

State-Space Control Fundamentals (00860721)

lecture no. 10

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Outline

H_∞ state feedback

H_∞ estimation and output feedback

H_∞ state feedback via LMIs

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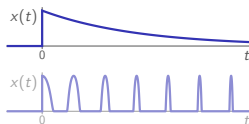
Preliminaries: L_2 signals and L_2 -stability

A signal $x : \mathbb{R} \rightarrow \mathbb{R}^n$ is said to belong to L_2 if its L_2 -norm is bounded, i.e.

$$\|x\|_2 := \left(\int_{-\infty}^{\infty} \|x(t)\|^2 dt \right)^{1/2} < \infty.$$

The quantity $\|x\|_2^2$ is known as the energy of x . Belonging to L_2 requires

- either sufficiently fast decay, like
- or sufficiently small support, like



A system $G : w \mapsto z$ is said to be L_2 -stable if

- $w \in L_2 \implies z \in L_2$
- there is $\gamma \geq 0$ such that $\|z\|_2 \leq \gamma \|w\|_2$ for all $w \in L_2$.

If G is L_2 -stable, then the infimal γ for which the inequality holds is termed the L_2 -induced norm of G . The L_2 -induced norm may be thought of as the (energy) gain of the system for its worst-case, i.e. most damaging, input.

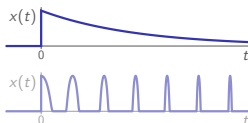
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H_∞ norm in the frequency domain

Let $\mathbb{C}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$. The Hardy space $H_\infty^{p \times m}$ (H_∞ , for simplicity) comprises functions $G : \mathbb{C}_0 \rightarrow \mathbb{C}^{p \times m}$ that are *holomorphic and bounded* on $\mathbb{C}_0$. If $G \in H_\infty$, then it is well defined on $j\mathbb{R} := \{s \in \mathbb{C} \mid \operatorname{Re} s = 0\}$ too and its H_∞ -norm

$$\|G\|_\infty = \operatorname{ess\,sup}_\omega \|G(j\omega)\|.$$

($\|M\|$ stands for the matrix spectral norm). If $G(s) = D + C(sI - A)^{-1}B$, then $G \in H_\infty \iff A$ is Hurwitz. Also, $\|G\|_\infty \geq \|D\| = \lim_{\omega \rightarrow \infty} \|G(j\omega)\|$.

A key fact is that

an LTI system is causal and L_2 -stable iff its transfer function is in H_∞ .

Moreover,

the L_2 -induced norm equals the H_∞ -norm of the transfer function, which is the peak value (the least upper bound) on a Bode magnitude plot. This is why we refer to the L_2 -induced norm of a system G as its H_∞ -norm, denoted $\|G\|_\infty$.

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H_∞ state-feedback problem

Plant:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t), & x(0) = 0 \\ z(t) = C_z x(t) + D_w w(t) + D_u u(t) \end{cases}$$

in which (in addition to the standard state x and control input u)

w is an exogenous input

may comprise disturbances, references

z is a regulated output

required to be “small” despite w

Control goal: assuming that x is measurable,

- stabilize the system
- ensure that $\|z\|_2 \leq \gamma \|w\|_2$ for all $w \in L_2$ and a given $\gamma \geq 0$

Requiring $\|z\|_2 \leq \gamma \|w\|_2$ for a smallest possible γ can be interpreted as

- minimizing $\|z\|_2$ under a worst-case $w \in L_2$ such that $\|w\|_2 = 1$
by linearity, $\|w\|_2 = 1$ can be assumed w.l.o.g., scaling of $w \implies$ same scaling of z

H_∞ state-feedback problem: assumptions and γ range

Like in the LQR case, we assume that

$\mathcal{A}_1$: (A, B_u) is stabilizable

$\mathcal{A}_2$: $D_u' D_u > 0$

counterpart of $R > 0$ in LQR

$\mathcal{A}_3$: $\begin{bmatrix} A - \lambda I & B_u \\ C_z & D_u \end{bmatrix}$ has full column rank for all $\lambda \in \{s \in \mathbb{C} \mid \operatorname{Re} s = 0\}$.

Some observations about γ :

- If $\mathcal{A}_1$ holds, then there exists K_u such that $A + B_u K_u$ is Hurwitz. The closed-loop $T_{zw}(s) = D_w + (C_z + D_u K_u)(sI - A - B_u K_u)^{-1} B_w$ belongs to H_∞ , so that the problem is solvable for $\gamma = \|T_{zw}\|_\infty \geq \|D_w\|$.
- If the problem is solvable for $\gamma = \gamma_0$, then it is solvable for all $\gamma \geq \gamma_0$.
- Thus, the problem is solvable whenever γ is sufficiently large and there is $\gamma_{\text{sf}} \geq \|D_w\|$ such that
 - the problem is solvable for all $\gamma > \gamma_{\text{sf}}$
 - the problem is unsolvable for all $\gamma < \gamma_{\text{sf}}$

We need to characterize γ_{sf} and find an admissible controller if $\gamma > \gamma_{\text{sf}}$.

Particular case: worst-case disturbance attenuation

Consider the familiar system

$$\begin{cases} \dot{x} = Ax + B_d d + Bu \\ y = Cx \end{cases}$$

with disturbance d . Suppose $d \in L_2$ but is otherwise arbitrary.

- the requirement $\|y\|_2^2 \leq \gamma^2 \|d\|_2^2$ for all admissible d corresponds to

$$\begin{bmatrix} A & B_w & B_u \\ C_z & D_w & D_u \end{bmatrix} = \begin{bmatrix} A & B_d & B \\ C & 0 & 0 \end{bmatrix}$$

- if $\|y\|_2^2 + r\|u\|_2^2 \leq \gamma^2 \|d\|_2^2$ for all admissible d is required, then choose

$$\begin{bmatrix} A & B_w & B_u \\ C_z & D_w & D_u \end{bmatrix} = \begin{bmatrix} A & B_d & B \\ C & 0 & 0 \\ 0 & 0 & \sqrt{r}I \end{bmatrix}$$

In both cases $w = d$.

Particular case: worst-case tracking

Now, let

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

and suppose that it is required to track a reference signal y_{ref} , of which we only know that $y_{\text{ref}} \in L_2$, by y . A possible formulation in this case is

$$\|y_{\text{ref}} - y\|_2^2 + r\|u\|_2^2 \leq \gamma^2 \|y_{\text{ref}}\|_2^2,$$

which corresponds to

$$\begin{bmatrix} A & B_w & B_u \\ C_z & D_w & D_u \end{bmatrix} = \left[\begin{array}{c|c|c} A & 0 & B \\ \hline -C & I & 0 \\ \hline 0 & 0 & \sqrt{r}I \end{array} \right]$$

and $w = y_{\text{ref}}$. Note that in this case it is not unreasonable to assume that we measure not only x , but also w . Such setting is known as full information.

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Cost

The relation $\|z\|_2 \leq \gamma \|w\|_2$ can be written as

$$\int_0^\infty \begin{bmatrix} x'(t) & u'(t) & w'(t) \end{bmatrix} \begin{bmatrix} C_z' C_z & C_z' D_u & C_z' D_w \\ D_u' C_z & D_u' D_u & D_u' D_w \\ D_w' C_z & D_w' D_u & D_w' D_w \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \\ w(t) \end{bmatrix} dt \leq \gamma^2 \int_0^\infty \|w(t)\|^2 dt$$

It becomes particularly familiar if $D_w = 0$, which is frequently the case,

$$\underbrace{\int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} C_z' C_z & C_z' D_u \\ D_u' C_z & D_u' D_u \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt}_{\text{LQR cost}} \leq \gamma^2 \underbrace{\int_0^\infty \|w(t)\|^2 dt}_{\text{energy of } w}$$

Still, this is not the same, the LQR $B_w w = x_0 \delta \notin L_2$.

Background: H_∞ CARE

Return to the CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0,$$

whose solution is said to be stabilizing if

$$A_K := A - BR^{-1}(S' + B'\bar{X})$$

is Hurwitz. We are interested now in the case when $B = \begin{bmatrix} B_u & B_w \end{bmatrix}$ and

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} C_z' \\ D_u' \\ D_w' \end{bmatrix} \begin{bmatrix} C_z & D_u & D_w \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma^2 I_{m_w} \end{bmatrix}$$

for some $\gamma \geq 0$ such that $\det R \neq 0$. A key difference from the LQR case is that we no longer ensure that $R > 0$. For example, if $D_w = 0$, then

$$R = \begin{bmatrix} D_u' D_u & 0 \\ 0 & -\gamma^2 I_{m_w} \end{bmatrix}$$

has m_u positive (assuming $\mathcal{A}_2$) and m_w negative eigenvalues.

Example

Let $A = a$, $B_u = b > 0$, $B_w = 1$, $C_z = \begin{bmatrix} \sqrt{q} \\ 0 \end{bmatrix}$, $D_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $D_w = 0$. For this choice $B = \begin{bmatrix} b & 1 \end{bmatrix}$, $Q = q$, $S = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix}$, so the CARE reads

$$2a\bar{X} + q - \alpha_\gamma \bar{X}^2 = 0, \quad \text{where } \alpha_\gamma := b^2 - \gamma^{-2} \in (-\infty, b^2)$$

- if $\alpha_\gamma = 0 \iff \gamma = 1/b$, then CARE reads $2a\bar{X} + q = 0$ and $A_K = a$
 - if $a \geq 0$, then no stabilizing solution exists
 - if $a < 0$, then $\bar{X} = -q/(2a) \geq 0$ is the stabilizing solution
- if $\alpha_\gamma \neq 0$, then the CARE is quadratic, solvable by

$$\bar{X} = \frac{a \pm \sqrt{a^2 + q\alpha_\gamma}}{\alpha_\gamma} \implies A_K = a - \alpha_\gamma \bar{X} = \mp \sqrt{a^2 + q\alpha_\gamma}$$

- if $\alpha_\gamma = -a^2/q \iff \gamma = 1/\sqrt{b^2 + a^2}/q$, then $A_K = 0$ is not Hurwitz
- if $\alpha_\gamma < -a^2/q$, then $\bar{X}$ is complex and A_K is not Hurwitz either
- if $\alpha_\gamma > -a^2/q$ (but $\alpha_\gamma \neq 0$), then $\bar{X} = (a + \sqrt{a^2 + q\alpha_\gamma})/\alpha_\gamma$ is stabilizing

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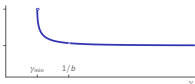
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Example (contd)

Denoting $\gamma_{\min} := 1/\sqrt{b^2 + a^2/q}$ and substituting $\alpha_\gamma = b^2 - \gamma^{-2}$, we have:
 $a < 0$ the stabilizing

$$\bar{X} = \frac{a + \sqrt{a^2 + q(b^2 - \gamma^{-2})}}{b^2 - \gamma^{-2}} =$$


exists for all $\gamma \in (\gamma_{\min}, \infty)$, including $\gamma = 1/b$, and $\bar{X} \geq 0$;

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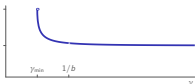
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Morals:

- the H_∞ CARE may have stabilizing solution only if γ is large enough
- the stabilizing solution to the H_∞ CARE is not necessarily nonnegative

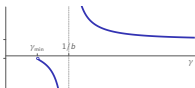
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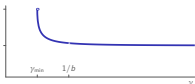
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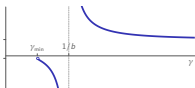
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Necessary conditions

Consider the CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0, \quad (\star)$$

where

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \left[\begin{array}{c|cc} C_z' C_z & C_z' D_u & C_z' D_w \\ \hline D_u' C_z & D_u' D_u & D_u' D_w \\ \hline D_w' C_z & D_w' D_u & D_w' D_w - \gamma^2 I \end{array} \right].$$

Its solution is said to be stabilizing if $A + B_u K_u + B_w K_w$ is Hurwitz, where

$$\begin{bmatrix} K_u \\ K_w \end{bmatrix} := - \begin{bmatrix} D_u' D_u & D_u' D_w \\ D_w' D_u & D_w' D_w - \gamma^2 I \end{bmatrix}^{-1} \left(\begin{bmatrix} D_u' \\ D_w' \end{bmatrix} C_z + \begin{bmatrix} B_u' \\ B_w' \end{bmatrix} \bar{X} \right)$$

(the right-hand side above is $-R^{-1}(S' + B'\bar{X})$).

Lemma 10.1

If $\mathcal{A}_{1-3}$ hold, then $\gamma > \gamma_{\text{sf}}$ only if $\gamma^2 I - D_w' D_w > 0$ and the CARE $(\star)$ has a stabilizing solution $\bar{X} \geq 0$, in which case $A + B_u K_u$ is Hurwitz as well.

Proof that $A + B_u K_u$ is Hurwitz

Tedious manipulations with $(\star)$, even more so than manipulations with the LQR Riccati that led to its Lyapunov form on Slide 14 of Lecture 4, yield

$$(A + B_u K_u)' \bar{X} + \bar{X} (A + B_u K_u) + (C_z + K_u D_u)' (C_z + D_u K_u) + \overbrace{K_w' (\gamma^2 I - D_w' D_w) K_w}^{>0} = 0.$$

Assume the contrary, that there is $\lambda \in \text{spec}(A + B_u K_u)$ such that $\text{Re } \lambda \geq 0$. Post- and pre-multiplying the **relation above** by a corresponding eigenvector η and its conjugate transpose, respectively, we get to the equality

$$2 \text{Re } \lambda \eta' \bar{X} \eta + \|(C_z + D_u K_u) \eta\|^2 + \|(\gamma^2 I - D_w' D_w)^{1/2} K_w \eta\|^2 = 0,$$

whose all terms are nonnegative (remember, $\bar{X} \geq 0$). Hence, $K_w \eta = 0$ and

$$(A + B_u K_u + B_w K_w) \eta = \lambda \eta \implies \lambda \in \text{spec}(A + B_u K_u + B_w K_w).$$

This contradicts the assumption that $\bar{X}$ is a stabilizing solution to $(\star)$. $\square$

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Completing the square

Remember the relation

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} - \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix},$$

holding true regardless of whether $R > 0$ or not. In the H_∞ case it reads

$$\begin{aligned} & \begin{bmatrix} C_z' \\ D_u' \\ D_w' \end{bmatrix} \begin{bmatrix} C_z & D_u & D_w \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma^2 I \end{bmatrix} \\ &= \begin{bmatrix} -K_u' & -K_w' \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_u' D_u & D_u' D_w \\ D_w' D_u & D_w' D_w - \gamma^2 I \end{bmatrix} \begin{bmatrix} -K_u & I & 0 \\ -K_w & 0 & I \end{bmatrix} \\ & \quad - \begin{bmatrix} \bar{X} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A & B_u & B_w \end{bmatrix} - \begin{bmatrix} A' \\ B_u' \\ B_w' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 & 0 \end{bmatrix} \end{aligned}$$

Completing the square (contd)

Pre- and post-multiplying by $\begin{bmatrix} x' & u' & w' \end{bmatrix}$ and $\begin{bmatrix} x \\ u \\ w \end{bmatrix}$, respectively, we get

$$\|z(t)\|^2 - \gamma^2 \|w(t)\|^2 = \left\| \begin{bmatrix} u(t) - K_u x(t) \\ w(t) - K_w x(t) \end{bmatrix} \right\|_R^2 - x'(t) \bar{X} \dot{x}(t) - \dot{x}'(t) \bar{X} x(t)$$

where $\|v\|_M^2 := v' M v$. If $\gamma > \gamma_{\text{crit}}$, then $\gamma^2 I > D_w' D_w$ and

$$\begin{aligned} R &= \begin{bmatrix} D_u' D_u & D_u' D_w \\ D_w' D_u & D_w' D_w - \gamma^2 I \end{bmatrix} \\ &= \begin{bmatrix} I & -D_u' D_w R_w^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} R_u & 0 \\ 0 & -R_w \end{bmatrix} \begin{bmatrix} I & 0 \\ -R_w^{-1} D_w' D_u & I \end{bmatrix} \end{aligned}$$

where $R_w := \gamma^2 I - D_w' D_w > 0$ and $R_u := D_u' (I + D_w R_w^{-1} D_w') D_u > 0$. So

$$\|z(t)\|^2 - \gamma^2 \|w(t)\|^2 = \|u(t) - u_w(t)\|_{R_u}^2 - \|w(t) - w_u(t)\|_{R_w}^2 - \frac{dx'(t) \bar{X} x(t)}{dt}$$

where $u_w := K_u x$ and $w_u := K_w x - R_w^{-1} D_w' D_u (u - u_w)$.

Completing the square (contd)

Pre- and post-multiplying by $\begin{bmatrix} x' & u' & w' \end{bmatrix}$ and $\begin{bmatrix} x \\ u \\ w \end{bmatrix}$, respectively, we get

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where $u_* := K_u x$ and $w_* := K_w x - R_w^{-1} D_w' D_u (u - u_*)$.

Admissible control law

Integrating on $[0, \infty)$, we end up with

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = \int_0^\infty \|u(t) - u_*(t)\|_{R_u}^2 dt - \int_0^\infty \|w(t) - w_*(t)\|_{R_w}^2 dt$$

(remember, $x(0) = 0$ and $\lim_{t \rightarrow \infty} x(t) = 0$ if u is stabilizing and $w \in L_2$).

Now,

our (i.e. of u) goal is to ensure that $\|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0$ for every w .

The choice

$$u = u_* = K_u x$$

is stabilizing and renders

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = - \int_0^\infty \|w(t) - w_*(t)\|_{R_w}^2 dt,$$

which is indeed non-positive for any w . Thus, $u = K_u x$ solves the problem, meaning that the necessary condition of Lemma 10.1 is sufficient as well.

Admissible control law

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Worst-case disturbance

Returning to

the only term affected by w directly

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = \int_0^\infty \|u(t) - u_*(t)\|_{R_u}^2 dt - \int_0^\infty \|w(t) - w_*(t)\|_{R_w}^2 dt$$

The best the disturbance could hope is to make $\|z\|_2^2 - \gamma^2 \|w\|_2^2 = 0$ via the selection $w = w_*$ (the closed-loop dynamics $\dot{x} = (A + B_u K_u + B_w K_w)x$ are stable then, so $w_* = K_w x - R_w^{-1} D_w' D_u (u - K_u x) = K_w x \in L_2$, as required). This is why

$$w_* = (K_w + R_w^{-1} D_w' D_u K_u)x - R_w^{-1} D_w' D_u u$$

is referred to as the **worst-case disturbance**. In fact, if $u \neq K_u x$, then it can be shown that $w = w_*$ would violate the H_∞ norm bound (more accurately, its slight alteration ensuring that $w \in L_2$).

Worst-case disturbance

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the only term affected by w directly

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H_∞ state-feedback problem: solution

Theorem 10.2

If $\mathcal{A}_{1-3}$ hold, then $\gamma > \gamma_{\text{sf}}$ iff $\gamma^2 I - D'_w D_w > 0$ and the CARE $(\star)$ has a stabilizing solution $\bar{X} \geq 0$. In this case the state feedback

$$u = K_u x$$

solves the problem, i.e. is stabilizing and ensuring $\|z\|_2 \leq \gamma \|w\|_2, \forall w \in L_2$.

Some properties:

- the solution $u = K_u x$ is not unique
- as $\gamma \rightarrow \infty$, the LQR control law is recovered
- as $\gamma \rightarrow \gamma_{\text{sf}}$, the CARE becomes ill-posed

Remark. If we measure also w , then we have the full-information setting. The solvability of the full-information problem requires $\|(I - D_u(D'_u D_u)^{-1} D'_u) D_w\| < \gamma$, instead of the more restrictive $\|D_w\| < \gamma$. A solution is $u = (D'_u D_u)^{-1}((B'_u \bar{X} + D'_u C_z)x + D'_u D_w w)$ then. If $D'_u D_w = 0$, then measuring w offers no advantages.

Outline

H_∞ state feedback

H_∞ estimation and output feedback

H_∞ state feedback via LMIs

H_∞ estimation: formulation

Let

$$\begin{cases} \dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t), & x(0) = 0 \\ y(t) = Cx(t) + D_w w(t) \end{cases}$$

The goal is to reconstruct $z = C_z x$ from y so that

$$\|z - \hat{z}\|_2 \leq \gamma \|w\|_2$$

for a given $\gamma > 0$. The problem is solvable for all $\gamma > \gamma_{\text{est}}$, where $\gamma_{\text{est}} \geq 0$ is the lower bound on the attainable performance, to be found.

Assumptions are like in the Kalman–Bucy filtering case,

$\mathcal{A}_4$: (C, A) is detectable

$\mathcal{A}_5$: $D_w D_w' > 0$

i.e. all measurement channels are noisy

$\mathcal{A}_6$: $\begin{bmatrix} A - \lambda I & B_w \\ C & D_w \end{bmatrix}$ has full row rank for all $\lambda \in \{z \in \mathbb{C} \mid \text{Re } z = 0\}$.

H_∞ estimation: solution

Theorem 10.3

If $\mathcal{A}_{4-6}$ hold, then $\gamma > \gamma_{\text{est}}$ if and only if the CARE

$$\begin{aligned} \bar{Y}A' + A\bar{Y} + B_w B_w' + \gamma^{-2} \bar{Y} C_z' C_z \bar{Y} \\ - (B_w D_w' + \bar{Y} C') (D_w D_w')^{-1} (D_w B_w' + C \bar{Y}) = 0 \end{aligned}$$

has a stabilizing solution $\bar{Y} \geq 0$. In this case, an admissible observer is

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + B_u u(t) \\ \quad + (B_w D_w' + \bar{Y} C') (D_w D_w')^{-1} (y(t) - C\hat{x}(t)), \quad \hat{x}(0) = 0 \\ \hat{z}(t) = C_z \hat{x}(t) \end{cases}$$

Comments:

- full-order observer with the gain $L = -(B_w D_w' + \bar{Y} C') (D_w D_w')^{-1}$
- the attainable performance γ_{est} and the gain L depend on C_z
- the solution recovers the Kalman–Bucy filter as $\gamma \rightarrow \infty$

H_∞ output feedback: formulation

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t), & x(0) = 0 \\ z(t) = C_z x(t) + D_{zw} w(t) + D_{zu} u(t) \\ y(t) = C_y x(t) + D_{yw} w(t) \end{cases}$$

with the same meaning of the signals as in the LQG case and assume that

$\mathcal{A}_7$: (C_y, A, B_u) is stabilizable and detectable

$\mathcal{A}_8$: $D'_{zu} D_{zu} > 0$ and $D_{yw} D'_{yw} > 0$

$\mathcal{A}_9$: $\begin{bmatrix} A - \lambda I & B_u \\ C_z & D_{zu} \end{bmatrix}$ has full column rank for all $\lambda \in j\mathbb{R}$

$\mathcal{A}_{10}$: $\begin{bmatrix} A - \lambda I & B_w \\ C_y & D_{yw} \end{bmatrix}$ has full row rank for all $\lambda \in j\mathbb{R}$

The problem is to design a controller $R : y \mapsto u$ that

- stabilizes the system
- ensures that $\|z\|_2 \leq \gamma \|w\|_2$ for all $w \in L_2$ for a given $\gamma > 0$

H_∞ output feedback: CAREs

We need two H_∞ algebraic Riccati equations

$$A'\bar{X} + \bar{X}A + C_z' C_z - (\bar{X} \begin{bmatrix} B_u & B_w \end{bmatrix} + C_z' \begin{bmatrix} D_{zu} & D_{zw} \end{bmatrix}) \\ \times \begin{bmatrix} D_{zu}' D_{zu} & D_{zu}' D_{zw} \\ D_{zw}' D_{zu} & D_{zw}' D_{zw} - \gamma^2 I \end{bmatrix}^{-1} \left(\begin{bmatrix} B_u' \\ B_w' \end{bmatrix} \bar{X} + \begin{bmatrix} D_{zu}' \\ D_{zw}' \end{bmatrix} C_z \right) = 0$$

(control CARE) and

$$A\bar{Y} + \bar{Y}A' + B_w B_w' - (\bar{Y} \begin{bmatrix} C_y' & C_z' \end{bmatrix} + B_w \begin{bmatrix} D_{yw}' & D_{zw}' \end{bmatrix}) \\ \times \begin{bmatrix} D_{yw} D_{yw}' & D_{yw} D_{zw}' \\ D_{zw} D_{yw}' & D_{zw} D_{zw}' - \gamma^2 I \end{bmatrix}^{-1} \left(\begin{bmatrix} C_y \\ C_z \end{bmatrix} \bar{Y} + \begin{bmatrix} D_{yw} \\ D_{zw} \end{bmatrix} B_w' \right) = 0$$

(filtering CARE). As in the LQG case, their solutions $\bar{X}$ and $\bar{Y}$ are dubbed stabilizing if $A + B_u K_u + B_w K_w$ and $A + L_y C_y + L_z C_z$ are Hurwitz, where B_u , B_w , L_y , and L_z are the corresponding gains.

H_∞ output feedback: solution / Matlab: hinfsyn(G,p,m)

Theorem 10.4

If $\mathcal{A}_{7-10}$ hold true, then the problem is solvable iff

1. $\|(I - D_{zu}(D'_{zu}D_{zu})^{-1}D'_{zu})D_{zw}\| < \gamma$ and stabilizing $\bar{X} \geq 0$ exists
2. $\|D_{zw}(I - D'_{yw}(D_{yw}D'_{yw})^{-1}D_{yw})\| < \gamma$ and stabilizing $\bar{Y} \geq 0$ exists
3. $\lambda_{\max}(\bar{Y}\bar{X}) < \gamma^2$ coupling condition

In this case an admissible controller is given by

$$R(s) = - \left[\frac{A + B_w K_w + B_u K_u + Z_\gamma L_y (C_y + D_{yw} K_w)}{K_u} \mid \frac{Z_\gamma L_y}{0} \right],$$

where $Z_\gamma := (I - \gamma^{-2} \bar{Y} \bar{X})^{-1}$ is well defined.

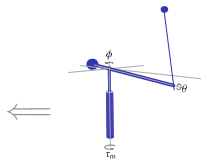
Comments:

- as $\gamma \rightarrow \infty$, R becomes the LQR controller $Z_\gamma \rightarrow I$ and $K_w \rightarrow 0$
- as γ approaches its infimum value, the order of R typically reduces
- not obvious, but R is still observer based, whose observer reconstructs the full-information $u = (D'_{zu}D_{zu})^{-1}((B'_u\bar{X} + D'_{zu}C_z)x + D'_{zu}D_{zw}w)$

Example: stability margins in inverted pendulum

Remember (Lecture 1),

$$P(s) = \frac{97.637(s + 5.382)(s - 5.27)}{s(s + 0.7318)(s + 6.351)(s - 5.958)}$$



where only the shaft angle, ϕ , is measured. The presence of both pole and zero in the RHP imposes severe limitations on stability margins.

The modulus margin μ_m is defined as the

– distance from the critical point to the Nyquist plot,

i.e.

$$\mu_m = \inf_{\omega} |1 + L(j\omega)| = \frac{1}{\sup_{\omega} |S(j\omega)|} = \frac{1}{\|S\|_\infty},$$

where $S(s) = (1 + L(s))^{-1}$ is the (stable) sensitivity function. Therefore,

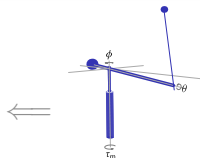
– maximizing μ_m is equivalent to minimizing $\|S\|_\infty$,

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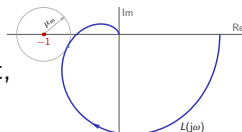
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where $S(s) = (1 + L(s))^{-1}$ is the (stable) sensitivity function. Therefore,

- maximizing μ_m is equivalent to minimizing $\|S\|_{\infty}$,

which is an H_∞ optimization problem.

Example: stability margins in inverted pendulum (contd)

The sensitivity function $S : y_{\text{ref}} \mapsto e = y_{\text{ref}} - y$ in the unity feedback setup. Hence, the H_∞ problem should have $w = y_{\text{ref}}$, $z = e$, and

$$\begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -C & 1 & 0 \\ -C & 1 & \end{bmatrix}$$

However, its D_{zu} does not satisfy **A8**. A slight alternation is

$$\begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -C & 1 & 0 \\ 0 & 0 & \epsilon_u \\ -C & 1 & \end{bmatrix}$$

for a sufficiently small control weight $\epsilon_u > 0$. This is the very realization to be fed to Matlab's `hinfsyn`.

Example: stability margins in inverted pendulum (contd)

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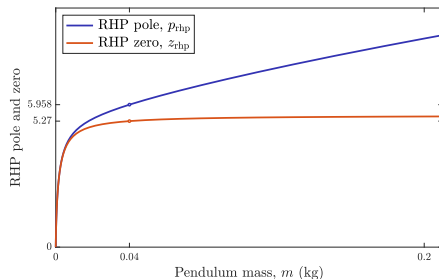
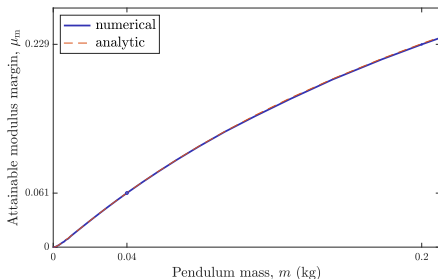
However, its D_{zu} does not satisfy $\mathcal{A}_8$. A slight alternation is

$$\begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -C & 1 & 0 \\ 0 & 0 & \epsilon_u \\ -C & 1 & \end{bmatrix}$$

for a sufficiently small control weight $\epsilon_u > 0$. This is the very realization to be fed to Matlab's `hinfsyn`.

Example: stability margins in inverted pendulum (contd)

Calculating the infimal γ for various pendulum masses with $\epsilon_u = 10^{-5}$,



These plots support the classical rule of thumbs that

- systems having close RHP poles and zeros are very hard to control.

Remark. It can be shown that the infimal modulus margin in this case is

$$\mu_m = \frac{p_{rhp} - z_{rhp}}{p_{rhp} + z_{rhp}}$$

where p_{rhp} and z_{rhp} are the single pole and zero of $P(s)$ in the open RHP.

Outline

H_∞ state feedback

H_∞ estimation and output feedback

H_∞ state feedback via LMIs

KYP (Kalman–Yakubovich–Popov lemma)

Lemma 10.5

Let $G(s) = D + C(sI - A)^{-1}B \in \mathbb{C}^{p \times m}$ and $M_{\text{kyp}} = M'_{\text{kyp}} \in \mathbb{R}^{(m+p) \times (m+p)}$. If $\text{spec}(A) \cap j\mathbb{R} = \emptyset$, then

$$\begin{bmatrix} G'(j\omega) & I_m \end{bmatrix} M_{\text{kyp}} \begin{bmatrix} G(j\omega) \\ I_m \end{bmatrix} < 0, \quad \forall \omega \in \mathbb{R} \cup \{\pm\infty\}$$

iff there is $P = P' \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} C' & 0 \\ D' & I_m \end{bmatrix} M_{\text{kyp}} \begin{bmatrix} C & D \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} I_n & A' \\ 0 & B' \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ A & B \end{bmatrix} < 0.$$

If (A, B) is controllable, then the equivalence holds for the non-strict version of inequalities as well.

KYP: proof of sufficiency

Note that

$$\begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} = \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}$$

and

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} = \begin{bmatrix} I \\ j\omega I \end{bmatrix} (j\omega I - A)^{-1}B$$

(remember, $(sI - A)^{-1} + I = s(sI - A)^{-1}$). Also,

$$\begin{bmatrix} I & -j\omega I \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I \\ j\omega I \end{bmatrix} = 0, \quad \forall \omega$$

The frequency-domain inequality follows then by post- and pre-multiplying the LMI by $\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$ and its conjugate transpose, respectively. $\square$

The proof of necessity is substantially more technical...

KYP: special case

If

$$M_{\text{kyp}} = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix},$$

then the frequency-dependent part of KYP reads

$$\begin{bmatrix} G'(j\omega) & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} = G'(j\omega)G(j\omega) - \gamma^2 I < 0.$$

Because (cf. Property 4 on Slide 28 of Lecture 1)

$$M'M < \gamma^2 I \iff \lambda_{\max}(M'M) < \gamma^2 \iff \|M\| < \gamma,$$

KYP implies that if $\text{spec}(A) \cap j\mathbb{R} = \emptyset$, then

$$\sup_{\omega \in \mathbb{R}} \|G(j\omega)\| < \gamma \iff \begin{bmatrix} PA + A'P + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix} < 0$$

for some $P = P'$

KYP: special case (contd)

Example: Let

$$G(s) = \frac{1}{s-a} \implies |G(j\omega)|^2 = \frac{1}{\omega^2 + a^2} \leq \frac{1}{a^2}.$$

The LMI condition requires the existence of P such that

$$\begin{bmatrix} 2aP + 1 & P \\ P & -\gamma^2 \end{bmatrix} < 0 \iff 2aP + 1 + \frac{P^2}{\gamma^2} < 0 \iff P^2 + 2a\gamma^2 P + \gamma^2 < 0$$

The minimum of the parabola $P^2 + 2a\gamma^2 P + \gamma^2$, attained by $P = -a\gamma^2$, is $\gamma^2(1 - \gamma^2 a^2)$. Hence,

$$\exists P \text{ such that } \begin{bmatrix} 2aP + 1 & P \\ P & -\gamma^2 \end{bmatrix} < 0 \iff \gamma^2 > \frac{1}{a^2},$$

agreeing with the frequency-domain bound.

KYP: special case (contd)

Because condition

$$\sup_{\omega \in \mathbb{R}} \|G(j\omega)\| < \gamma \iff \begin{bmatrix} PA + A'P + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix} < 0$$

holds for unstable G as well, it is not readily applicable for verifying whether $\|G\|_\infty < \gamma$. We need to complement it with the Hurwitz requirement to A , which is equivalent to the existence of $\tilde{P} = \tilde{P}' > 0$ such that $\tilde{P}A + A'\tilde{P} < 0$ (by Lemma 6.2 from Lecture 6).

$$PA + A'P + C'C < 0 \implies PA + A'P < 0 \iff PA + A'P + Q = 0$$

for some $Q = Q' > 0$. Hence,

- if $P > 0$, then A is necessarily Hurwitz
- if A is Hurwitz, then $P = \int_0^\infty e^{A't} Q e^{At} dt$ is necessarily positive definite

Thus, all we need is to complement the LMI by requiring $P = P' > 0$.

KYP: special case (contd)

Because condition

$$\sup_{\omega \in \mathbb{R}} \|G(j\omega)\| < \gamma \iff \begin{bmatrix} PA + A'P + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix} < 0$$

holds for unstable G as well, it is not readily applicable for verifying whether $\|G\|_\infty < \gamma$. We need to complement it with the Hurwitz requirement to A , which is equivalent to the existence of $\tilde{P} = \tilde{P}' > 0$ such that $\tilde{P}A + A'\tilde{P} < 0$ (by Lemma 6.2 from Lecture 6). The LMI from KYP requires

$$PA + A'P + C'C < 0 \implies PA + A'P < 0 \iff PA + A'P + Q = 0$$

for some $Q = Q' > 0$. Hence,

- if $P > 0$, then A is necessarily Hurwitz
- if A is Hurwitz, then $P = \int_0^\infty e^{A't} Q e^{At} dt$ is necessarily positive definite

Thus, all we need is to complement the LMI by requiring $P = P' > 0$.

Bounded Real Lemma

Lemma 10.6

Let $G(s) = D + C(sI - A)^{-1}B$. The following conditions are equivalent

1. $G \in H_\infty$ and $\|G\|_\infty < \gamma$
2. there is $P = P' > 0$ such that
$$\begin{bmatrix} PA + A'P + C'C & PB + C'D \\ B'P + D'C & D'D - \gamma^2 I \end{bmatrix} < 0$$
3. there is $P = P' > 0$ such that
$$\begin{bmatrix} PA + A'P & PB & C' \\ B'P & -\gamma I & D' \\ C & D & -\gamma I \end{bmatrix} < 0$$

Proof of 2 $\iff$ 3: Pre- and post-multiplying by $\text{diag}\{\sqrt{\gamma}, \sqrt{\gamma}, 1/\sqrt{\gamma}\}$,

$$\begin{bmatrix} PA + A'P & PB & C' \\ B'P & -\gamma I & D' \\ C & D & -\gamma I \end{bmatrix} < 0 \iff \left[\begin{array}{cc|c} \gamma(PA + A'P) & \gamma PB & C' \\ \gamma B'P & -\gamma^2 I & D' \\ \hline C & D & -I \end{array} \right] < 0$$

and 2. is its Schur complement via substituting $P \rightarrow \gamma^{-1}P$. □

H_∞ state feedback via Bounded Real Lemma

Given K_u , the state-feedback law $u = K_u x$ renders the closed-loop dynamics

$$T_{zw} : \begin{cases} \dot{x} = (A + B_u K_u)x + B_w w \\ z = (C_z + D_u K_u)x + D_w w \end{cases}$$

By the Bounded Real Lemma, the controller is stabilizing and $\|T_{zw}\|_\infty < \gamma$ iff there is $P = P' > 0$ such that

$$\begin{bmatrix} P(A + B_u K_u) + (A' + K_u' B_u')P & PB_w & C_z' + K_u' D_u' \\ B_w' P & -\gamma I & D_w' \\ C_z + D_u K_u & D_w & -\gamma I \end{bmatrix} < 0.$$

By already familiar trick (Lecture 6, Slide 26), this is equivalent to the LMI

$$\begin{bmatrix} AM + B_u N + M A' + N' B_u' & B_w & M C_z' + N' D_u' \\ B_w' & -\gamma I & D_w' \\ C_z M + D_u N & D_w & -\gamma I \end{bmatrix} < 0$$

for the decision variables $M = M' > 0$ and N . If feasible, then $K_u = NM^{-1}$.

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By already familiar trick (Lecture 6, Slide 26), this is equivalent to the LMI

$$\begin{bmatrix} AM + B_u N + MA' + N' B_u' & B_w & MC_z' + N' D_u' \\ B_w' & -\gamma I & D_w' \\ C_z M + D_u N & D_w & -\gamma I \end{bmatrix} < 0$$

for the decision variables $M = M' > 0$ and N . If feasible, then $K_u = NM^{-1}$.