BK)x \\ v_a = -M_d x_o + (K + M_d L)x \end{cases}$$

whose transfer function

$$R(s) = \left[\begin{array}{c|c} 0 & L(A + BK) \\ \hline -M_d & K + M_d L \end{array} \right] = \overbrace{K + M_d L}^{\text{PD}} + \overbrace{M_d L \underbrace{(-A - BK)}_{sI - A - BK|_{s=0}}}^{\text{PI}} \frac{1}{s}$$

cf.

$$\tilde{K}x = \tilde{k}_1 \theta_m + \tilde{k}_2 \dot{\theta}_m \quad \text{and} \quad \tilde{K} \int x = \tilde{k}_2 \theta_m + \tilde{k}_1 \int \theta_m$$

Closed-loop dynamics

Combining the plant and controller, the closed-loop dynamics are

$$\begin{bmatrix} \dot{x} \\ \dot{\tau}_d \\ \dot{x}_o \end{bmatrix} = \begin{bmatrix} A + BK + B_d L & B_d & -B_d \\ 0 & 0 & 0 \\ L(A + BK) & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \tau_d \\ x_o \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} \dot{\tau}_d \\ \dot{x} \\ \dot{\epsilon}_\tau \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A + BK & B_d \\ 0 & 0 & LB_d \end{bmatrix} \begin{bmatrix} \tau_d \\ x \\ \epsilon_\tau \end{bmatrix}$$

where $\epsilon_\tau := \tau_d + Lx - x_o = \tau_d - \hat{\tau}_d$. As expected,

- the dynamics of τ_d are isolated from the rest of the system
- the error dynamics are stable

Simulation setup

Parameter	K_m [N m/A]	R_a [Ω]	J [kg m ²]	b [N m s/rad]	τ_0 [N m]
Value	0.126	2.08	0.008	0.005	0.235

The state-feedback gain K is designed via LQR, with

$$\mathcal{J} = \int_0^\infty (\lambda \theta_m^2(t) + (1 - \lambda) \dot{\theta}_m^2(t) + r v_a^2(t)) dt$$

for $\lambda \in (0, 1]$ and $r > 0$. In other words,

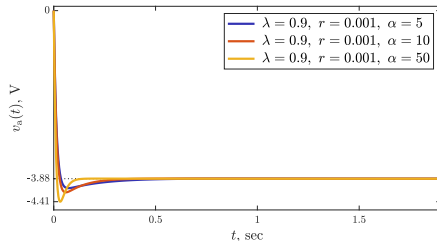
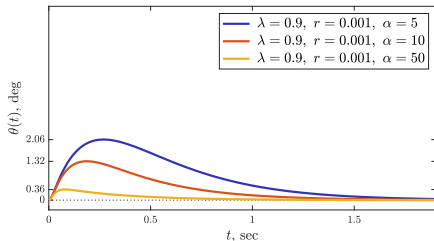
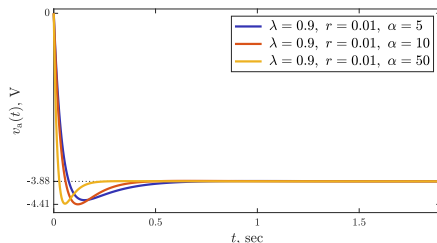
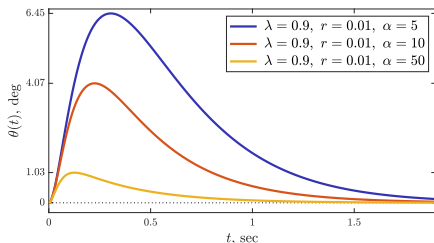
$$Q = \begin{bmatrix} \lambda & 0 \\ 0 & 1 - \lambda \end{bmatrix} \quad \text{and} \quad R = r.$$

The observer gain L is designed as

$$L = -\alpha (B_d' B_d)^{-1} B_d' \implies L B_d = -\alpha$$

for $\alpha > 0$ (larger $\alpha \implies$ faster reconstruction of τ_d).

Simulation results



Note that the

- control amplitude doesn't grow as r decreases (not a regulator problem)

Problem discretization

Assuming that $\tau_d = \mathcal{H}\bar{\tau}_d$ is also piecewise constant (true if $\tau_d = \text{const}$),

$$\begin{bmatrix} A & B_d & B \end{bmatrix} \rightarrow \begin{bmatrix} \bar{A} & \bar{B}_d & \bar{B} \end{bmatrix} = \begin{bmatrix} e^{Ah} & \int_0^h e^{As} ds B_d & \int_0^h e^{As} ds B \end{bmatrix}$$

and we still have $\bar{B}_d = \bar{B}M_d$. Discrete exosystem:

$$\bar{\tau}_d[t+1] = \bar{\tau}_d[t], \quad \bar{\tau}_d[0] = \tau_0$$

generates $\bar{\tau}_d[t] = \tau_0$ for all t . The regulator equation

$$\begin{bmatrix} \bar{A} & \bar{B} \\ I & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} + \begin{bmatrix} \bar{B}_d \\ 0 \end{bmatrix} = 0 \implies \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} = \begin{bmatrix} 0 \\ -M_d \end{bmatrix}$$

and the full-information $\bar{v}_a = K\bar{x} - M_d\bar{\tau}_d$, again.

Reduced-order observer in discrete time

Discrete reduced-order observer can be designed repeating continuous-time steps mechanically. It reads

$$\begin{cases} x_o[t+1] = (C_z + LC_{\text{aug}})A_{\text{aug}}T_1x_o[t] \\ \quad + (C_z + LC_{\text{aug}})(A_{\text{aug}}(T_2 - T_1L)\bar{x}[t] + B_{\text{aug}}\bar{v}_a[t]) \\ \hat{\tau}_d[t] = x_o[t] - L\bar{x}[t] \end{cases}$$

where

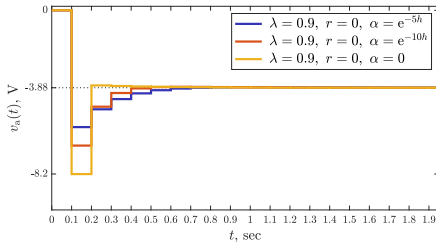
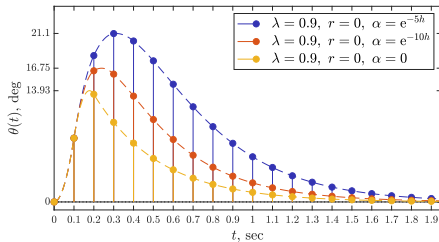
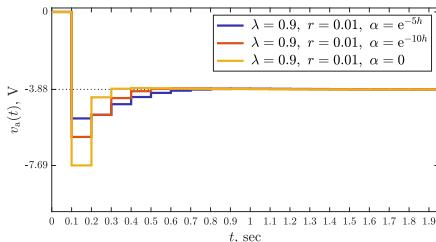
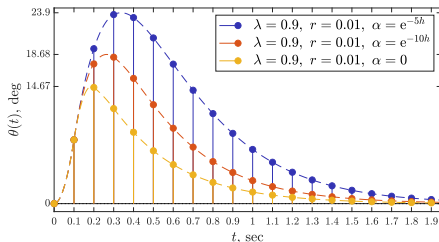
$$A_{\text{aug}} = \begin{bmatrix} \bar{A} & \bar{B}_d \\ 0 & 1 \end{bmatrix}, \quad B_{\text{aug}} = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix}, \quad C_{\text{aug}} = \begin{bmatrix} I & 0 \end{bmatrix}$$

In this case $C_z A_{\text{aug}} T_1 = 1$ and $C_{\text{aug}} A_{\text{aug}} T_1 = \bar{B}_d$, so that

$$\begin{cases} (C_z + LC_{\text{aug}})A_{\text{aug}}T_1 = 1 + L\bar{B}_d \\ (C_z + LC_{\text{aug}})A_{\text{aug}}(T_2 - T_1L) = L(\bar{A} - \bar{B}_dL - I) \\ (C_z + LC_{\text{aug}})B_{\text{aug}} = L\bar{B} \end{cases}$$

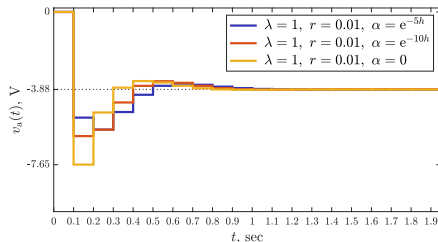
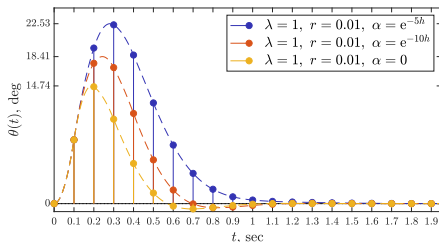
and $1 + L\bar{B}_d$ must be Schur.

Simulation results with $1 + LB_d = \alpha$ and $\lambda = 0.9$



In the discrete case, smaller $|\alpha| \implies$ faster reconstruction of τ_d . Selecting $\alpha = 0$ yields deadbeat (one step) reconstruction.

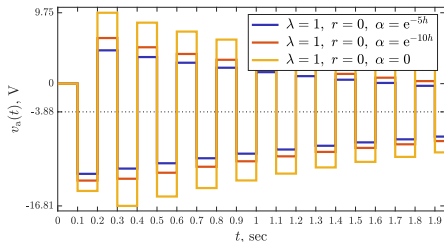
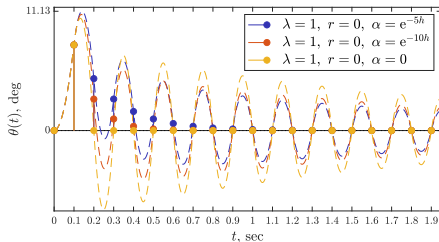
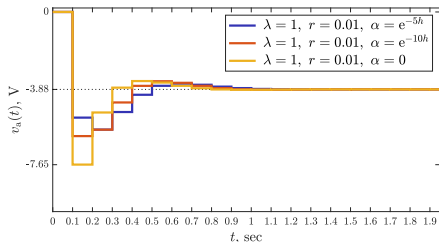
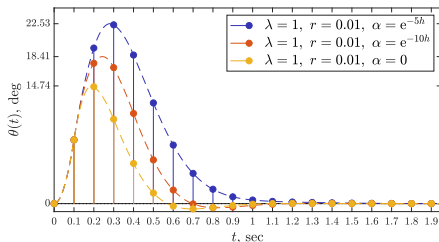
Simulation results with $1 + LB_d = \alpha$ and $\lambda = 1$



Again hidden oscillations. So beware,

→ stroboscopic models might not be faithful to analog behaviors.

Simulation results with $1 + LB_d = \alpha$ and $\lambda = 1$



Again hidden oscillations. So beware,

- stroboscopic models might not be faithful to analog behaviors.