

State-Space Control Fundamentals (00860721)

lecture no. 8

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Outline

Tracking and 2DOF architecture

Disturbances and their modeling

Asymptotic disturbance rejection in the full-information setting

Asymptotic disturbance rejection in the output-feedback setting

Outline

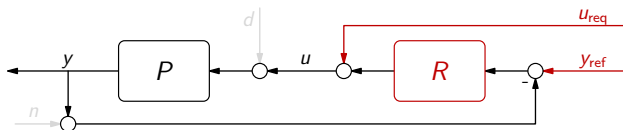
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Asymptotic disturbance rejection in the output-feedback setting

2DOF architecture in classical control



With $d = n = 0$,

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P \\ 1 \end{bmatrix} \frac{1}{1 + PR} (u_{\text{req}} + R y_{\text{ref}}).$$

If bounded u_{req} and y_{ref} are **consistent**, in the sense that

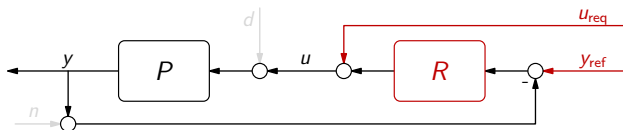
$$y_{\text{ref}} = P u_{\text{req}}$$

(i.e. y_{ref} can be generated by the plant if its input is u_{req}), then

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P \\ 1 \end{bmatrix} \frac{1}{1 + PR} (1 + PR) u_{\text{req}} = \begin{bmatrix} P \\ 1 \end{bmatrix} u_{\text{req}} = \begin{bmatrix} y_{\text{ref}} \\ u_{\text{req}} \end{bmatrix},$$

so $y = y_{\text{ref}}$ regardless the choice of R (provided it is stabilizing, of course).

Example: set-point tracking by double integrator

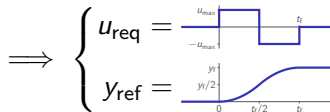


Let $P(s) = 1/s^2$ (acceleration $\mapsto$ position) and the goal be

- attain $y(t) = y_f$ for all $t \geq t_f$ from $y(0) = 0$ in minimum time t_f
- with a bounded input (force) $|u(t)| \leq u_{\max}$.

The solution (for which $t_f = 2\sqrt{y_f/u_{\max}}$):

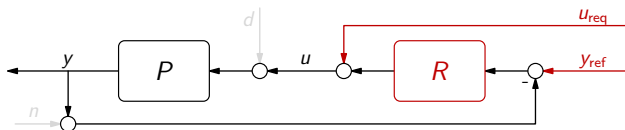
- maximal acceleration until $y(t) = y_f/2$
- maximal deceleration until $y(t) = y_f$
- rest afterwards



Note that

- both u_{req} and $y_{\text{ref}} = Pu_{\text{req}}$ are bounded, even though P is unstable.

2DOF architecture: pros and cons



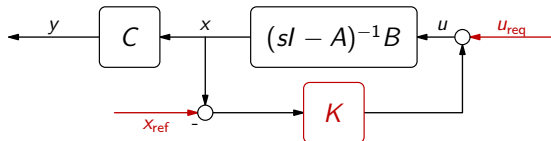
Separating tracking (can be done in open loop) from feedback properties,

- ☺ simplifies tracking design
only the nominal plant model is involved
- ☺ simplifies feedback design
less tasks for R , can concentrate on what only feedback can handle (uncertainty)
- ☺ add flexibility into design
 R can be tuned / adapted without affecting tracking properties, and vice versa

Consistency requirement $y_{\text{ref}} = Pu_{\text{req}}$

- ☹ might be an obstacle if y_{ref} is measured online
think of $P(s) = 1/s^2$, for which consistency is $u_{\text{req}} = \ddot{y}_{\text{ref}}$, under noisy measurements of y_{ref} ; it is even more complicated if $P(s)$ is nonminimum phase

State-feedback version of the 2DOF architecture



If x_{ref} and u_{req} are bounded and satisfy (state consistency)

$$\dot{x}_{\text{ref}} = Ax_{\text{ref}} + Bu_{\text{req}}, \quad x_{\text{ref}}(0) = x(0)$$

then $\dot{x} = Ax + B(u_{\text{req}} + K(x - x_{\text{ref}})) = (A + BK)x + Bu_{\text{req}} - BKx_{\text{ref}}$. Thus, $e = x - x_{\text{ref}}$ satisfies

$$\dot{e} = (A + BK)e, \quad e(0) = 0 \quad \implies \quad e \equiv 0$$

(if $x_{\text{ref}}(0) \neq x(0)$, then $e(t) = e^{(A+BK)t}e(0)$ is still exponentially decaying).

Remark. Works smoothly if the choice of y_{ref} and u_{req} is a part of one optimization setup. If we measure y_{ref} , then the sequence is $y_{\text{ref}} \rightarrow u_{\text{req}} \rightarrow x_{\text{ref}}$, if feasible.

Example: set-point tracking by double integrator (contd)

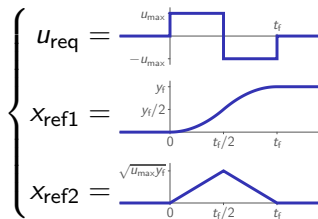
In the double-integrator case,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and

$$x_1(t) = y(t) \quad \text{and} \quad x_2(t) = \dot{x}_1(t) = \int_0^t u(s) ds.$$

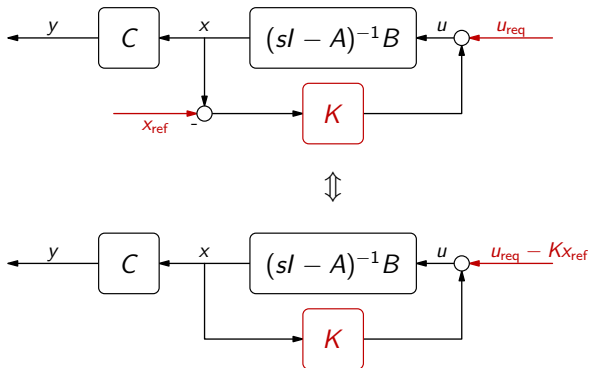
Hence, we end up with



for $t_f = 2\sqrt{y_f/u_{\text{max}}}$.

State-feedback version of the 2DOF architecture (contd)

Routine block-diagram manipulation rules yield



which is effectively the standard state feedback $u = Kx + v$ with

$$v = u_{req} - Kx_{req}$$

Remark. This v depends on K . If we prefer exogenous signals independent of the feedback gain, in the spirit of the classical 2DOF, then the upper diagram shall be used.

Minimum-energy set-point tracking

Let x_0 be given and x_f be such that there is u_f such that $Ax_f + Bu_f = 0$ (i.e. x_f corresponds to an equilibrium). Given $t_f > 0$, the goal to

- steer $x_0 = x(0) \rightarrow x(t_f) = x_f$ with minimum energy of u and stay there.

This could be a way to design x_{ref} and u_{req} for tracking. From Lecture 3,

$$u_{\text{req}}(t) = \begin{cases} B'\Gamma_{22}(t)\Gamma_{12}^{-1}(t_f)(x_f - \Gamma_{11}(t_f)x_0) & \text{if } t \leq t_f \\ u_f & \text{if } t > t_f \end{cases}$$

which generates (so that $\dot{x}_{\text{ref}} = Ax_{\text{ref}} + Bu_{\text{req}}$ with $x_{\text{ref}}(0) = x_0$, as required)

$$x_{\text{ref}}(t) = \begin{cases} \Gamma_{11}(t)x_0 + \Gamma_{12}(t)\Gamma_{12}^{-1}(t_f)(x_f - \Gamma_{11}(t_f)x_0) & \text{if } t \leq t_f \\ x_f & \text{if } t \geq t_f \end{cases}$$

where

$$\begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ 0 & \Gamma_{22}(t) \end{bmatrix} = \exp\left(\begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} t\right).$$

Minimum-energy set-point tracking (contd)

Note that for $t \in [0, t_f]$

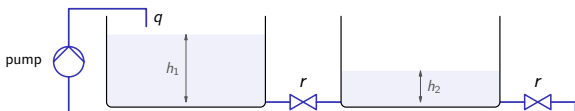
$$\begin{aligned} \begin{bmatrix} x_{\text{ref}}(t) \\ u_{\text{req}}(t) \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & B' \end{bmatrix} \begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ 0 & \Gamma_{22}(t) \end{bmatrix} \begin{bmatrix} x_0 \\ \Gamma_{12}^{-1}(t_f)(x_f - \Gamma_{11}(t_f)x_0) \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & B' \end{bmatrix} \exp\left(\begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} t\right) \begin{bmatrix} I & 0 \\ \Gamma_{11}(t_f) & \Gamma_{12}(t_f) \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ x_f \end{bmatrix}. \end{aligned}$$

Hence, $v = u_{\text{req}} - Kx_{\text{ref}}$ could be generated via the differential equation

$$\begin{cases} \dot{\xi}(t) = \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} \xi(t), & \xi(0) = \begin{bmatrix} I & 0 \\ \Gamma_{11}(t_f) & \Gamma_{12}(t_f) \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ x_f \end{bmatrix} \\ v(t) = \begin{bmatrix} -K & B' \end{bmatrix} \xi(t) \end{cases}$$

Remark. It can be shown that the initial conditions above are equivalent to the conditions $\xi_1(0) = x_0$ and $\xi_1(t_f) = x_f$. This is a particular case of *two-point boundary conditions*, the notion that can be seen in numerous optimal control problems.

Example: filling 2 tanks



As before, $a_1 = a_2 = 1$ and $r_1 = r_2 = r = 1$ (so $\rho = r_2/r_1 = 1$), so that

$$- h_{eq1} = 0.5 \implies h_{eq2} = 0.5 h_{eq1} = 0.25 \text{ and } q_{eq} = \sqrt{0.5 h_{eq1}} = 0.5$$

Linearized dynamics

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad x(0) = x_0$$

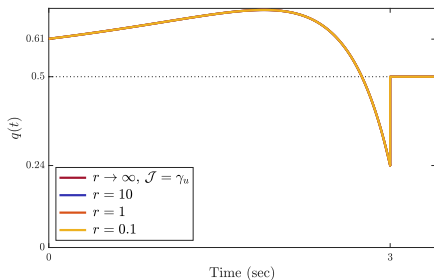
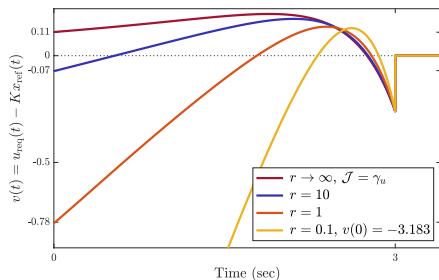
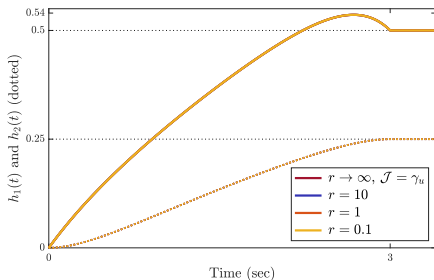
for $x = h - h_{eq}$ and $u = q - q_{eq}$. The goal is to fill up the tanks to $h = h_{eq}$ from $h(0) = 0$ with a minimum-energy deviation of the input flow from q_{eq} . Hence,

$$- x_0 = -h_{eq} \quad \text{i.e. } h(0) = 0$$

$$- x_f = 0 \implies u_f = 0 \quad \text{i.e. } h(t_f) = h_{eq} \text{ and } q(t_f) = q_{eq}$$

Example: filling 2 tanks (contd)

Taking $t_f = 3$ and K via LQR minimizing $\mathcal{J} = \gamma_1 + 16\gamma_2 + r\gamma_u$, we have



with

$$K = - \begin{bmatrix} 0 & 0 \\ 0.2502 & 0.2316 \\ 1.1431 & 1.2964 \\ 3.7346 & 5.7082 \end{bmatrix}.$$

We see that $h = x_{\text{ref}} + h_{\text{eq}}$ and $q = u_{\text{req}} + q_{\text{eq}}$ irrespective of K . Naturally, $v = u_{\text{req}} - Kx_{\text{ref}}$ depends on K .

Outline

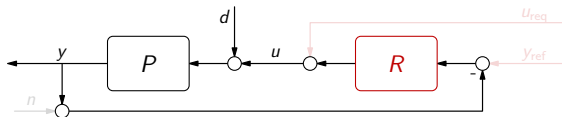
Tracking and 2DOF architecture

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Asymptotic disturbance rejection in the output-feedback setting

Disturbances and their treatment in classical control



The signal d

- represents effects of the environment independent of our actions.

Classical control is

- mostly concerned with load disturbances (i.e. acting via the u -channel)
- aiming at high-gain feedback to attenuate their effects

think of

$$|T_d(j\omega)| = \left| \frac{P(j\omega)}{1 + P(j\omega)R(j\omega)} \right| = \frac{1}{|1/P(j\omega) + R(j\omega)|}$$

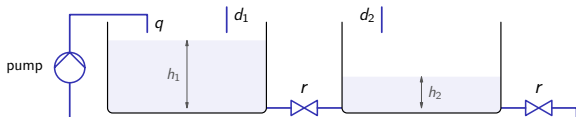
so either of

- small $|P(j\omega)|$
- large $|R(j\omega)|$

frequencies where the plant itself filters d out
frequencies where feedback works

is enough to have a small $|T_d(j\omega)|$.

Example: 2 tanks with disturbances



Linearized dynamics with old parameters and disturbances are

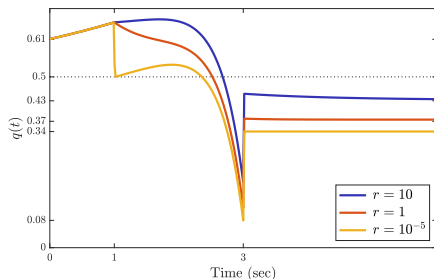
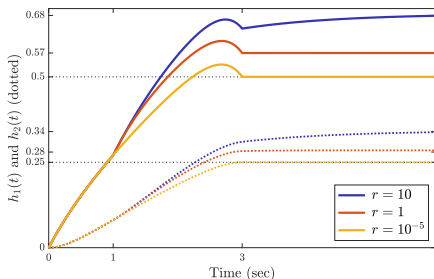
$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d_2(t), \quad x(0) = x_0$$

with

- load disturbance d_1
- unmatched disturbance d_2

Example: 2 tanks with disturbances (contd)

Tracking with v and K as before (i.e. $t_f = 3$ and $\mathcal{J} = \gamma_1 + 16\gamma_2 + r\gamma_u$), yet now with the (load) disturbances $d_1(t) = 0.161(t - 1)$ and $d_2(t) = 0$:



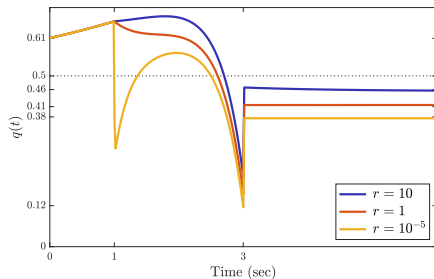
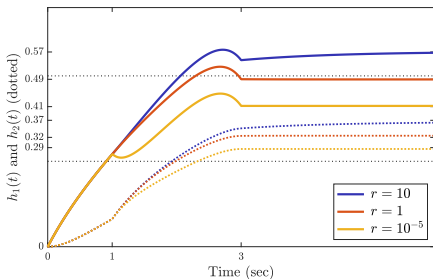
with

$$K = - \begin{bmatrix} 0.2502 & 0.2316 \\ 1.1431 & 1.2964 \\ 317.6796 & 777.8546 \end{bmatrix}.$$

High-gain feedback compensates for the effect of d_1 ($q \rightarrow u_{\text{req}} + q_{\text{eq}} - d_1$).

Example: 2 tanks with disturbances (contd)

Tracking with v and K as before (i.e. $t_f = 3$ and $\mathcal{J} = \gamma_1 + 16\gamma_2 + r\gamma_u$), yet now with the disturbances $d_1(t) = 0$ and $d_2(t) = 0.161(t - 1)$:

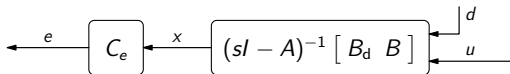


with

$$K = - \begin{bmatrix} 0.2502 & 0.2316 \\ 1.1431 & 1.2964 \\ 317.6796 & 777.8546 \end{bmatrix}.$$

Yet now high-gain feedback does not compensate for the effect of d_2 ! This suggests that it needs more than just high feedback to handle disturbances.

Disturbances in state-space settings



where

- e is a regulated signal, which is required to be small
not necessarily the same as the measured output y , although we only consider $y = e$

The signal d

- still represents effects of the environment independent of our actions.

But disturbances are treated more specifically and quantitatively. There are basically two directions¹:

- rejecting persistent disturbances from a particular class asymptotically
disturbances are modeled via exosystems, the problem reduces to output regulation
- attenuating worst-case disturbances from a wide class
mostly of transients, if the H_∞ formalism is adopted

¹Also for handling other exogenous signals, like references or measurement noises.

Modeling persistent bounded signals via exosystems

A possible model of (unmeasurable) d is

$$\begin{cases} \dot{x}_{\text{ex}}(t) = A_{\text{ex}}x_{\text{ex}}(t), & x_{\text{ex}}(0) = x_{\text{ex},0}, \\ d(t) = C_{\text{ex}}x_{\text{ex}}(t), \end{cases}$$

for

- given $A_{\text{ex}} \in \mathbb{R}^{n_d \times n_d}$ and $C_{\text{ex}} \in \mathbb{R}^{m_d \times n_d}$, with observable $(C_{\text{ex}}, A_{\text{ex}})$ reflects our knowledge about the waveform of d
- unknown $x_{\text{ex},0} \in \mathbb{R}^{n_d}$ reflects uncertainty in d , like amplitude or phase

describing the family of signals

$$d(t) = C_{\text{ex}}e^{A_{\text{ex}}t}x_{\text{ex},0} \iff D(s) = C_{\text{ex}}(sI - A_{\text{ex}})^{-1}x_{\text{ex},0}.$$

This system is

- called disturbance generator or **exosystem**

and typically $\text{spec}(A_{\text{ex}}) \subset j\mathbb{R}$, generating persistent signals.

Exosystem for a step

If

$$d(t) = d_0 \mathbb{1}(t) = \begin{array}{c} d_0 \\ | \\ \text{---} \\ 0 \end{array} \quad \begin{array}{c} t \end{array}$$

for some unknown d_0 , then

$$D(s) = \frac{d_0}{s}.$$

The corresponding signal generator is

$$\begin{cases} \dot{x}_{\text{ex}}(t) = 0 \cdot x_{\text{ex}}(t), & x_{\text{ex},0} = d_0, \\ d(t) = 1 \cdot x_{\text{ex}}(t), \end{cases}$$

i.e. $A_{\text{ex}} = 0$, with $\text{spec}(A_{\text{ex}}) = \{0\} \subset j\mathbb{R}$, and $C_{\text{ex}} = 1$.

Exosystem for a ramp

If

$$d(t) = (d_0 + d_r t) \mathbb{1}(t) =$$
A graph showing a signal d(t) as a function of time t. The horizontal axis is labeled t and has an origin marked 0. The vertical axis has a point marked d_0. The signal is zero for t < 0. At t = 0, it jumps to d_0 and then increases linearly with a constant positive slope.

for some unknown d_0 and d_r , then

$$D(s) = \frac{d_0 s + d_r}{s^2},$$

The corresponding signal generator is

$$\begin{cases} \dot{x}_{\text{ex}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_{\text{ex}}(t), & x_{\text{ex},0} = \begin{bmatrix} d_0 \\ d_r \end{bmatrix}, \\ d(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{\text{ex}}(t), \end{cases}$$

$$\text{i.e. } A_{\text{ex}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ with } \text{spec}(A_{\text{ex}}) = \{0\} \subset j\mathbb{R}, \text{ and } C_{\text{ex}} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

If

$$d(t) = a \sin(\omega t + \phi) \mathbb{1}(t) =$$


The graph shows a sine wave starting at the origin (0,0) and oscillating with amplitude 'a'. The horizontal axis is labeled 't' and the vertical axis is labeled 'a'.

for some known ω and unknown a and ϕ , then

$$D(s) = \frac{a \sin(\phi) s + a\omega \cos(\phi)}{s^2 + \omega^2},$$

The corresponding signal generator² is

$$\begin{cases} \dot{x}_{\text{ex}}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x_{\text{ex}}(t), & x_{\text{ex},0} = \begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix} a, \\ d(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{\text{ex}}(t), \end{cases}$$

i.e. $A_{\text{ex}} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, with $\text{spec}(A_{\text{ex}}) = \{\pm j\omega\} \subset j\mathbb{R}$, and $C_{\text{ex}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

²Take the observer form and apply the similarity transformation with $T = \begin{bmatrix} 1 & 0 \\ 0 & 1/\omega \end{bmatrix}$.

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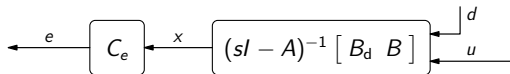
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Asymptotic disturbance rejection problem



Assuming d is generated by the exosystem

$$\begin{cases} \dot{x}_{\text{ex}}(t) = A_{\text{ex}}x_{\text{ex}}(t), & x_{\text{ex}}(0) = x_{\text{ex},0}, \\ d(t) = C_{\text{ex}}x_{\text{ex}}(t), \end{cases}$$

for given observable $(C_{\text{ex}}, A_{\text{ex}})$ and unknown $x_{\text{ex},0}$, design u such that

$$\lim_{t \rightarrow \infty} e(t) = 0$$

Classification:

- **full-information problem** if both x and x_{ex} (hence, d) are measured
- **state-feedback problem** if x , but not x_{ex} , is measured
- **output-feedback problem** if only a part of x , e.g. e , is measured

Asymptotic disturbance rejection as output regulation

Combining the plant and exosystem,

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_{\text{ex}}(t) \end{bmatrix} = \begin{bmatrix} A & B_d C_{\text{ex}} \\ 0 & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x(t) \\ x_{\text{ex}}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), & \begin{bmatrix} x(0) \\ x_{\text{ex}}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_{\text{ex},0} \end{bmatrix} \\ e(t) = \begin{bmatrix} C_e & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_{\text{ex}}(t) \end{bmatrix} \end{cases}$$

Thus,

- asymptotic rejection of d is equivalent to an *output regulator* problem, viz. ensuring $\lim_{t \rightarrow \infty} e(t) = 0$ from all initial conditions x_0 and $x_{\text{ex},0}$.

is qualitatively different from the state regulator problem because

all modes of A_{ex} are *unstabilizable* in the system, as

$$\begin{bmatrix} 0 & \eta \end{bmatrix} \begin{bmatrix} A - \lambda I & B_d C_{\text{ex}} & B \\ 0 & A_{\text{ex}} - \lambda I & 0 \end{bmatrix} = 0 \iff \eta A_{\text{ex}} = \eta \lambda I$$

is solvable in $\eta \neq 0$ for all all $\lambda \in \text{spec}(A_{\text{ex}}) \subset j\mathbb{R}$.

Asymptotic disturbance rejection as output regulation

Combining the plant and exosystem,

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_{\text{ex}}(t) \end{bmatrix} = \begin{bmatrix} A & B_d C_{\text{ex}} \\ 0 & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x(t) \\ x_{\text{ex}}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), & \begin{bmatrix} x(0) \\ x_{\text{ex}}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_{\text{ex},0} \end{bmatrix} \\ e(t) = \begin{bmatrix} C_e & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_{\text{ex}}(t) \end{bmatrix} \end{cases}$$

Thus,

- asymptotic rejection of d is equivalent to an *output regulator* problem, viz. ensuring $\lim_{t \rightarrow \infty} e(t) = 0$ from all initial conditions x_0 and $x_{\text{ex},0}$. Yet it is qualitatively different from the state regulator problem because
- all modes of A_{ex} are *unstabilizable* in the system, as

$$\begin{bmatrix} 0 & \eta' \end{bmatrix} \begin{bmatrix} A - \lambda I & B_d C_{\text{ex}} & B \\ 0 & A_{\text{ex}} - \lambda I & 0 \end{bmatrix} = 0 \iff \eta' A_{\text{ex}} = \eta' \lambda$$

is solvable in $\eta \neq 0$ for all $\lambda \in \text{spec}(A_{\text{ex}}) \subset j\mathbb{R}$.

Full-information control

Assume first that

- both x and x_{ex} are measurable,

which is the most favorable, albeit hardly realistic, scenario. The control law

$$u = Kx + K_d x_{\text{ex}} + v$$

(i.e. of the full-information type) results in the closed-loop system

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A + BK & B_d C_{\text{ex}} + BK_d \\ 0 & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x \\ x_{\text{ex}} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v \\ e = \begin{bmatrix} C_e & 0 \end{bmatrix} \begin{bmatrix} x \\ x_{\text{ex}} \end{bmatrix} \end{cases}$$

whose modes are the union of $\text{spec}(A + BK)$ and $\text{spec}(A_{\text{ex}}) \subset j\mathbb{R}$. Because the controller has no effect on the modes of A_{ex} ,

- the whole state cannot be regulated and x_{ex} will not decay.

The only way to reject it in e is to make it unobservable via a choice of K_d .

Full-information control

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(i.e. of the full-information type) results in the closed-loop system

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A + BK & B_d C_{\text{ex}} + BK_d \\ 0 & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x \\ x_{\text{ex}} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v \\ e = \begin{bmatrix} C_e & 0 \end{bmatrix} \begin{bmatrix} x \\ x_{\text{ex}} \end{bmatrix} \end{cases}$$

whose modes are the union of $\text{spec}(A + BK)$ and $\text{spec}(A_{\text{ex}}) \subset j\mathbb{R}$. Because the controller has no effect on the modes of A_{ex} ,

- the whole state cannot be regulated and x_{ex} will not decay.

The only way to reject it in e is to make it unobservable via a choice of K_d .

Closed-loop dynamics: steady state and transients

If $A + BK$ Hurwitz, then $\text{spec}(A + BK) \cap \text{spec}(A_{\text{ex}}) = \emptyset$ and the Sylvester equation

$$(A + BK)\Pi - \Pi A_{\text{ex}} + B_d C_{\text{ex}} + BK_d = 0$$

is solvable in $\Pi \in \mathbb{R}^{n \times n_d}$ for every K_d . Applying the similarity transformation with $T = \begin{bmatrix} I & -\Pi \\ 0 & I \end{bmatrix}$ (recall Roth's removal rule), the controlled dynamics read

$$\begin{cases} \begin{bmatrix} \dot{x} - \Pi \dot{x}_{\text{ex}} \\ \dot{x}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A + BK & 0 \\ 0 & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x - \Pi x_{\text{ex}} \\ x_{\text{ex}} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v \\ e = \begin{bmatrix} C_e & C_e \Pi \end{bmatrix} \begin{bmatrix} x - \Pi x_{\text{ex}} \\ x_{\text{ex}} \end{bmatrix} \end{cases}$$

If $v = 0$,

$$e(t) = C_e e^{(A+BK)t} (x_0 - \Pi x_{\text{ex},0}) + C_e \Pi e^{A_{\text{ex}} t} x_{\text{ex},0}$$

and $\underbrace{C_e e^{(A+BK)t} (x_0 - \Pi x_{\text{ex},0})}_{\text{transients}} + \underbrace{C_e \Pi e^{A_{\text{ex}} t} x_{\text{ex},0}}_{\text{steady state}}$

– $C_e \Pi = 0$ is sufficient for y to decay for every $x_{\text{ex},0}$
 (in fact, necessary too, as all modes of A_{ex} must be unobservable via $C_e \Pi$)

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If $v = 0$,

$$e(t) = \underbrace{C_e e^{(A+BK)t} (x_0 - \Pi x_{\text{ex},0})}_{\text{transients (decaying)}} + \underbrace{C_e \Pi e^{A_{\text{ex}}t} x_{\text{ex},0}}_{\text{steady state}}$$

and

— $C_e \Pi = 0$ is sufficient for y to decay for every $x_{\text{ex},0}$

(in fact, necessary too, as all modes of A_{ex} must be unobservable via $C_e \Pi$).

Regulator equation

Shifting $K_d = \Gamma - K\Pi$, conditions

$$(A + BK)\Pi - \Pi A_{\text{ex}} + B_d C_{\text{ex}} + BK_d = 0 \quad \text{and} \quad C_e \Pi = 0$$

can be equivalently written in problem data as

$$\begin{bmatrix} A & B \\ C_e & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} A_{\text{ex}} + \begin{bmatrix} B_d C_{\text{ex}} \\ 0 \end{bmatrix} = 0.$$

This relation is known the **regulator equation** (aka Francis' equation). If it's solvable in $\Pi \in \mathbb{R}^{n \times n_d}$ and $\Gamma \in \mathbb{R}^{m \times n_d}$, then the control law

$$u = Kx + (\Gamma - K\Pi)x_{\text{ex}} = K(x - \Pi x_{\text{ex}}) + \Gamma x_{\text{ex}}$$

solves the output regulation problem and thus rejects d asymptotically.

Example: 2 tanks with disturbances (contd)

Two cases under $C_e = I_2$ (here $n_d = 1$ with $A_{ex} = 0$ and $C_{ex} = 1$):

- if $B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (disturbance in the first tank), then

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & -2 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \begin{bmatrix} \pi_1 \\ \pi_2 \\ \gamma \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \implies \begin{bmatrix} \pi \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

and $u = Kx - x_{ex} = Kx - d$ solves the problem for any stabilizing K ;

- if $B_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (disturbance in the second tank), then

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & -2 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \begin{bmatrix} \pi_1 \\ \pi_2 \\ \gamma \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \implies \text{no solutions}$$

(from the last two rows $\pi_1 = \pi_2 = 0$, so the second row reads $1 = 0$).

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Example: 2 tanks with disturbances (contd)

If only the liquid level in the second tank is of interest, i.e. if $C_e = \begin{bmatrix} 0 & 1 \end{bmatrix}$, and step disturbances affect both tanks, then $n_d = 2$ and

$$B_d = I_2, \quad A_{\text{ex}} = 0, \quad \text{and} \quad C_{\text{ex}} = I_2.$$

The regulator equation

$$\begin{bmatrix} -1 & 1 & | & 1 \\ 1 & -2 & | & 0 \\ \hline 0 & 1 & | & 0 \end{bmatrix} \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \\ \hline \gamma_1 & \gamma_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{bmatrix} = 0 \implies \begin{bmatrix} \pi \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

and

$$u = Kx + \left(-\begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & k_1 \end{bmatrix} \right) x_{\text{ex}} = Kx - d_1 + (k_1 - 1)d_2$$

The controlled dynamics are then

$$\dot{x} = \begin{bmatrix} k_1 - 1 & k_2 + 1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & k_1 - 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \implies T_d(0) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Example: 2 tanks with disturbances (contd)

If only the liquid level in the second tank is of interest, i.e. if $C_e = \begin{bmatrix} 0 & 1 \end{bmatrix}$, and step disturbances affect both tanks, then $n_d = 2$ and

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The regulator equation

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & -2 & 0 \\ \hline 0 & 1 & 0 \end{array} \right] \left[\begin{array}{cc} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \\ \hline \gamma_1 & \gamma_2 \end{array} \right] + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array} \right] = 0 \implies \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ \hline -1 & -1 \end{bmatrix}$$

and

$$u = Kx + \left(- \begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & k_1 \end{bmatrix} \right) x_{\text{ex}} = Kx - d_1 + (k_1 - 1)d_2$$

The controlled dynamics are then

$$\dot{x} = \begin{bmatrix} k_1 - 1 & k_2 + 1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 & k_1 - 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \implies T_d(0) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Solvability of regulator equation

Lemma 8.1

Equation

$$\begin{bmatrix} A & B \\ C_e & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} A_{\text{ex}} + \begin{bmatrix} B_d C_{\text{ex}} \\ 0 \end{bmatrix} = 0$$

is solvable in $(\Pi, \Gamma) \in \mathbb{R}^{n \times n_d} \times \mathbb{R}^{m \times n_d}$ for every $B_d C_{\text{ex}}$ iff $\begin{bmatrix} A - \lambda I & B \\ C_e & 0 \end{bmatrix}$ has full row rank for all $\lambda \in \text{spec}(A_{\text{ex}})$.

The condition of Lemma 8.1 says that the plant

- is not underactuated (i.e. $\dim u \geq \dim e$ and the rank is full)
- has no zeros at the modes of the disturbance model

Remark. The condition of Lemma 8.1 might no longer be necessary for specific choices of $B_d C_{\text{ex}}$. For example, if $B_d = B M_d$ (matched disturbances), then $\Pi = 0$ and $\Gamma = -M_d C_{\text{ex}}$ solve the equation regardless other properties (we just saw that in the example). The control signal $u = Kx - M_d C_{\text{ex}} x_{\text{ex}} = Kx - M_d d$ then, not quite surprisingly.

Outline

Tracking and 2DOF architecture

Disturbances and their modeling

Asymptotic disturbance rejection in the full-information setting

Asymptotic disturbance rejection in the output-feedback setting

Setup

Let

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_{\text{ex}}(t) \end{bmatrix} = \begin{bmatrix} A & B_d C_{\text{ex}} \\ 0 & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x(t) \\ x_{\text{ex}}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), & \begin{bmatrix} x(0) \\ x_{\text{ex}}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_{\text{ex},0} \end{bmatrix} \\ e(t) = \begin{bmatrix} C_e & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_{\text{ex}}(t) \end{bmatrix} \end{cases}$$

assuming that

$\mathcal{A}_1$: (A, B) is stabilizable

$\mathcal{A}_2$: $\left(\begin{bmatrix} C_e & 0 \end{bmatrix}, \begin{bmatrix} A & B_d C_{\text{ex}} \\ 0 & A_{\text{ex}} \end{bmatrix} \right)$ is detectable

$\mathcal{A}_3$: $\begin{bmatrix} A & B \\ C_e & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} A_{\text{ex}} + \begin{bmatrix} B_d C_{\text{ex}} \\ 0 \end{bmatrix} = 0$ is solvable in (Π, Γ)

The control goal is to

- ensure $\lim_{t \rightarrow \infty} e(t) = 0$ measuring only the regulated signal e

Observer-based controller

Under $\mathcal{A}_3$, the full-information controller $u = Kx + (\Gamma - K\P)x_{\text{ex}} + v$ would solve the problem under any stabilizing K . Yet x and x_{ex} are not measured. So we use its observer-based version

$$u = K\hat{x} + (\Gamma - K\P)\hat{x}_{\text{ex}} + v,$$

where

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{x}}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A & B_d C_{\text{ex}} \\ 0 & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_{\text{ex}} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u - \begin{bmatrix} L \\ L_{\text{ex}} \end{bmatrix} \left(e - \begin{bmatrix} C_e & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_{\text{ex}} \end{bmatrix} \right)$$

for any $L \in \mathbb{R}^{n \times p}$ and $L_{\text{ex}} \in \mathbb{R}^{n_d \times p}$ such that

$$\begin{bmatrix} A & B_d C_{\text{ex}} \\ 0 & A_{\text{ex}} \end{bmatrix} + \begin{bmatrix} L \\ L_{\text{ex}} \end{bmatrix} \begin{bmatrix} C_e & 0 \end{bmatrix} = \begin{bmatrix} A + LC_e & B_d C_{\text{ex}} \\ L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix}$$

is Hurwitz (exist by $\mathcal{A}_2$).

Observer-based controller (contd)

Substituting u into the observer equation, we end up with

$$\begin{cases} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{x}}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A + BK + LC_e & B_d C_{\text{ex}} + B(\Gamma - K\Pi) \\ L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_{\text{ex}} \end{bmatrix} \\ u = [K \quad \Gamma - K\Pi] \begin{bmatrix} \hat{x} \\ \hat{x}_{\text{ex}} \end{bmatrix} + v - \begin{bmatrix} L \\ L_{\text{ex}} \end{bmatrix} e + \begin{bmatrix} B \\ 0 \end{bmatrix} v \end{cases}$$

Alternative realization: introduce the variable

$$x_R := \begin{bmatrix} \hat{x} - \Pi \hat{x}_{\text{ex}} \\ \hat{x}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} I & -\Pi \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_{\text{ex}} \end{bmatrix},$$

in terms of which (mind A_3) the controller can be rewritten as

$$\begin{cases} \dot{x}_R = \begin{bmatrix} A + BK + (L - \Pi L_{\text{ex}}) C_e & 0 \\ L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix} x_R - \begin{bmatrix} L - \Pi L_{\text{ex}} \\ L_{\text{ex}} \end{bmatrix} e + \begin{bmatrix} B \\ 0 \end{bmatrix} v \\ u = [K \quad \Gamma] x_R \end{cases}$$

(this form will be discussed later on).

Observer-based controller (contd)

Substituting u into the observer equation, we end up with

$$\begin{cases} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{x}}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A + BK + LC_e & B_d C_{\text{ex}} + B(\Gamma - K\Pi) \\ L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_{\text{ex}} \end{bmatrix} \\ u = [K \quad \Gamma - K\Pi] \begin{bmatrix} \hat{x} \\ \hat{x}_{\text{ex}} \end{bmatrix} + v - \begin{bmatrix} L \\ L_{\text{ex}} \end{bmatrix} e + \begin{bmatrix} B \\ 0 \end{bmatrix} v \end{cases}$$

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$$x_R := \begin{bmatrix} \hat{x} - \Pi \hat{x}_{\text{ex}} \\ \hat{x}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} I & -\Pi \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_{\text{ex}} \end{bmatrix},$$

in terms of which (mind **A₃**) the controller can be rewritten as

$$\begin{cases} \dot{x}_R = \begin{bmatrix} A + BK + (L - \Pi L_{\text{ex}})C_e & 0 \\ L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix} x_R - \begin{bmatrix} L - \Pi L_{\text{ex}} \\ L_{\text{ex}} \end{bmatrix} e + \begin{bmatrix} B \\ 0 \end{bmatrix} v \\ u = [K \quad \Gamma] x_R \end{cases}$$

(this form will be discussed later on).

Closed-loop dynamics

Combining plant and controller dynamics and setting for brevity $v = 0$, the closed-loop dynamics read

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{\text{ex}} \\ \dot{\hat{x}} \\ \dot{\hat{x}}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A & B_d C_{\text{ex}} & BK & B(\Gamma - K\Pi) \\ 0 & A_{\text{ex}} & 0 & 0 \\ -LC_e & 0 & A + BK + LC_e & B_d C_{\text{ex}} + B(\Gamma - K\Pi) \\ -L_{\text{ex}} C_e & 0 & L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x \\ x_{\text{ex}} \\ \hat{x} \\ \hat{x}_{\text{ex}} \end{bmatrix}$$

or, transforming coordinates “observed states $\rightarrow$ observation errors,”

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{\text{ex}} \\ \dot{\epsilon} \\ \dot{\epsilon}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A + BK & B_d C_{\text{ex}} + B(\Gamma - K\Pi) & BK & B(\Gamma - K\Pi) \\ 0 & A_{\text{ex}} & 0 & 0 \\ 0 & 0 & A + LC_e & B_d C_{\text{ex}} \\ 0 & 0 & L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x \\ x_{\text{ex}} \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix}$$

where $\epsilon := x - \hat{x}$ and $\epsilon_{\text{ex}} := x_{\text{ex}} - \hat{x}_{\text{ex}}$. This is the standard separation, but

- the closed-loop dynamics still include unstable modes, those of A_{ex} .

Closed-loop dynamics (contd)

Introducing yet another change of coordinates,

$$\begin{bmatrix} x \\ x_{\text{ex}} \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix} \rightarrow x_{\text{cl}} := \begin{bmatrix} x_{\text{ex}} \\ \xi \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix} := \begin{bmatrix} x_{\text{ex}} \\ x - \Pi x_{\text{ex}} \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ I & -\Pi & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ x_{\text{ex}} \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix},$$

we end up with the closed-loop system (in this stage v is added as well)

$$\begin{cases} \begin{bmatrix} \dot{x}_{\text{ex}} \\ \dot{\xi} \\ \dot{\epsilon} \\ \dot{\epsilon}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A_{\text{ex}} & 0 & 0 & 0 \\ 0 & A + BK & BK & B(\Gamma - K\Pi) \\ 0 & 0 & A + LC_e & B_d C_{\text{ex}} \\ 0 & 0 & L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x_{\text{ex}} \\ \xi \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix} + \begin{bmatrix} 0 \\ B \\ 0 \\ 0 \end{bmatrix} v \\ e = [0 \quad C_e \quad 0 \quad 0] x_{\text{cl}} \end{cases}$$

(to derive it $\mathcal{A}_3$ was used).

— disturbance dynamics are isolated from the other states and from e .

Closed-loop dynamics (contd)

Introducing yet another change of coordinates,

$$\begin{bmatrix} x \\ x_{\text{ex}} \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix} \rightarrow x_{\text{cl}} := \begin{bmatrix} x_{\text{ex}} \\ \xi \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix} := \begin{bmatrix} x_{\text{ex}} \\ x - \Pi x_{\text{ex}} \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ I & -\Pi & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ x_{\text{ex}} \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix},$$

we end up with the closed-loop system (in this stage v is added as well)

$$\begin{cases} \begin{bmatrix} \dot{x}_{\text{ex}} \\ \dot{\xi} \\ \dot{\epsilon} \\ \dot{\epsilon}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} A_{\text{ex}} & 0 & 0 & 0 \\ 0 & A + BK & BK & B(\Gamma - K\Pi) \\ 0 & 0 & A + LC_e & B_d C_{\text{ex}} \\ 0 & 0 & L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix} \begin{bmatrix} x_{\text{ex}} \\ \xi \\ \epsilon \\ \epsilon_{\text{ex}} \end{bmatrix} + \begin{bmatrix} 0 \\ B \\ 0 \\ 0 \end{bmatrix} v \\ e = [0 \mid C_e \ 0 \ 0] x_{\text{cl}} \end{cases}$$

(to derive it $\mathcal{A}_3$ was used). Important is that the

- disturbance dynamics are isolated from the other states and from e .

Observer-based controller and the internal model principle

Return to the controller in the form

$$R : \begin{cases} \dot{x}_R = \begin{bmatrix} A + BK + (L - \Pi L_{\text{ex}})C_e & 0 \\ L_{\text{ex}}C_e & \textcolor{red}{A}_{\text{ex}} \end{bmatrix} x_R - \begin{bmatrix} L - \Pi L_{\text{ex}} \\ L_{\text{ex}} \end{bmatrix} e + \begin{bmatrix} B \\ 0 \end{bmatrix} v \\ u = \begin{bmatrix} K & \Gamma \end{bmatrix} x_R \end{cases}$$

Remarkable property is that the

- controller has all modes of the disturbance model as its own modes.

This is a manifestation of the **internal model principle**, whose essence is that

- the controller should contain a model of persistent exogenous signals to be capable of rejecting them. A classical example of its utilization is the use of an integral action in PI controllers, aiming at constant disturbances.

To have the modes of A_{ex} as poles of the transfer function of the controller $R : e \mapsto u$, we need their controllability and observability.

Observer-based controller and the internal model principle

Return to the controller in the form

$$R : \begin{cases} \dot{x}_R = \begin{bmatrix} A + BK + (L - \Pi L_{\text{ex}})C_e & 0 \\ L_{\text{ex}}C_e & \textcolor{red}{A}_{\text{ex}} \end{bmatrix} x_R - \begin{bmatrix} L - \Pi L_{\text{ex}} \\ L_{\text{ex}} \end{bmatrix} e + \begin{bmatrix} B \\ 0 \end{bmatrix} v \\ u = \begin{bmatrix} K & \Gamma \end{bmatrix} x_R \end{cases}$$

Remarkable property is that the

- controller has all modes of the disturbance model as its own modes.

This is a manifestation of the **internal model principle**, whose essence is that

- the controller should contain a model of persistent exogenous signals to be capable of rejecting them. A classical example of its utilization is the use of an integral action in PI controllers, aiming at constant disturbances.

To have the modes of A_{ex} as poles of the transfer function of the controller $R : e \mapsto u$, we need their controllability and observability.

Exosystem modes in controller: controllability

The corresponding PBH matrix

$$\begin{aligned}
 & \begin{bmatrix} A + BK + (L - \Pi L_{\text{ex}})C_e - \lambda I & 0 & L - \Pi L_{\text{ex}} \\ L_{\text{ex}}C_e & A_{\text{ex}} - \lambda I & L_{\text{ex}} \end{bmatrix} \quad \overbrace{\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ C_e & 0 & I \end{bmatrix}}^{\text{full-rank}} \\
 &= \left[\begin{array}{c|cc} A + BK - \lambda I & 0 & L - \Pi L_{\text{ex}} \\ \hline 0 & A_{\text{ex}} - \lambda I & L_{\text{ex}} \end{array} \right] \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ C_e & 0 & I \end{bmatrix}
 \end{aligned}$$

Hence, the uncontrollable modes of A_{ex} are those of $(A_{\text{ex}}, L_{\text{ex}})$. Assume the contrary, that there are $\omega \in \mathbb{R}$ and $\eta \neq 0$ such that

$$\eta' \begin{bmatrix} A_{\text{ex}} - j\omega I & L_{\text{ex}} \end{bmatrix} = 0 \implies \begin{bmatrix} 0 & \eta' \end{bmatrix} \begin{bmatrix} A + LC_e - j\omega I & B_d C_{\text{ex}} \\ L_{\text{ex}} C_e & A_{\text{ex}} - j\omega I \end{bmatrix} = 0$$

(remember, $\text{spec}(A_{\text{ex}}) \subset j\mathbb{R}$). In other words,

$$-j\omega \text{ is an eigenvalue of } \begin{bmatrix} A + LC_e & B_d C_{\text{ex}} \\ L_{\text{ex}} C_e & A_{\text{ex}} \end{bmatrix},$$

which contradicts the Hurwitz assumption about it (observer dynamics).

Exosystem modes in controller: observability

The corresponding PBH matrix now is

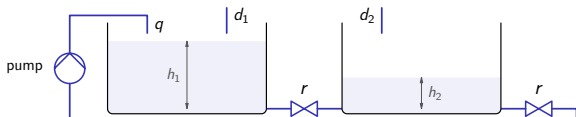
$$\left[\begin{array}{c|c} A + BK + (L - \Pi L_{\text{ex}})C_e - \lambda I & 0 \\ \hline L_{\text{ex}}C_e & A_{\text{ex}} - \lambda I \\ K & \Gamma \end{array} \right]$$

by which the unobservable modes of A_{ex} are those of (Γ, A_{ex}) . Assume the contrary, that there are $\omega \in \mathbb{R}$ and $\eta \neq 0$ such that $A_{\text{ex}}\eta = j\omega\eta$ and $\Gamma\eta = 0$. Post-multiplying the regulator equation by η yields

$$\begin{bmatrix} A - j\omega I & B_d C_{\text{ex}} \\ C_e & 0 \end{bmatrix} \begin{bmatrix} \Pi\eta \\ \eta \end{bmatrix} = 0 \implies \begin{bmatrix} A - j\omega I & B_d C_{\text{ex}} \\ C_e & A_{\text{ex}} - j\omega I \\ & 0 \end{bmatrix} \begin{bmatrix} \Pi\eta \\ \eta \end{bmatrix} = 0,$$

which, in turn, implies that $([C_e \ 0], [A \ B_d C_{\text{ex}}; 0 \ A_{\text{ex}}])$ has an unobservable mode at the imaginary axis, thus contradicting assumption $\mathcal{A}_2$.

Example



As before, $a_1 = a_2 = 1$ and $r_1 = r_2 = r = 1$ (so $\rho = r_2/r_1 = 1$), so that

$$- h_{\text{eq}1} = 0.5 \implies h_{\text{eq}2} = 0.5 h_{\text{eq}1} = 0.25 \text{ and } q_{\text{eq}} = \sqrt{0.5 h_{\text{eq}1}} = 0.5$$

Linearized dynamics

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad x(0) = x_0$$

for $x = h - h_{\text{eq}}$ and $u = q - q_{\text{eq}}$. The goal is to fill up the second tank to $h_2 = h_{\text{eq}2}$ from $h(0) = 0$ despite disturbances of the form

- $d_1(t) = d_{10} \mathbb{1}(t - 1)$
- $d_2(t) = a_2 \sin(\omega_2 t + \phi_2) \mathbb{1}(t - 1) + 0.5 d_1(t)$

for a given $\omega_2 > 0$ and arbitrary d_{10} , a_2 , and ϕ_2 (assuming $d_{10} \geq 2a_2$).

Example: exosystem

The signal

$$d(t) = \begin{bmatrix} d_{10} \mathbb{1}(t) \\ a_2 \sin(\omega_2 t + \phi_2) \mathbb{1}(t) + 0.5 d_{10} \mathbb{1}(t) \end{bmatrix}$$

can be generated by the exosystem with

$$A_{\text{ex}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_2 \\ 0 & -\omega_2 & 0 \end{bmatrix}, \quad C_{\text{ex}} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \end{bmatrix}, \quad x_{\text{ex},0} = \begin{bmatrix} d_{10} \\ a_2 \sin(\phi_2) \\ a_2 \cos(\phi_2) \end{bmatrix}$$

for which

$$x_{\text{ex}}(t) = \begin{bmatrix} d_{10} \mathbb{1}(t) \\ a_2 \sin(\omega_2 t + \phi_2) \mathbb{1}(t) \\ a_2 \cos(\omega_2 t + \phi_2) \mathbb{1}(t) \end{bmatrix}.$$

Example: $\mathcal{A}_2$

With $e = x_2$ (second tank), so that $C_e = \begin{bmatrix} 0 & 1 \end{bmatrix}$,

$$\left(\begin{bmatrix} C_e & 0 \end{bmatrix}, \begin{bmatrix} A & B_d C_{ex} \\ 0 & A_{ex} \end{bmatrix} \right) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0.5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_2 \\ 0 & 0 & 0 & -\omega_2 & 0 \end{bmatrix} \right)$$

The observability matrix

$$M_o = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -2 & 0.5 & 1 & 0 \\ -3 & 5 & 0 & -2 & \omega_2 \\ 8 & -13 & -0.5 & -\omega_2^2 + 5 & -2\omega_2 \\ -21 & 34 & 1.5 & 2\omega_2^2 - 13 & -\omega_2(\omega_2^2 - 5) \end{bmatrix}$$

and $\det M_o = -1.5\omega_2^3(\omega_2^2 + 1) \neq 0$ for all $\omega_2 \neq 0$. Hence, $\mathcal{A}_2$ holds.

Example: $\mathcal{A}_3$

The regulator equation

$$\begin{bmatrix} -1 & 1 & | & 1 \\ 1 & -2 & | & 0 \\ \hline 0 & 1 & | & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} - \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ \hline 0 & 0 & | & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega_2 \\ 0 & -\omega_2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ \hline 0 & 0 & 0 \end{bmatrix} = 0$$

is solved by

$$\Pi = - \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Gamma = - \begin{bmatrix} 1.5 & 1 & \omega_2 \end{bmatrix},$$

Hence, $\mathcal{A}_3$ holds. The corresponding full-information control law would be

$$u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} x - \begin{bmatrix} 1.5 - 0.5k_1 & 1 - k_1 & \omega_2 \end{bmatrix} x_{\text{ex}}$$

for any stabilizing $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ (i.e. such that $k_1 < 3$ and $k_2 < 1 - 2k_1$).

Example: design

The overall system:

$$\left\{ \begin{array}{l} \begin{bmatrix} \dot{x} \\ \dot{x}_{\text{ex}} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0.5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_2 \\ 0 & 0 & 0 & -\omega_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_{\text{ex}} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \\ e = [0 \ 1 \ 0 \ 0 \ 0] \begin{bmatrix} x \\ x_{\text{ex}} \end{bmatrix} \end{array} \right.$$

Two elements of observer-based controller:

- state-feedback gain K is designed by LQR for $\mathcal{J} = \gamma_1 + 16\gamma_2 + \gamma_u$ because all modes of A_{ex} are uncontrollable, LQR is only concerned with the plant
- observer for the 5-order augmented system is selected via Ackermann's formula placing all 5 observer modes to $-1/\tau$ for some $\tau > 0$.

Example: controllers in simulations

Simulations are carried out for $\omega_2 = \pi$, $d_{10} = 0.16$, $a_2 = 0.05$, $\phi_2 = 1.466$. Depending on the choice of τ (time constant of the observer), the transfer functions of the controllers $R : e \mapsto u$ are:

- if $\tau = 0.2$, then

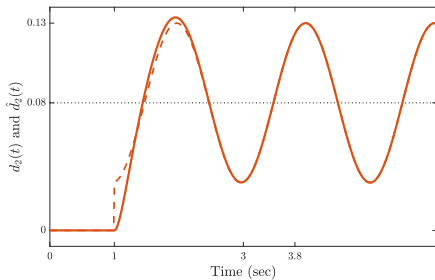
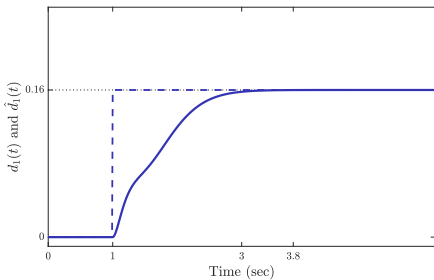
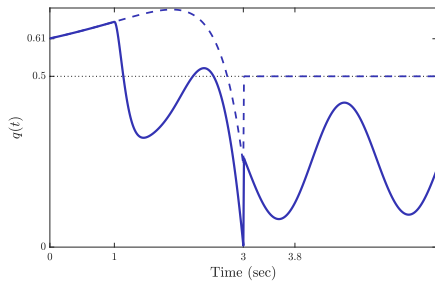
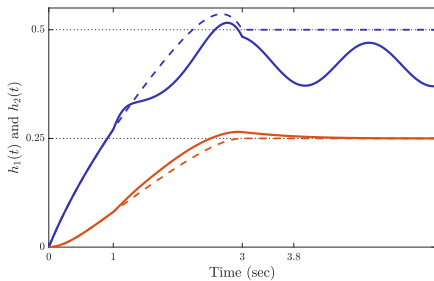
$$R(s) = -\frac{1280(s + 1.965)(s + 171.9)(s^2 + 0.7646s + 3.124)}{s(s^2 + 9.87)(s + 2.898)(s + 240.4)}$$

- if $\tau = 0.1$, then

$$R(s) = -\frac{10644(s + 1.985)(s + 843)(s^2 + 4.234s + 10.53)}{s(s^2 + 9.87)(s + 2.946)(s + 990.3)}$$

In both cases the controllers have poles matching the modes of A_{ex} , viz. in $\{0, \pm j\pi\}$.

Example: simulations for $\tau = 0.2$



Example: simulations for $\tau = 0.1$

