

State-Space Control Fundamentals (00860721)

lecture no. 5

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Outline

Linear-Quadratic Regulator (LQR) problem: solution

LQR: return difference equality

LQR: root-locus insight

LQR: dynamic weights

Outline

Linear-Quadratic Regulator (LQR) problem: solution

LQR: return difference equality

LQR: root-locus insight

LQR: dynamic weights

Problem formulation

Given

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

find u that

- stabilizes the systems (in the sense that $\lim_{t \rightarrow \infty} x(t) = 0$),
- minimizes

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

Assumptions:

$\mathcal{A}_1$: (A, B) is stabilizable

necessary for a stabilizing controller to exist

$\mathcal{A}_2$: $R > 0$ and $Q - SR^{-1}S' \geq 0$

guarantees that the optimal u is bounded and that $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0 \implies \mathcal{J} \geq 0$

$\mathcal{A}_3$: $\begin{bmatrix} A - \lambda I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank for all $\lambda \in \{z \in \mathbb{C} \mid \operatorname{Re} z = 0\}$.

guarantees that the optimal u exists and is unique

Solution / Matlab: `lqr(A,B,Q,R,S)` or `lqr(A,B,Q,R)`

Theorem 5.1

If $\mathcal{A}_{1-3}$ hold, then the unique stabilizing controller that minimizes

$$\mathcal{J} = \int_0^{\infty} \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

is

$$u(t) = -R^{-1}(S' + B'\bar{X})x(t),$$

where $\bar{X} = \bar{X}' \geq 0$ is the stabilizing solution to the CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0.$$

The optimal cost $\mathcal{J}_{opt} = x_0'\bar{X}x_0$.

Remark. The optimal control law is a state feedback, with the gain $K = -R^{-1}(S' + B'\bar{X})$. The closed-loop “A” matrix $A_K = A + BK$ is **Hurwitz** because $\bar{X}$ is the stabilizing solution.

Remark. Matlab assumes negative feedback, $u = -Kx$, so computes $K = R^{-1}(S' + B'\bar{X})$.

Proof

Denote $K := -R^{-1}(S' + B'\bar{X})$. Because the CARE can be rewritten as

$$Q + \bar{X}A + A'\bar{X} = (S + \bar{X}B)R^{-1}RR^{-1}(S' + B'\bar{X}) = K'RK$$

and

$$S' + B'\bar{X} = -RK,$$

we can write

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} + \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix}$$

for every Riccati solution $\bar{X}$. Therefore,

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} - \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix}$$

which is a key relation.

Proof (contd)

Now, the quadratic form in the cost $\mathcal{J}$ is

$$\begin{aligned}
 & \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
 &= \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
 &\quad - \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \left(\begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix} \right) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
 &= (u(t) - Kx(t))' R (u(t) - Kx(t)) - x'(t) \bar{X} \dot{x}(t) - \dot{x}'(t) \bar{X} x(t) \\
 &= \|R^{1/2}(u(t) - Kx(t))\|^2 - \frac{d}{dt} (x'(t) \bar{X} x(t))
 \end{aligned}$$

where $R^{1/2} > 0$ is the matrix square root (Lecture 1, Slide 29, Property 10) of R .

Proof (contd)

Thus,

$$\begin{aligned}\mathcal{J} &= \int_0^\infty \left(\|R^{1/2}(u(t) - Kx(t))\|^2 - \frac{d}{dt}(x'(t)\bar{X}x(t)) \right) dt \\ &= \int_0^\infty \|R^{1/2}(u(t) - Kx(t))\|^2 dt + x_0'\bar{X}x_0 - \lim_{t \rightarrow \infty} x'(t)\bar{X}x(t).\end{aligned}$$

If u is stabilizing, then $\lim_{t \rightarrow \infty} x'(t)\bar{X}x(t) = 0$. Because $R^{1/2} > 0$,

$$\mathcal{J} = \int_0^\infty \|R^{1/2}(u(t) - Kx(t))\|^2 dt + x_0'\bar{X}x_0 \geq x_0'\bar{X}x_0$$

whenever the system is asymptotically stable and the equality is attained by the unique $u = Kx$, which *is* stabilizing ($\bar{X}$ is the stabilizing solution). $\square$

Suboptimal solutions

Corollary 5.2

Let $\mathcal{A}_{1-3}$ hold and the cost function

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

The control law

$$u(t) = -R^{-1}(S' + B'\bar{X})x(t) + R^{-1/2}v(t),$$

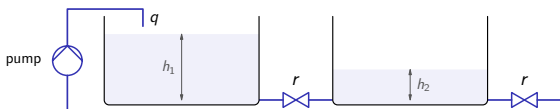
where $\bar{X} = \bar{X}' \geq 0$ is the stabilizing solution to the CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0,$$

renders

$$\mathcal{J} = x_0'\bar{X}x_0 + \int_0^\infty \|v(t)\|^2 dt.$$

Example



Objectives:

1. level deviation from its steady state, $\gamma_i := \int_0^\infty \|x_i(t)\|^2 dt$, for $i = 1, 2$

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} C_i' C_i & 0 \\ 0 & 0 \end{bmatrix}$$

2. control effort, $\gamma_u := \int_0^\infty \|u(t)\|^2 dt$

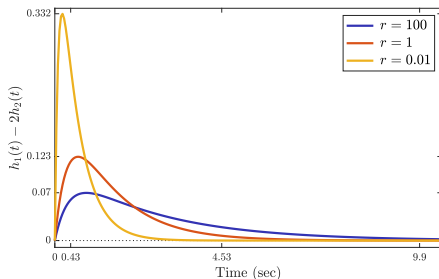
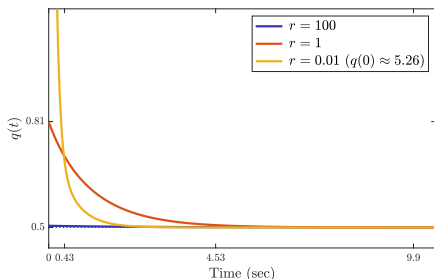
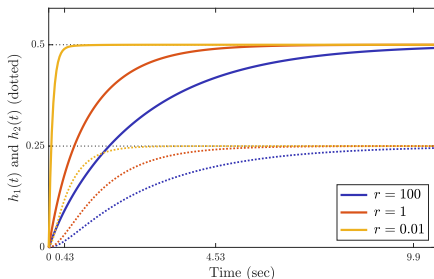
$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3. level asynchronization, $\gamma_{12} := \int_0^\infty \|x_1(t) - 2x_2(t)\|^2 dt$

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} (C_1 - 2C_2)'(C_1 - 2C_2) & 0 \\ 0 & 0 \end{bmatrix}$$

Example: design 1

If $\mathcal{J} = \gamma_1 + r\gamma_u$, then



with

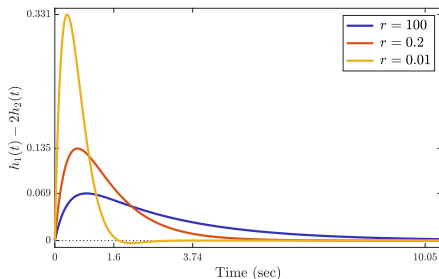
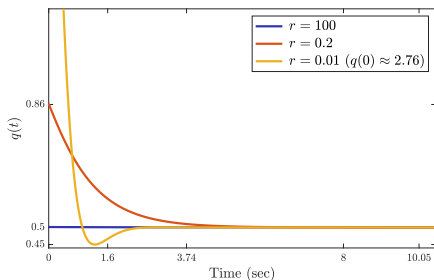
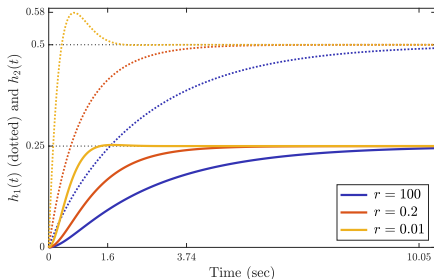
$$K = - \begin{bmatrix} 0.0083 & 0.0033 \\ 0.5316 & 0.1729 \\ 9.1264 & 0.7722 \end{bmatrix}$$

and

$$\text{spec}(A + BK) = \left\{ \begin{array}{l} -0.3894, -2.6188 \\ -0.8267, -2.7049 \\ -1.9721, -10.154 \end{array} \right\}$$

Example: design 2

If $\mathcal{J} = \gamma_2 + r\gamma_u$, then



with

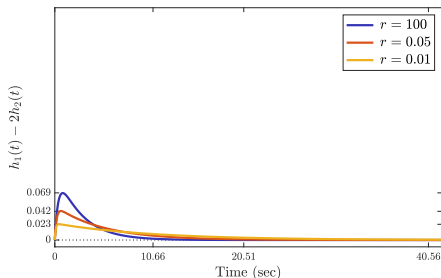
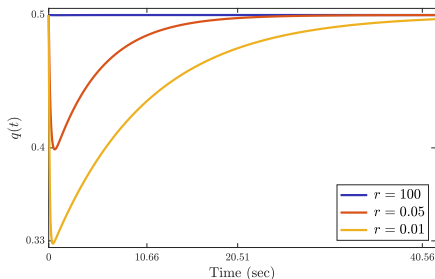
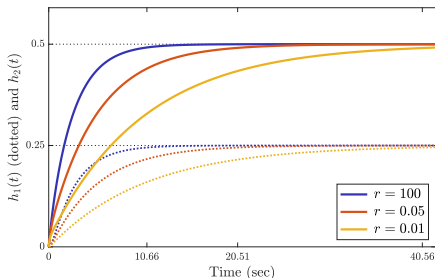
$$K = - \begin{bmatrix} 0.0017 & 0.0017 \\ 0.4495 & 0.5505 \\ 2.2057 & 4.6384 \end{bmatrix}$$

and

$$\text{spec}(A + BK) = \left\{ \begin{array}{l} -0.3839, -2.6177 \\ -1.5434, -2.1490 \\ -2.6029 \pm j1.8097 \end{array} \right\}$$

Example: design 3

If $\mathcal{J} = \gamma_{12} + r\gamma_u$, then



with

$$K = \begin{bmatrix} -0.0017 & 0.0033 \\ -2.3852 & 4.7703 \\ -7.4403 & 14.8806 \end{bmatrix}$$

and

$$\text{spec}(A + BK) = \left\{ \begin{array}{l} -0.3817, -2.6200 \\ -0.1926, -5.1926 \\ -0.0967, -10.344 \end{array} \right\}$$

Uniform scaling the cost $\mathcal{J}$

Optimal solutions for

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \quad \text{and} \quad \gamma \begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$$

coincide. Indeed, the CARE for the latter reads

$$A'\bar{X}_\gamma + \bar{X}_\gamma A + \gamma Q - (\gamma S + \bar{X}_\gamma B)(\gamma R)^{-1}(\gamma S' + B'\bar{X}_\gamma) = 0$$

or, equivalently,

$$A'(\gamma^{-1}\bar{X}_\gamma) + (\gamma^{-1}\bar{X}_\gamma)A + Q - (S + (\gamma^{-1}\bar{X}_\gamma)B)R^{-1}(S' + B'(\gamma^{-1}\bar{X}_\gamma)) = 0$$

Hence, $\bar{X}_\gamma = \gamma\bar{X}$ and

$$K_\gamma = -(\gamma R)^{-1}(\gamma S' + B'\bar{X}_\gamma) = -R^{-1}(S' + B'\bar{X})$$

is independent of γ .

but the optimal cost is proportional to γ

Exponential scaling the cost $\mathcal{J}$

Consider now the cost

$$\mathcal{J}_\alpha = \int_0^\infty e^{2\alpha t} \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

for some $\alpha \geq 0$. To render $\mathcal{J}_\alpha$ finite,

- x and u must decay faster than $e^{-\alpha t}$.

Key observation: $\mathcal{J}_\alpha$ is the standard LQR cost

$$\mathcal{J}_\alpha = \int_0^\infty \begin{bmatrix} x'_\alpha(t) & u'_\alpha(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x_\alpha(t) \\ u_\alpha(t) \end{bmatrix} dt$$

in terms of signals x_α and u_α having $x_\alpha(t) = e^{\alpha t} x(t)$ and $u_\alpha(t) = e^{\alpha t} u(t)$.

These scaled signals satisfy

$$\dot{x}_\alpha(t) = \alpha e^{\alpha t} x(t) + e^{\alpha t} \dot{x}(t) = (\alpha I + A)x_\alpha(t) + Bu_\alpha(t)$$

Thus, we end up with a standard LQR modulo the substitution $A \rightarrow \alpha I + A$.

Exponential scaling the cost $\mathcal{J}$

Consider now the cost

$$\mathcal{J}_\alpha = \int_0^\infty e^{2\alpha t} \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

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in terms of signals x_α and u_α having $x_\alpha(t) = e^{\alpha t}x(t)$ and $u_\alpha(t) = e^{\alpha t}u(t)$. These scaled signals satisfy

$$\dot{x}_\alpha(t) = \alpha e^{\alpha t}x(t) + e^{\alpha t}\dot{x}(t) = (\alpha I + A)x_\alpha(t) + Bu_\alpha(t)$$

Thus, we end up with a **standard LQR** modulo the substitution $A \rightarrow \alpha I + A$.

Exponential scaling the cost $\mathcal{J}$ (contd)

Thus, the control law minimizing $\mathcal{J}_\alpha$ is

$$u_\alpha(t) = K_\alpha x_\alpha(t) \implies u(t) = K_\alpha x(t),$$

where $K_\alpha := -R^{-1}(S' + B'\bar{X}_\alpha)$ and $\bar{X}_\alpha = \bar{X}'_\alpha \geq 0$ is the stabilizing solution (i.e. such that $\alpha I + A + BK_\alpha$ is Hurwitz) to

$$(\alpha I + A)'\bar{X}_\alpha + \bar{X}_\alpha(\alpha I + A) + Q - (S + \bar{X}_\alpha B)R^{-1}(S' + B'\bar{X}_\alpha) = 0.$$

Moreover, because $\lambda \in \text{spec}(\alpha I + A) \iff \lambda - \alpha \in \text{spec}(A)$,

- $\alpha I + A + BK_\alpha$ is Hurwitz iff $\text{spec}(A + BK_\alpha) \subset \{z \in \mathbb{C} \mid \text{Re } z < -\alpha\}$ and we end up with the property that
- minimizing $\mathcal{J}_\alpha$ ensures that closed-loop eigenvalues have $\text{Re } \lambda_i < -\alpha$.

Exponential scaling the cost $\mathcal{J}$ (contd)

Thus, the control law minimizing $\mathcal{J}_\alpha$ is

$$u_\alpha(t) = K_\alpha x_\alpha(t) \implies u(t) = K_\alpha x(t),$$

where $K_\alpha := -R^{-1}(S' + B'\bar{X}_\alpha)$ and $\bar{X}_\alpha = \bar{X}'_\alpha \geq 0$ is the stabilizing solution (i.e. such that $\alpha I + A + BK_\alpha$ is Hurwitz) to

$$(\alpha I + A)'\bar{X}_\alpha + \bar{X}_\alpha(\alpha I + A) + Q - (S + \bar{X}_\alpha B)R^{-1}(S' + B'\bar{X}_\alpha) = 0.$$

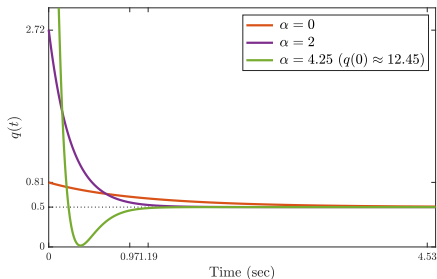
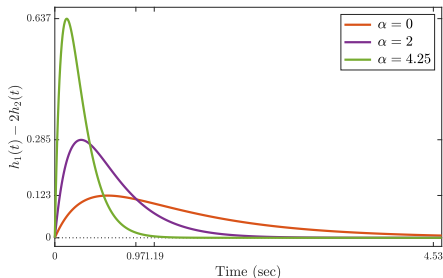
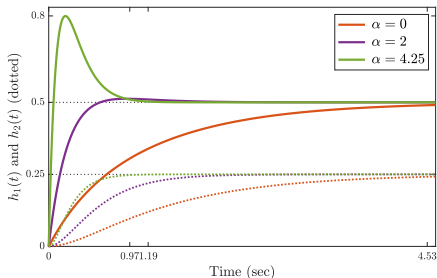
Moreover, because $\lambda \in \text{spec}(\alpha I + A) \iff \lambda - \alpha \in \text{spec}(A)$,

- $\alpha I + A + BK_\alpha$ is Hurwitz iff $\text{spec}(A + BK_\alpha) \subset \{z \in \mathbb{C} \mid \text{Re } z < -\alpha\}$ and we end up with the property that
- minimizing $\mathcal{J}_\alpha$ ensures that **closed-loop eigenvalues have $\text{Re } \lambda_i < -\alpha$** .

Remark. Exponential scaling with $\alpha < 0$ is known as the *discounted LQR*. It may be useful in proofs and becomes popular in learning applications.

Example: design 1 with $r = 1$ and various α 's

If $\mathcal{J} = \gamma_1 + \gamma_u$, then its α -modifications yield



with

$$K_\alpha = - \begin{bmatrix} 0.5316 & 0.1729 \\ 3.4495 & 2 \\ 11.1593 & 25.4975 \end{bmatrix}$$

and

$$\text{spec}(A + BK_\alpha) = \begin{cases} -0.8267, -2.7049 \\ -2.5176, -3.9319 \\ -5.9371, -8.2222 \end{cases}$$

So beware, “more stable” $\nRightarrow$ “better”.

Outline

Linear-Quadratic Regulator (LQR) problem: solution

LQR: return difference equality

LQR: root-locus insight

LQR: dynamic weights

Preliminaries

Because $A(sl - A)^{-1} + I = s(sl - A)^{-1}$ (cf. Lecture 2, Slide 17),

$$\begin{aligned} \begin{bmatrix} B'(-sl - A')^{-1} & I \end{bmatrix} \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} (sl - A)^{-1}B \\ I \end{bmatrix} \\ = B'(-sl - A')^{-1}\bar{X}(A(sl - A)^{-1} + I)B \\ = sB'(-sl - A')^{-1}\bar{X}(sl - A)^{-1}B \end{aligned}$$

and because $(-sl - A')^{-1}A' + I = -s(-sl - A')^{-1}$,

$$\begin{aligned} \begin{bmatrix} B'(-sl - A')^{-1} & I \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix} \begin{bmatrix} (sl - A)^{-1}B \\ I \end{bmatrix} \\ = B'((-sl - A')^{-1}A' + I)\bar{X}(sl - A)^{-1}B \\ = -sB'(-sl - A')^{-1}\bar{X}(sl - A)^{-1}B. \end{aligned}$$

Therefore,

$$\begin{bmatrix} B'(-sl - A')^{-1} & I \end{bmatrix} \left(\begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix} \right) \begin{bmatrix} (sl - A)^{-1}B \\ I \end{bmatrix} = 0.$$

Key relation

Remember,

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} - \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix}.$$

Taking into account the relation in the previous slide,

$$\begin{aligned} \begin{bmatrix} B'(-sI - A')^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} \\ = (I - B'(-sI - A')^{-1}K')R(I - K(sI - A)^{-1}B). \end{aligned}$$

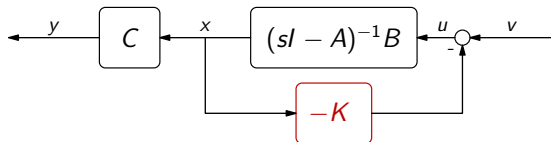
Equivalently,

$$\begin{bmatrix} B'(-sI - A')^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} = (I + L'_{\text{sf}}(-s))R(I + L_{\text{sf}}(s)),$$

where

$$L_{\text{sf}}(s) := -K(sI - A)^{-1}B = \left[\begin{array}{c|c} A & B \\ \hline -K & 0 \end{array} \right].$$

State-feedback loop and return-difference transfer function



$L_{\text{sf}}(s) = -K(sI - A)^{-1}B$ is the (strictly proper) loop transfer function and

$$I + L_{\text{sf}}(s) = \left[\begin{array}{c|c} A & B \\ \hline -K & I \end{array} \right]$$

is the return-difference transfer function (inverse of the sensitivity S_{sf}), with

$$S_{\text{sf}}(s) := (I + L_{\text{sf}}(s))^{-1} = \left[\begin{array}{c|c} A + BK & B \\ \hline K & I \end{array} \right].$$

In the SISO case ($m = 1$)

$$- 1 + L_{\text{sf}}(s) = \frac{\chi_{\text{cl}}(s)}{\chi_{\text{ol}}(s)}$$

key relation to derive the Nyquist criterion

Return difference equality for $m = 1$ and $S = 0$

Denote $r = R$ (to emphasize that it is scalar) and rewrite the key relation as

$$(1 + L_{\text{sf}}(-s))(1 + L_{\text{sf}}(s)) = 1 + \frac{1}{r} B'(-sI - A')^{-1} Q(sI - A)^{-1} B,$$

If $s = j\omega$, then $1 + L_{\text{sf}}(-j\omega) = \overline{1 + L_{\text{sf}}(j\omega)}$ and the above reads

$$|1 + L_{\text{sf}}(j\omega)|^2 = 1 + \frac{1}{r} B'(-j\omega I - A')^{-1} Q(j\omega I - A)^{-1} B$$

known as the **return difference equality**. Because $Q \geq 0$, we have that

$$B'(-j\omega I - A')^{-1} Q(j\omega I - A)^{-1} B = [(j\omega I - A)^{-1} B]' Q [(j\omega I - A)^{-1} B] \geq 0$$

Hence, the return difference equality ensures that

$$|1 + L_{\text{sf}}(j\omega)|^2 \geq 1, \quad \forall \omega \in \mathbb{R}.$$

Return difference equality for $m = 1$ and $S = 0$

Denote $r = R$ (to emphasize that it is scalar) and rewrite the key relation as

$$(1 + L_{\text{sf}}(-s))(1 + L_{\text{sf}}(s)) = 1 + \frac{1}{r} B'(-sI - A')^{-1} Q(sI - A)^{-1} B,$$

If $s = j\omega$, then $1 + L_{\text{sf}}(-j\omega) = \overline{1 + L_{\text{sf}}(j\omega)}$ and the above reads

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known as the **return difference equality**. Because $Q \geq 0$, we have that

$$B'(-j\omega I - A')^{-1} Q(j\omega I - A)^{-1} B = [(j\omega I - A)^{-1} B]' Q[(j\omega I - A)^{-1} B] \geq 0$$

Hence, the return difference equality ensures that

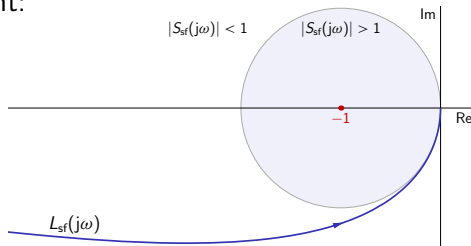
$$|1 + L_{\text{sf}}(j\omega)|^2 \geq 1, \quad \forall \omega \in \mathbb{R}.$$

Return difference equality: implication

The inequality

$$|1 + L_{sf}(j\omega)|^2 \geq 1 \iff |S_{sf}(j\omega)| \leq 1, \quad \forall \omega \in \mathbb{R}$$

implies that the polar plot of $L_{sf}(j\omega)$ is outside the open unit disk centered at the critical point:

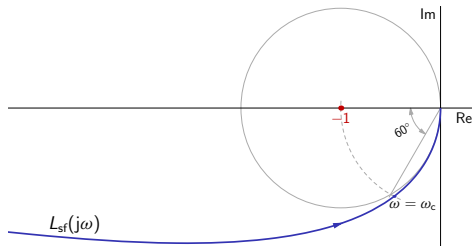


Remark. By Bode's sensitivity integral,

$$\int_0^\infty \ln |S_{sf}(j\omega)| d\omega = \pi \left(\sum_{i=1}^{n_{\text{rhp}}} \operatorname{Re} p_i - \frac{1}{2} \lim_{s \rightarrow \infty} sL(s) \right),$$

the equality above means that the pole excess of the LQR $L_{sf}(s)$ under $S = 0$ is always 1.

Stability margins of LQR ($m = 1$ and $S = 0$)



An immediate consequence is that **LQR** optimal loop **guarantees**

- gain margin $\mu_g = \infty$ $L_{sf}(j\omega)$ does not cross the real axis in $(-1, 0)$
- phase margin $\mu_{ph} \geq 60^\circ$ $L_{sf}(j\omega)$ is further from -1 than $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$
- modulus margin $\mu_m = 1$ $\mu_m = \inf_{\omega} |1 + L_{sf}(j\omega)|$

whenever the LQR parameter $S = 0$.

When is state feedback optimal?

Theorem 5.3

If $m = 1$, (A, B) is controllable, and $K \in \mathbb{R}^{1 \times n}$ satisfies

1. $A + BK$ is stable,
2. $|1 - K(j\omega I - A)^{-1}B| \geq 1$ for all $\omega \in \mathbb{R}$,

then there is $0 \leq Q = Q' \in \mathbb{R}^{n \times n}$ such that this K is LQR optimal for

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

If (A, B) is not controllable, then there exists Q for which $-K(sl - A)^{-1}B$ is the optimal loop.

This result shows that LQR optimization is

→ sufficiently rich, so that we won't miss any "good" controller by using Q and r as design parameter under $S = 0$.

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This result shows that LQR optimization is

- sufficiently rich, so that we won't miss any "good" controller by using Q and r as design parameter under $S = 0$.

When is state feedback optimal? (contd)

The condition $S = 0$ is essential. If S can also be chosen freely, then

— all stabilizing state-feedback gains are LQR-optimal, regardless the second condition. For example, the cost

$$\mathcal{J} = \int_0^\infty \|u(t) - K_\star x(t)\|^2 dt$$

for any $K_\star$ such that $A + BK_\star$ is Hurwitz, which corresponds to

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K'_\star \\ I \end{bmatrix} \begin{bmatrix} -K_\star & I \end{bmatrix} = \begin{bmatrix} K'_\star K_\star & -K'_\star \\ -K_\star & I \end{bmatrix} \geq 0,$$

produces the optimal $u = K_\star x$, for which $\mathcal{J}_{\text{opt}} = 0$, the obvious minimum.

Remark. Can also be checked by brute-force calculations as well. It is readily seen that $\bar{X} = 0$ for this choice of Q , S , and R , and then $K = K_\star$. BTW, the matrix in $\mathcal{A}_3$ reads

$$\begin{bmatrix} A - sI & B \\ -K_\star & I \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} A + BK_\star - sI & 0 \\ -K_\star & I \end{bmatrix}$$

so its rank drops iff $s \in \text{spec}(A + BK_\star)$ (cf. condition $2'$ in Theorem 4.4).

Outline

Linear-Quadratic Regulator (LQR) problem: solution

LQR: return difference equality

LQR: root-locus insight

LQR: dynamic weights

Closed-loop poles ($m = 1$ and $S = 0$)

Because $1 + L_{sf}(s) = \chi_{cl}(s)/\chi_{ol}(s)$, the return-difference equality reads

$$\frac{\chi_{cl}(-s)\chi_{cl}(s)}{\chi_{ol}(-s)\chi_{ol}(s)} = 1 + \frac{1}{r}G_*(s),$$

where

$$G_*(s) := B'(-sI - A')^{-1}Q(sI - A)^{-1}B = \left[\begin{array}{cc|c} A & 0 & B \\ Q & -A' & 0 \\ \hline 0 & -B' & 0 \end{array} \right]$$

is such that $G_*(-s) = G_*(s)$, so that its poles / zeros are symmetric about the imaginary axis. If $x'Qx = y^2$, then $Q = C'C$ and $G_*(s) = P(-s)P(s)$.

The solutions of

$$1 + \frac{1}{r}G_*(s) = 0 \iff -r = G_*(s),$$

which is a root-locus form under $k = 1/r$, determine then both the roots of $\chi_{cl}(s)$ and those of $\chi_{cl}(-s)$.

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which is a **root-locus form** under $k = 1/r$, determine then both the roots of $\chi_{cl}(s)$ (closed-loop LQR poles) and those of $\chi_{cl}(-s)$ (their reflection about the $j\omega$ -axis).

Root locus with respect to $k = 1/r$

- has $2n$ loci symmetric about both real and imaginary axes
- starts for $r = \infty$ at roots of $\chi_{ol}(-s)\chi_{ol}(s)$
- ends as $r \downarrow 0$ at zeros of $G_*(s)$ or ∞ (depending on the cost)
- has an even number of asymptotes, whose centroid is at $\sigma_c = 0$
- if $G_*(s)$ is strictly proper (even pole excess), then asymptotes angles

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8
2	0	π						
4	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$				
6	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$		
8	$\pi/8$	$3\pi/8$	$5\pi/8$	$7\pi/8$	$9\pi/8$	$11\pi/8$	$13\pi/8$	$15\pi/8$

half of them are in the LHP (and none on the imaginary axis).

- there are exactly n roots in the open LHP at each $r \in (0, \infty)$
(because $G_*(j\omega) \geq 0$, $-x = G_*(j\omega)$ cannot be solved in $r > 0$)

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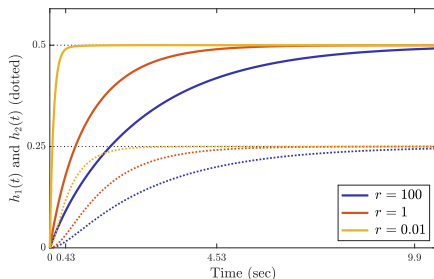
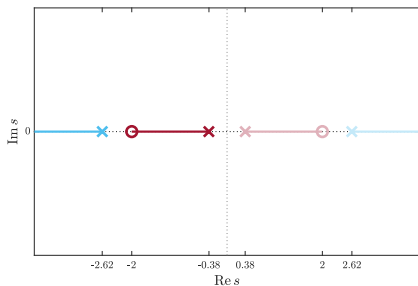
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(because $G_*(j\omega) \geq 0$, $-r = G_*(j\omega)$ cannot be solved in $r > 0$)

Two-tank example: design 1

LQR for $\mathcal{J} = \gamma_1 + r\gamma_u$ has $S = 0$ and

$$G_*(s) = \frac{(s+2)(-s+2)}{\chi_{ol}(s)\chi_{ol}(-s)} = P_1(-s)P_1(s)$$

(here $\chi_{ol}(s) = (s+2.618)(s+0.382)$). Its root locus with respect to $1/r$ is



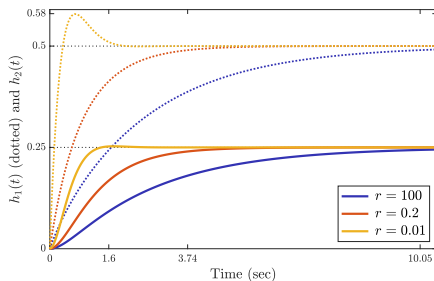
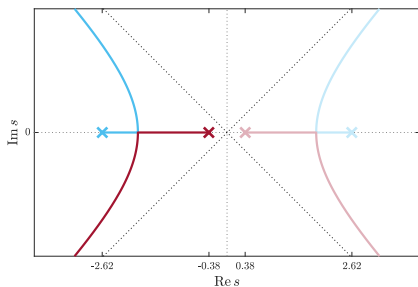
Closed-loop poles were at $\{-0.39, -2.62\}$, $\{-0.83, -2.7\}$, and $\{-1.97, -10.15\}$.

Two-tank example: design 2

LQR for $\mathcal{J} = \gamma_2 + r\gamma_u$ has $S = 0$ and

$$G_*(s) = \frac{1}{\chi_{ol}(s)\chi_{ol}(-s)} = P_2(-s)P_2(s)$$

(here $\chi_{ol}(s) = (s + 2.618)(s + 0.382)$). Its root locus with respect to $1/r$ is



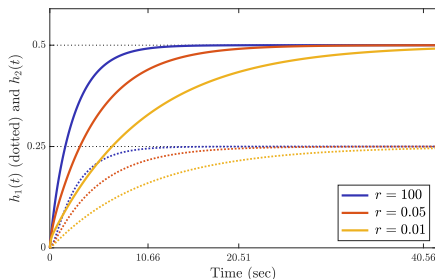
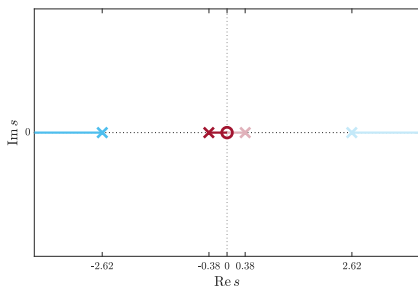
Closed-loop poles were at $\{-0.38, -2.62\}$, $\{-1.54, -2.15\}$, and $\{-2.6 \pm j1.81\}$.

Two-tank example: design 3

LQR for $\mathcal{J} = \gamma_{12} + r\gamma_u$ has $S = 0$ and

$$G_*(s) = \frac{s^2}{\chi_{ol}(s)\chi_{ol}(-s)} = (P_1(-s) - 2P_2(-s))(P_1(s) - 2P_2(s))$$

(here $\chi_{ol}(s) = (s + 2.618)(s + 0.382)$). Its root locus with respect to $1/r$ is



Closed-loop poles were at $\{-0.38, -2.62\}$, $\{-0.19, -5.19\}$, and $\{-0.1, -10.34\}$.

Limiting behavior: cheap control, $r \rightarrow 0$

The LQR problem with the cost

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \quad r \rightarrow 0$$

is known as the **cheap control** LQR problem. By root-locus arguments, the roots of $\chi_{cl}(s)$ for $1/r \rightarrow \infty$

- either approach the stable zeros of $G_*(s)$
- or go to $\operatorname{Re} s = -\infty$ along the LHP asymptotes
(if the pole excess of $G_*(s)$ is larger than 2, then the damping ratio of the “worst” asymptote is at most $1/\sqrt{2}$)

Limiting behavior: expensive control, $r \rightarrow \infty$

We already know that the solution to the LQR for

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

and

$$\mathcal{J}_r = \int_0^\infty \begin{bmatrix} x'(t) & u(t) \end{bmatrix} \begin{bmatrix} Q/r & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt = \frac{1}{r} \mathcal{J}$$

coincide. Thus, the LQR problem for the limiting case of $r \rightarrow \infty$ effectively corresponds to

$$\mathcal{J}_\infty = \int_0^\infty \|u(t)\|^2 dt \quad \text{or} \quad \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is the problem of

- stabilizing with the lowest control energy.

Limiting behavior: expensive control, $r \rightarrow \infty$ (contd)

Assumptions:

$\mathcal{A}_1$: (A, B) is stabilizable

$\mathcal{A}_2$: $R > 0$ and $Q - SR^{-1}S' \geq 0$

for $Q = 0$, $S = 0$, and $R = 1$ obviously holds

$\mathcal{A}_3$: $\begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank for all ω

for $Q = 0$, $S = 0$, and $R = 1$ is equivalent to $\text{spec}(A) \cap j\mathbb{R} = \emptyset$

By root-locus arguments, the roots of $\chi_d(s)$ for $1/r = 0$ are the stable roots of $\chi_{ol}(-s)\chi_{ol}(s)$. In other words,

- if λ is an OLHP root of $\chi_{ol}(s)$, then it is also a root of $\chi_d(s)$,
- if λ is an ORHP root of $\chi_{ol}(s)$, then $-\lambda$ is a root of $\chi_d(s)$.

Thus, the optimal controller for this problem

- reflects unstable poles about the $j\omega$ -axis and
- keeps stable poles untouched.

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Limiting behavior: expensive control, $r \rightarrow \infty$ (contd)

If $\mathcal{A}_3$ does not hold, i.e. if for some ω

$$\text{rank} \begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix} = \text{rank} \begin{bmatrix} A - j\omega I & B \\ 0 & 0 \\ 0 & r \end{bmatrix} = \text{rank}(A - j\omega I) + 1 < n + 1,$$

then

- no optimal stabilizing solution exists.

For example, let $\dot{x}(t) = u(t)$ and $u(t) = kx(t)$ for $k < 0$. In this case

$$\dot{x}(t) = kx(t) \implies x(t) = e^{kt}x_0 \implies u(t) = ke^{kt}x_0$$

and

$$\mathcal{J}_\infty = \int_0^\infty u^2(t) dt = \int_0^\infty k^2 e^{2kt} x_0^2 dt = \frac{|k|}{2} x_0^2.$$

It can be made arbitrarily small, but the optimal $k = 0$ is not stabilizing.

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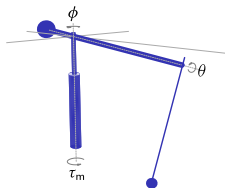
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It can be made arbitrarily small, but the optimal $k = 0$ is not stabilizing.

Exercise: Furuta pendulum damping



whose model is 4-order, with the state vector $x = (\phi, \dot{\phi}, \theta, \dot{\theta})$ and

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & J_b + mr^2 & 0 & mrl \\ 0 & 0 & 1 & 0 \\ 0 & mrl & 0 & J_p + ml^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 \\ 0 & -f & 0 & 0 & | & 1 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & -gml & -\zeta & | & 0 \end{bmatrix}.$$

Simulate with

- $R = r$ (then reduce r to something small)
- $Q = C_z' C_z$ with C_z chosen to assign zero locations of $C_z(sI - A)^{-1}B$

Outline

Linear-Quadratic Regulator (LQR) problem: solution

LQR: return difference equality

LQR: root-locus insight

LQR: dynamic weights

Smoothing the control signal

State-feedback controllers

- tend to have high spikes at the very beginning,

we saw that in all considered examples so far. To avoid that, we may think of penalizing derivative(s) of u in the cost function, like

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) & \dot{u}'(t) \end{bmatrix} \begin{bmatrix} Q & S_0 & S_1 \\ S'_0 & R_{00} & R_{01} \\ S'_1 & R'_{01} & R_{11} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \\ \dot{u}(t) \end{bmatrix} dt$$

This becomes a standard LQR for the augmented plant

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \dot{u}$$

The solution

$$\dot{u} = Kx + K_d u \iff \begin{cases} \dot{x}_d = K_d x_d + Kx \\ u = x_d \end{cases}$$

is a dynamic controller with the transfer function $(sI - K_d)^{-1}K$.

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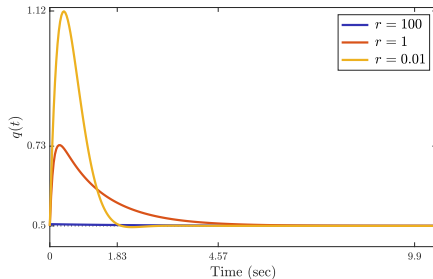
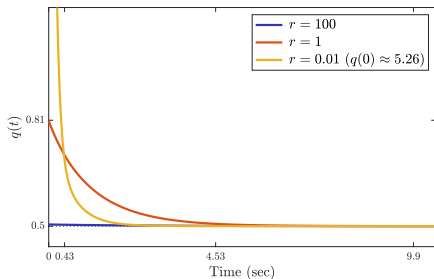
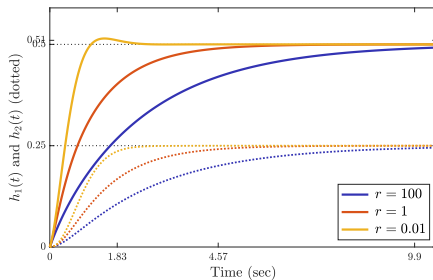
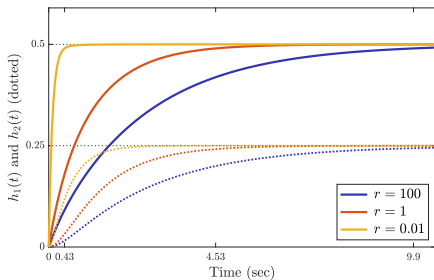
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is a dynamic controller with the transfer function $(sI - K_u)^{-1}K$.

Two-tank example: design 4

If $\mathcal{J} = \gamma_1 + r\gamma_u + \frac{1}{100}\gamma_{\dot{u}}$, then $Q = C_1' C_1$, $R_{00} = r$, and $R_{11} = 0.01$, so that



Extensions

- any $k \in \mathbb{N}$ derivatives can be penalized via augmenting the plant as

$$\begin{bmatrix} \dot{x} \\ \dot{u} \\ \vdots \\ u^{(k-1)} \\ u^{(k)} \end{bmatrix} = \begin{bmatrix} A & B & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x \\ u \\ \dot{u} \\ \vdots \\ u^{(k-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u^{(k)}$$

and adding higher-order dynamics into the controller,

$$\begin{bmatrix} \dot{u} \\ \vdots \\ u^{(k-1)} \\ u^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \\ K_{u,0} & K_{u,1} & \cdots & K_{u,k-1} \end{bmatrix} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(k-1)} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ K \end{bmatrix} x$$

- dynamic weights can also be added to x , also by augmenting dynamics of the plant with dynamics of weights