

State-Space Control Fundamentals (00860721)

lecture no. 7

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Outline

Observability

Example: 2-mass system (observability)

Minimality

State observers: pole placement and reduced-order observers

Observer-based feedback

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Observability: definition

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0, \\ y(t) = Cx(t) + Du(t). \end{cases}$$

This system, or the pair (C, A) , is said to be

- **observable** if any initial state x_0 can be reconstructed from time history of $u : [0, t_1] \rightarrow \mathbb{R}^m$ and $y : [0, t_1] \rightarrow \mathbb{R}^p$ for every $t_1 > 0$ and u .

Remark: Without loss of generality (w.l.o.g.) we can assume that $u = 0$. Indeed,

$$y(t) = Ce^{At}x_0 + Du(t) + C \int_0^t e^{A(t-s)}Bu(s)ds \iff \tilde{y}(t) = Ce^{At}x_0,$$

where

$$\tilde{y}(t) := y(t) - Du(t) - C \int_0^t e^{A(t-s)}Bu(s)ds.$$

Hence, reconstructing x_0 from y and $u \iff$ reconstructing x_0 from $\tilde{y}$ and $\tilde{u} = 0$.

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A possible algorithm

If $u = 0$, then $y(t) = Ce^{At}x_0$ and

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} = \overbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}^{M_o} e^{At}x_0.$$

Hence,

$$\begin{bmatrix} y(t_1) \\ \dot{y}(t_1) \\ \vdots \\ y^{(n-1)}(t_1) \end{bmatrix} = M_o e^{At_1}x_0$$

is solvable in any x_0 and for every $t_1 > 0$ whenever¹ $M_o \in \mathbb{R}^{(np) \times n}$ has full column rank. Thus,

- (C, A) is observable if $\text{rank } M_o = n$.

¹Remember, e^{At_1} is nonsingular, with $(e^{At_1})^{-1} = e^{-At_1}$, for every $t_1 \geq 0$.

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What if M_o is rank-deficient ?

If $\text{rank } M_o < n$, then there is $0 \neq \eta \in \mathbb{R}^n$ such that $M_o \eta = 0$, i.e. $CA^i \eta = 0$ for all $i \in \mathbb{Z}_{0..n-1}$. By Cayley–Hamilton, this implies that

$$CA^i \eta = 0, \quad \forall i \geq 0 \implies Ce^{At} \eta = 0, \quad \forall t \geq 0.$$

Hence,

$$\left. \begin{array}{l} x_0 = \eta \\ x_0 = 0 \end{array} \right\} \implies y(t) = Ce^{At} x_0 = 0, \quad \forall t \geq 0$$

and we can never tell whether it was $x_0 = \eta$ or $x_0 = 0$. Thus,

- (C, A) is not observable if $\text{rank } M_o < n$.

The matrix

$$M_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{(np) \times n}$$

called the **observability matrix** (M_o' is the controllability matrix for (A', C')).

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Criteria

Theorem 7.1

The following statements are equivalent:

1. (C, A) is observable
2. (A', C') is controllable
3. $M_o \in \mathbb{R}^{(np) \times n}$ has full column rank
4. $W_o(t) > 0$ for all $t > 0$, where $W_o(t) := \int_0^t e^{A's} C' C e^{As} ds \in \mathbb{R}^{n \times n}$
5. $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \in \mathbb{C}^{(n+p) \times n}$ has full column rank $\forall \lambda \in \mathbb{C}$ **PBH test**

If A is Hurwitz, then $W_o := \lim_{t \rightarrow \infty} W_o(t)$ exists and solves the Lyapunov

$$A' W_o + W_o A + C' C = 0$$

Matrix W_o is known as the **observability Gramian**.

sometimes $W_o(t)$

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Observability and similarity transformations

If $\tilde{A} = TAT^{-1}$ and $\tilde{C} = CT^{-1}$, then (remember, $(TAT^{-1})^k = TA^k T^{-1}$)

$$\tilde{M}_o = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \begin{bmatrix} CT^{-1} \\ CAT^{-1} \\ \vdots \\ CA^{n-1}T^{-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T^{-1} = M_o T^{-1}.$$

Likewise,

$$\tilde{W}_o(t) = \int_0^t e^{\tilde{A}'s} \tilde{C}' \tilde{C} e^{\tilde{A}s} ds = T^{-'} W_o(t) T^{-1}.$$

Hence,

- observability is not affected by similarity transformations.

If $p = 1$, then

$$T = \tilde{M}_o^{-1} M_o$$

is the transformation matrix connecting two similar observable realizations.

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Unobservable modes and detectability

If (C, A) is not observable, then PBH fails for some $\lambda_i \in \mathbb{C}$ such that

$$\text{rank} \begin{bmatrix} A - \lambda_i I \\ C \end{bmatrix} < n \implies \exists \eta_i \neq 0 \text{ such that } \begin{bmatrix} A - \lambda_i I \\ C \end{bmatrix} \eta_i = 0.$$

Thus, PBH fails at $\lambda_i \in \mathbb{C}$ only if $\lambda_i \in \text{spec}(A)$ and η_i is a right eigenvector of A for this λ_i .

Eigenvalues of A at which PBH fails are called

- **unobservable modes** of (C, A) .

Unobservable modes of (C, A) are eigenvalues of $A + LC$ for every L .

Pair (C, A) is said to be

- **detectable** if all its unobservable modes are stable,

i.e. in the open LHP $\{z \in \mathbb{C} \mid \text{Re } z < 0\}$.

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Outline

Observability

Example: 2-mass system (observability)

Minimality

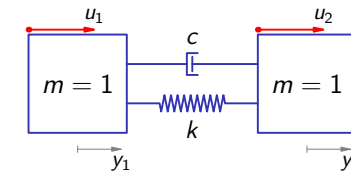
State observers: pole placement and reduced-order observers

Observer-based feedback

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Setup

Consider again



with

$$\begin{cases} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_C \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} \end{cases}$$

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Observability under $y = \gamma_1 y_1 + \gamma_2 y_2$

In this case

$$y(t) = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & 0 & 0 \end{bmatrix} x(t)$$

Observability matrix (denoting $\delta_\gamma := \gamma_1 - \gamma_2$):

$$M_o = \begin{bmatrix} \gamma_1 & \gamma_2 & 0 & 0 \\ 0 & 0 & \gamma_1 & \gamma_2 \\ -k\delta_\gamma & k\delta_\gamma & -c\delta_\gamma & c\delta_\gamma \\ 2ck\delta_\gamma & -2ck\delta_\gamma & -(k-2c^2)\delta_\gamma & (k-2c^2)\delta_\gamma \end{bmatrix}$$

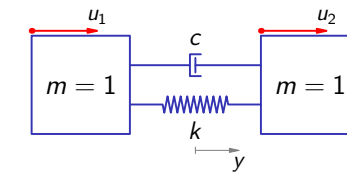
with $\det M_o = -k^2(\gamma_1^2 - \gamma_2^2)^2$. Thus, the system is

- unobservable for $\gamma_1 = \pm \gamma_2$.

Q: what could it mean?

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Observability with $\gamma_1 = \gamma_2$, e.g. $y = \frac{y_1 + y_2}{2}$



PBH test:

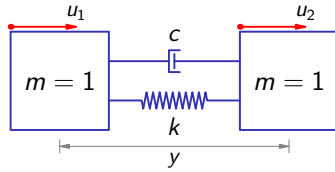
$$\text{rank} \left[\begin{array}{cccc} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -k & k & -c-\lambda & c \\ k & -k & c & -c-\lambda \\ 1 & 1 & 0 & 0 \end{array} \right] \bigg|_{\lambda = -c \pm \sqrt{c^2 - 2k}} = 3,$$

(rank lost at **unobservable modes** of A). This agrees with our intuition that

- oscillations cannot be seen via the center of mass.

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Observability with $\gamma_1 = -\gamma_2$, e.g. $y = y_1 - y_2$



PBH test:

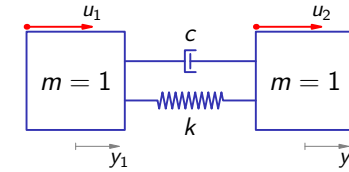
$$\text{rank} \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -k & k & -c-\lambda & c \\ k & -k & c & -c-\lambda \\ 1 & -1 & 0 & 0 \end{bmatrix} \bigg|_{\lambda=0} = 3,$$

(rank lost at **unobservable mode** of A). This agrees with our intuition that

- rigid body motion cannot be seen via relative position of the masses.

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Transfer functions for $y = \gamma_1 y_1 + \gamma_2 y_2$



Transfer function from u_1 to y :

$$P_1(s) = \frac{\gamma_1 s^2 + c(\gamma_1 + \gamma_2)s + k(\gamma_1 + \gamma_2)}{s^2(s^2 + 2cs + 2k)}$$

and transfer function from u_2 to y :

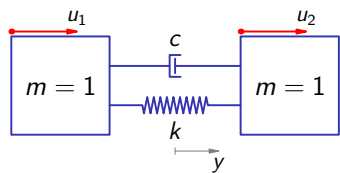
$$P_2(s) = \frac{\gamma_2 s^2 + c(\gamma_1 + \gamma_2)s + k(\gamma_1 + \gamma_2)}{s^2(s^2 + 2cs + 2k)}$$

(both obtained via $C(sI - A)^{-1}B$).

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Transfer functions for $y = \gamma_1 y_1 + \gamma_2 y_2$ (contd)

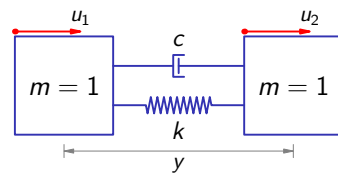
$\gamma_1 = \gamma_2$:



then

$$P_1(s) = P_2(s) = \frac{\gamma_1}{s^2}.$$

$\gamma_1 = -\gamma_2$:



then

$$P_1(s) = -P_2(s) = \frac{\gamma_1}{s^2 + 2cs + 2k}.$$

In both cases we have **pole/zero cancellations** (of different modes though).

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Minimal state-space realization

Example 7.2

Let $G(s) = 1/(s - 1)$. The following are its state-space realizations:

$$\begin{cases} \dot{x} = x + u \\ y = x \end{cases} \quad \text{and} \quad \begin{cases} \dot{\tilde{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}. \end{cases}$$

The first of them has the state dimension $n = 1$, the second one has $n = 2$. This indicates that there is *redundancy* in $\tilde{x}$.

We may be interested to detect redundancy. To this end, the notion of

- minimal state-space realization, i.e. a realization of a minimal possible order,

plays a key role.

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Minimality criterion

Theorem 7.3

Realization

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = 0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is minimal iff it is both controllable and observable.

Explanations:

- uncontrollable part of x cannot be affected by input u ,
- unobservable part of x is invisible from output y .

Remark: Uncontrollable modes of (A, B) and unobservable modes of (C, A) are collectively known as *hidden modes* of the realization (A, B, C, D) . They do not affect i/o properties of the system, although could affect the response of other variables and / or be excited by other inputs or by initial conditions.

Theorem 7.4

Every two minimal realizations of the same system are similar.

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Minimality and poles / zeros

Theorem 7.5

Let

$$G(s) = D + C(sI - A)^{-1}B = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

for some $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$. If the realization (A, B, C, D) is minimal, then

- $\lambda \in \mathbb{C}$ is a pole of $G(s)$ iff it is an eigenvalue of A .
- $\lambda \in \mathbb{C}$ is a zero of $G(s)$ iff $\text{rank} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = n - \min\{p, m\}$.

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State observer: choice of L

Let (C, A) be observable, then for an arbitrary polynomial

$$\hat{\chi}_{cl}(s) = s^n + \hat{\chi}_{n-1}s^{n-1} + \dots + \hat{\chi}_1s + \hat{\chi}_0$$

there exists observer gain L such that $\hat{\chi}_{cl}(s)$ is characteristic polynomial of observer error, i.e. $\hat{\chi}_{cl}(s) = \det(sI - A_L)$.

The gain L leading to a required $\hat{\chi}_{cl}(s)$ can be chosen by the counterpart of Ackermann's formula

$$L = -\hat{\chi}_{cl}(A)M_o^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

(can be derived by applying Ackermann's formula to $(A + LC)' = A' + C'L'$ and then transposing the result).

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Transfer functions of (full-order) observer

The observer is a dynamic system $(u, y) \mapsto \hat{x}$,

$$\dot{\hat{x}}(t) = A_L \hat{x}(t) + Bu(t) - Ly(t) = A_L \hat{x}(t) + \begin{bmatrix} B & -L \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$$

where $A_L := A + LC$. The corresponding $n \times (m + p)$ transfer function

$$\begin{bmatrix} G_{\hat{x}u}(s) & G_{\hat{x}y}(s) \end{bmatrix} = (sI - A_L)^{-1} \begin{bmatrix} B & -L \end{bmatrix} = \left[\begin{array}{c|c} A_L & B - L \\ \hline I & 0 \end{array} \right],$$

which is strictly proper and normally has the same order as the plant itself.

This system is always stable, because

- A_L must be Hurwitz to have the reconstruction error ϵ decaying.

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State is a key

- If we are interested to reconstruct only a part of the state, e.g.

$$z(t) = C_z x(t) \quad \text{for some } C_z \in \mathbb{R}^{p_z \times n},$$

then we observe x , as $\hat{x}$, and the reconstruction of z is $\hat{z} = C_z \hat{x}$.

- If we are interested in derivative of y , then the signal to reconstruct is

$$z(t) = \dot{y}(t) = C\dot{x}(t) = CAx(t) + CBu(t).$$

Hence, all we need is to observe x and then the reconstruction of $\dot{y}$ is

$$\hat{z}(t) = CA\hat{x}(t) + CBu(t).$$

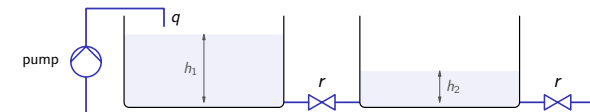
- A reconstruction of the second derivative of y , i.e. $z = \ddot{y}$, is (check it)

$$\hat{z}(t) = CA^2\hat{x}(t) + CABu(t) + CB\dot{u}(t)$$

(especially useful if $CB = 0$, i.e. the pole excess of $u \mapsto y$ is at least 2).

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Example: two-tank system



With $a_1 = a_2 = 1$, $r = 1$, $h_{eq} = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$, & $q_{eq} = 0.5$, suppose that the liquid level of only the first tank can be measured, i.e.

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), & \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases} \quad \text{thus, } y = h_1 - h_{eq1}$$

To reconstruct x_2 and $\dot{x}_1$ we build an observer (virtual sensor) in the form

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (y(t) - \hat{x}_1(t))$$

for some L .

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Example: two-tank system (contd)

Choosing any Hurwitz $\hat{\chi}_{cl}(s) = s^2 + \hat{\chi}_1 s + \hat{\chi}_0$ (i.e. with $\hat{\chi}_0 > 0$ and $\hat{\chi}_1 > 0$)

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = -\hat{\chi}_{cl}(A) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} \hat{\chi}_1 - 3 \\ \hat{\chi}_0 - 2\hat{\chi}_1 + 5 \end{bmatrix}.$$

With some boring math,

$$G_{\hat{x}_2 u}(s) = \frac{2\hat{\chi}_1 - \hat{\chi}_0 - 4}{\hat{\chi}_{cl}(s)}, \quad G_{\hat{x}_2 y}(s) = \frac{(5 - 2\hat{\chi}_1 + \hat{\chi}_0)s + 2 - \hat{\chi}_1 + \hat{\chi}_0}{\hat{\chi}_{cl}(s)}$$

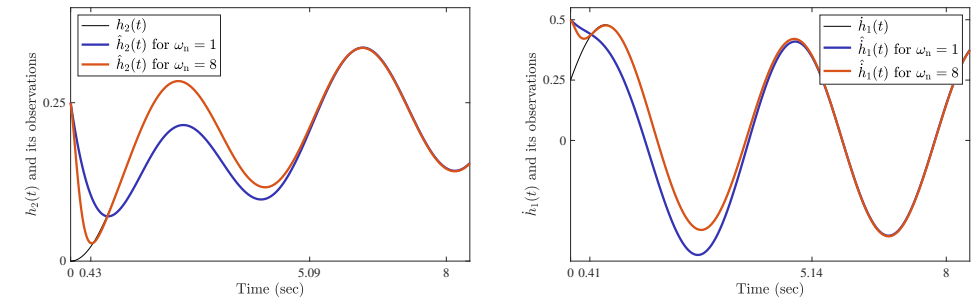
and

$$G_{\hat{x}_1 u}(s) = \frac{(s+2)(s+\hat{\chi}_1-3)}{\hat{\chi}_{cl}(s)}, \quad G_{\hat{x}_1 y}(s) = \frac{(8-3\hat{\chi}_1+\hat{\chi}_0)s+3-\hat{\chi}_1}{\hat{\chi}_{cl}(s)}$$

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Example: simulations with $\hat{\chi}_{cl}(s) = s^2 + 1.6\hat{\omega}_n s + \hat{\omega}_n^2$

With $x(0) = -h_{eq}$, $u(t) = q(t) - q_{eq} = q_{eq} \sin(0.5\pi t)$ and $\hat{x}(0) = -\begin{bmatrix} h_{eq1} \\ 0 \end{bmatrix}$:

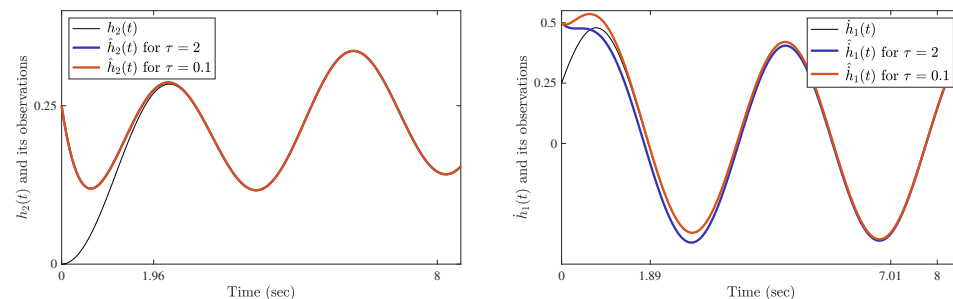


and observer gains $L = - \begin{bmatrix} -1.4 \\ 2.8 \end{bmatrix}$ and $L = - \begin{bmatrix} 9.8 \\ 43.4 \end{bmatrix}$.

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Example: simulations with $\hat{\chi}_{cl}(s) = (s+2)(s+1/\tau)$

With $x(0) = -h_{eq}$, $u(t) = q(t) - q_{eq} = q_{eq} \sin(0.5\pi t)$ and $\hat{x}(0) = -\begin{bmatrix} h_{eq1} \\ 0 \end{bmatrix}$:



and observer gains $L = \begin{bmatrix} 1 - 1/\tau \\ -1 \end{bmatrix}$. Curiously, in this case

$$G_{\hat{x}_2 u}(s) = 0 \quad \text{and} \quad G_{\hat{x}_2 y}(s) = \frac{1}{s+2}$$

regardless of τ , cf. $P_2(s) = P_1(s)/(s+2)$.

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Reduced-order observer: rationale

Return to

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0, \\ y(t) = Cx(t). \end{cases}$$

with $C \in \mathbb{R}^{p \times n}$ and $\text{rank } C = p < n$. A key observation is that

- we effectively directly measure p components of x .

Hence, we may only need to reconstruct its remaining $n - p$ components.

Example: In the two-tank system

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), & \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = x_1(t) \end{cases}$$

we directly measure x_1 and only need to reconstruct the 1-dimensional x_2 .

Q: can $n - p$ components of x be reconstructed by a lower-order system?

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Reduced-order observer: signal to reconstruct

Let

$$z := C_z x,$$

where $C_z \in \mathbb{R}^{(n-p) \times n}$ is any matrix such that $\begin{bmatrix} C_z \\ C \end{bmatrix}$ is nonsingular. Partition

$$\begin{bmatrix} C_z \\ C \end{bmatrix}^{-1} = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \implies \begin{bmatrix} C_z \\ C \end{bmatrix} \begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} I_{n-p} & 0 \\ 0 & I_p \end{bmatrix}$$

(i.e. $T_1 \in \mathbb{R}^{n \times (n-p)}$ and $T_2 \in \mathbb{R}^{n \times p}$), so that

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} C_z \\ C \end{bmatrix} x \implies x = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}.$$

In other words,

- if we measure y and reconstruct z , then we reconstruct $x = T_1 z + T_2 y$.

We shall thus aim at reconstructing the $(n-p)$ -dimensional z .

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Reduced-order observer: signal to reconstruct (contd)

Example 1: In the two-tank system,

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \implies C_z = \begin{bmatrix} 0 & 1 \end{bmatrix} \implies \begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example 2: In the second spring-mass-damper system from Lecture 2,

$$y = \begin{bmatrix} k/m & c/m \end{bmatrix} x \implies C = \begin{bmatrix} k/m & c/m \end{bmatrix}.$$

The choice of C_z is not unique, e.g. if $k \neq 0$, then a possible choice is

$$C_z = \begin{bmatrix} 0 & 1 \end{bmatrix} \implies \begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ k/m & c/m \end{bmatrix}^{-1} = \begin{bmatrix} -c/k & m/k \\ 1 & 0 \end{bmatrix}$$

(alternatively, $C_z = \begin{bmatrix} c & -k \end{bmatrix}$ can be used whenever $k^2 + c^2 \neq 0$), so that

$$x = \begin{bmatrix} -c/k \\ 1 \end{bmatrix} z + \begin{bmatrix} m/k \\ 0 \end{bmatrix} y \implies \dot{y} = C(Ax + Bu) = \frac{k}{m} z - \frac{k}{m} y + \frac{c}{m} u$$

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Reduced-order observer: transformation

Transform $x \rightarrow \begin{bmatrix} C_z \\ C \end{bmatrix} x = \begin{bmatrix} z \\ y \end{bmatrix}$, so that

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} C_z A T_1 & C_z A T_2 \\ \textcolor{red}{C} A T_1 & \textcolor{red}{C} A T_2 \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} C_z B \\ \textcolor{red}{C} B \end{bmatrix} u \quad (*)$$

To reconstruct z , we need to imitate its dynamics, like

$$\dot{\hat{z}} = C_z A T_1 \hat{z} + C_z A T_2 y + C_z B u$$

with the innovations signal somehow based on the **second relation**.

Motivation: in the full-order observer case, the system can be written as

$$\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

or, pre-multiplying by the non-singular $\begin{bmatrix} I & L \\ 0 & I \end{bmatrix}$, as

$$\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} A + LC & -L \\ C & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

whose first row, $\dot{x} = (A + LC)x - Ly + Bu$, is the basis for the observer.

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Reduced-order observer

Similarity transformation of $(*)$ with $\begin{bmatrix} I & L \\ 0 & I \end{bmatrix}$ for any $L \in \mathbb{R}^{n \times (n-p)}$ yields

$$\begin{bmatrix} \dot{\xi} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} (C_z + LC)AT_1 & (C_z + LC)A(T_2 - T_1L) \\ \textcolor{red}{C}AT_1 & \textcolor{red}{C}A(T_2 - T_1L) \end{bmatrix} \begin{bmatrix} \xi \\ y \end{bmatrix} + \begin{bmatrix} C_z + LC \\ C \end{bmatrix} Bu,$$

where $\xi := z + Ly \implies z = \xi - Ly$. Hence, we may consider

$$\begin{cases} \dot{\hat{\xi}} = (C_z + LC)AT_1 \hat{\xi} + (C_z + LC)A(T_2 - T_1L)y + Bu, & \hat{\xi}(0) = \hat{\xi}_0 \\ \hat{z} = \hat{\xi} - Ly \\ \hat{x} = T_1 \hat{\xi} + (T_2 - T_1L)y \end{cases}$$

as an observer for z and x (note that $C\hat{x} = y$). Its transfer function

$$\begin{bmatrix} G_{\hat{x}u}(s) & G_{\hat{z}y}(s) \end{bmatrix} = \begin{bmatrix} \frac{(C_z + LC)AT_1}{T_1} & \frac{(C_z + LC)B}{0} & \frac{(C_z + LC)A(T_2 - T_1L)}{T_2 - T_1L} \end{bmatrix}$$

is at most of the order $n-p$, as aimed, but is no longer strictly proper.

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Reduced-order observer: error system

Define $\epsilon_z := z - \hat{z} = \zeta - L y - (\hat{\zeta} - L y) = \zeta - \hat{\zeta}$.

$$\begin{aligned}\dot{\epsilon}_z &= (C_z + LC)AT_1\zeta + (C_z + LC)(A(T_2 - T_1L)y + Bu) \\ &\quad - (C_z + LC)AT_1\hat{\zeta} - (C_z + LC)(A(T_2 - T_1L)y + Bu) \\ &= (C_z + LC)AT_1\epsilon_z\end{aligned}$$

This is again an autonomous system, driven only by the mismatch between $\hat{z}(0)$ and $z(0)$, with

$$\epsilon_z(t) = e^{(C_z + LC)AT_1 t} (z_0 - \hat{z}_0).$$

Q: what freedom we have in assigning $\text{spec}((C_z + LC)AT_1)$ by L ?

This is the observability / detectability question for the pair (CAT_1, C_zAT_1) .

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Reduced-order observer: observability

Lemma 7.6

Unobservable modes of (CAT_1, C_zAT_1) coincide with those of (C, A) .

Proof: Because $C_z T_1 = I_{n-p}$ and $CT_1 = 0$,

$$\begin{bmatrix} C_zAT_1 - \lambda I_{n-p} \\ CAT_1 \end{bmatrix} = \begin{bmatrix} C_z(A - \lambda I_n)T_1 \\ C(A - \lambda I_n)T_1 \end{bmatrix} = \begin{bmatrix} C_z \\ C \end{bmatrix} \overset{\text{full-rank}}{(A - \lambda I_n)T_1}.$$

Because $T_1 C_z + T_2 C = I_n$,

$$\begin{bmatrix} A - \lambda I_n \\ C \end{bmatrix} = \begin{bmatrix} (A - \lambda I_n)T_1 & (A - \lambda I_n)T_2 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} C_z \\ C \end{bmatrix} \overset{\text{full-rank}}{.}$$

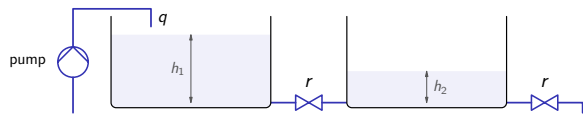
Therefore,

$$\text{rank} \begin{bmatrix} A - \lambda I_n \\ C \end{bmatrix} = \text{rank} [(A - \lambda I_n)T_1] + p = \text{rank} \begin{bmatrix} C_zAT_1 - \lambda I_{n-p} \\ CAT_1 \end{bmatrix} + p$$

for every $\lambda \in \mathbb{C}$. □

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Example: two-tank system



With $a_1 = a_2 = 1$, $r = 1$, $h_{eq} = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$, and $q_{eq} = 0.5$, we have

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), & \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases} \quad \text{thus, } y = h_1 - h_{eq1}$$

Because $C_z = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have

$$\overbrace{\begin{bmatrix} C_zAT_1 = -2 \\ CAT_1 = 1 \end{bmatrix}}^{\text{observable pair}} \Rightarrow \begin{cases} (C_z + LC)AT_1 = L - 2 \\ (C_z + LC)B = L \\ (C_z + LC)A(T_2 - T_1L) = 1 + L - L^2 \end{cases}$$

and stability requires $L < 2$.

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Example: two-tank system (contd)

As reduced-order observers are 1-order for this system, choose

$$\hat{\chi}_{cl}(s) = s + 1/\tau \implies L = 2 - 1/\tau,$$

so that

$$\begin{cases} \dot{\hat{\zeta}} = -(1/\tau)\hat{\zeta} + (2 - 1/\tau)u - (1 - 3/\tau + 1/\tau^2)y \\ \dot{\hat{x}}_2 = \hat{\zeta} - (2 - 1/\tau)y \end{cases}$$

with the transfer functions

$$G_{\hat{x}_2 u}(s) = -\frac{1 - 2\tau}{\tau s + 1} \quad \text{and} \quad G_{\hat{x}_2 y}(s) = \frac{(1 - 2\tau)s + 1 - \tau}{\tau s + 1}$$

(and see what happens as $\tau \rightarrow 1/2$). If we need to reconstruct $\dot{x}_1$, then

$$\hat{\dot{x}}_1 = CAT_1\hat{\zeta} + CBu + CA(T_2 - T_1L)y = \hat{\zeta} + u - (3 - 1/\tau)y$$

and

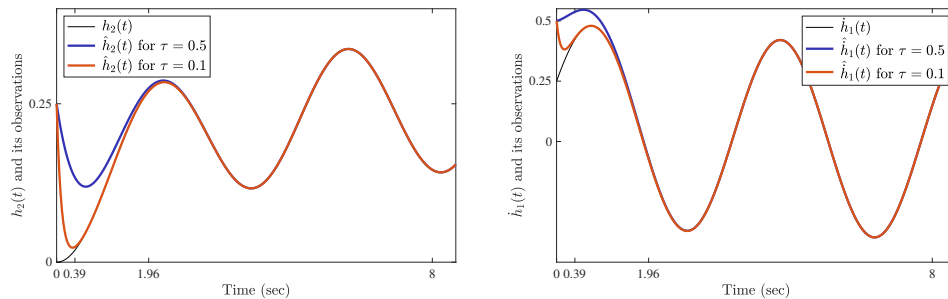
$$G_{\hat{\dot{x}}_1 u}(s) = \frac{\tau(s + 2)}{\tau s + 1} \quad \text{and} \quad G_{\hat{\dot{x}}_1 y}(s) = \frac{(1 - 3\tau)s - \tau}{\tau s + 1}$$

(and see what happens as $\tau \rightarrow 0$).

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Example: simulations with $\hat{\chi}_{cl}(s) = s + 1/\tau$

With $x(0) = -h_{eq}$, $u(t) = q_{eq} \sin(0.5\pi t)$ and $\hat{\zeta}(0) = 0 \implies \hat{x}_2(0) = 0$:



Note that

- the error ϵ_z is monotonically decaying, since its dynamics are 1-order.

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Outline

Observability

Example: 2-mass system (observability)

Minimality

State observers: pole placement and reduced-order observers

Observer-based feedback

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Amalgamating state feedback and observer

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) \end{cases}$$

in which the state vector x is not measured. Thus, state feedback cannot be used. Under such circumstances, we may think of combining state feedback and state observer, in the sense that we may

- use observed state $\hat{x}$ in control law as if it was the true state x .

This results in

$$R : \begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)), & \hat{x}(0) = \hat{x}_0 \\ u(t) = K\hat{x}(t) + v(t) \end{cases}$$

which is dubbed the **observer-based controller**.

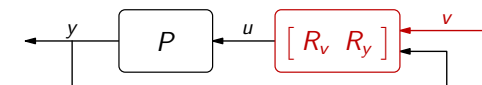
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Observer-based controller

Observer-based control law can be rewritten as

$$R : \begin{cases} \dot{\hat{x}}(t) = (A + BK + LC)\hat{x}(t) - Ly(t) + Bv(t), & \hat{x}(0) = \hat{x}_0 \\ u(t) = K\hat{x}(t) + v(t) \end{cases}$$

which is a system having v and y as its inputs and u as its output:



with

$$R(s) = \begin{bmatrix} R_v(s) & R_y(s) \end{bmatrix} = \left[\begin{array}{c|c} \frac{A + BK + LC}{K} & \begin{array}{c} B & -L \\ I & 0 \end{array} \end{array} \right]$$

where $R_v : v \mapsto u$ and $R_y : y \mapsto u$. Note that

- controller $-R_y$ is the state-space counterpart of the feedback controller in the standard unity-feedback case with negative feedback.

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Closed-loop system

State equation of the closed-loop system $v \mapsto y = Cx$ is:

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v(t), & \begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \end{cases}$$

Q: what can we say about its modes / stability?

A key observation is that

- dynamics of the observer error $\epsilon = x - \hat{x}$ do not depend on u .

This suggests changing the state vector to

$$\begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix},$$

i.e. using the similarity transformation with $T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = T^{-1}$.

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Closed-loop system: similarity transformation

We have:

$$\tilde{A}_{cl} = T A_{cl} T^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix}$$

$$\tilde{B}_{cl} = T B_{cl} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\tilde{C}_{cl} = C_{cl} T^{-1} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} C & 0 \end{bmatrix}$$

Note that

- the pair $(\tilde{A}_{cl}, \tilde{B}_{cl})$ has all modes of A_L uncontrollable in it.

Indeed,

$$\begin{bmatrix} 0 & \tilde{\eta}'_2 \end{bmatrix} \begin{bmatrix} A_K - \lambda I & -BK & B \\ 0 & A_L - \lambda I & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tilde{\eta}'_2(A_L - \lambda I) & 0 \end{bmatrix} = 0$$

for every right (nonzero) eigenvector $\tilde{\eta}_2$ of A_L , so PBH yields the conclusion.

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The separation

Thus, we end up with the closed-loop system

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t), & \begin{bmatrix} x(0) \\ \epsilon(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_0 - \hat{x}_0 \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} \end{cases}$$

The roots of the closed-loop characteristic polynomial

$$\chi_{cl}(s) = \det(sI - A_K) \det(sI - A_L)$$

are the union of the state-feedback and observer modes. Thus, all we need to do to stabilize the system is to

- design stabilizing state feedback i.e. its gain K
- design stable observer i.e. its gain L

separately. This is known as the **separation principle**.

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Closed-loop system $v \mapsto y$

The “ ϵ ” part of the closed-loop behavior

$$\dot{\epsilon}(t) = A_L \epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0 \implies \epsilon(t) = e^{A_L t} (x_0 - \hat{x}_0)$$

The “ x ” part is then

$$\dot{x}(t) = A_K x(t) + B v(t) - BK \epsilon(t) = A_K x(t) + B(v(t) - K e^{A_L t} (x_0 - \hat{x}_0))$$

i.e. including an observer is

- equivalent to adding an **exponentially decaying signal** to v .

Moreover, if $x_0 = \hat{x}_0$, then $\epsilon = 0$ and

$$\dot{x}(t) = A_K x(t) + B v(t), \quad x(0) = x_0$$

exactly like in the case of measured state.

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