

State-Space Control Fundamentals (00860721)

lecture no. 4

Leonid Mirkin

The Stephen B. Klein Faculty of Aerospace Engineering
Technion—IIT



Optimization-based control design

Underlying idea:

- express (transient) control requirements in terms of a **cost function** whose minimization is the ultimate objective (but not goal) of the design.

Important to remember:

Optimization-based control design

Underlying idea:

- express (transient) control requirements in terms of a **cost function** whose minimization is the ultimate objective (but not goal) of the design.

Important to remember:

- every control problem has several, often conflicting, objectives
e.g. fast convergence, smooth convergence, low control effort, ...
- the cost function is a blend of those objectives, each with its weight
- as the cost function J is a plus ∞ if $\|u\|_2$ is large, the outcome depends on A and B
- a right balance is sought via a trial and error process of tuning weights
- hence, optimization is merely a means, not a goal, in control design

Optimization-based control design

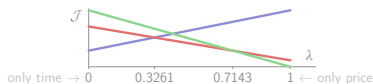
Underlying idea:

- express (transient) control requirements in terms of a **cost function** whose minimization is the ultimate objective (but not goal) of the design.

Important to remember:

- every control problem has several, often conflicting, objectives
e.g. fast convergence, smooth convergence, low control effort, ...
- the cost function is a blend of those objectives, each with its weight
options for commuting from Qiryat Tiv'on to the Technion:

	taxi	bus	bike
price	70	8	0
time	20	50	70



for the cost function $\mathcal{J} = \lambda \text{ price} + (1 - \lambda) \text{ time}$, the outcome depends on $\lambda \in [0, 1]$

- a right balance is sought via a trial and error process of tuning weights
- hence, optimization is merely a means, not a goal, in control design

Optimization-based control design

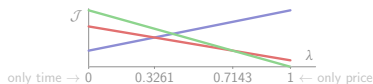
Underlying idea:

- express (transient) control requirements in terms of a **cost function** whose minimization is the ultimate objective (but not goal) of the design.

Important to remember:

- every control problem has several, often conflicting, objectives
e.g. fast convergence, smooth convergence, low control effort, ...
- the cost function is a blend of those objectives, each with its weight
options for commuting from Qiryat Tiv'on to the Technion:

	taxi	bus	bike
price	70	8	0
time	20	50	70



for the cost function $\mathcal{J} = \lambda \text{ price} + (1 - \lambda) \text{ time}$, the outcome depends on $\lambda \in [0, 1]$

- a right balance is sought via a trial and error process of tuning weights
- hence, optimization is merely a means, not a goal, in control design

Optimization-based control design

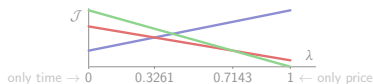
Underlying idea:

- express (transient) control requirements in terms of a **cost function** whose minimization is the ultimate objective (but not goal) of the design.

Important to remember:

- every control problem has several, often conflicting, objectives
e.g. fast convergence, smooth convergence, low control effort, ...
- the cost function is a blend of those objectives, each with its weight
options for commuting from Qiryat Tiv'on to the Technion:

	taxi	bus	bike
price	70	8	0
time	20	50	70



for the cost function $\mathcal{J} = \lambda \text{ price} + (1 - \lambda) \text{ time}$, the outcome depends on $\lambda \in [0, 1]$

- a right balance is sought via a trial and error process of tuning weights
- hence, optimization is merely a means, **not a goal**, in control design

Outline

Linear-Quadratic Regulator (LQR) problem: formulation

Technical background: continuous-time algebraic Riccati equation

The LQR problem

Given

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, find u that

- stabilizes the systems (in the sense that $\lim_{t \rightarrow \infty} x(t) = 0$),
- minimizes the cost function

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

for given parameters (weights) $Q = Q' \in \mathbb{R}^{n \times n}$, $R = R' \in \mathbb{R}^{m \times m}$, and $S \in \mathbb{R}^{n \times m}$ such that $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$.

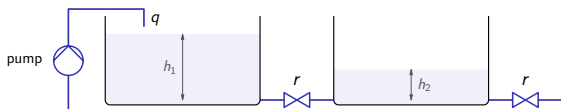
Here

L is for the linear plant

Q is for the quadratic cost a choice, mostly because the math is neat

R is for the regulator setup

Example 1: two-tank system



With $a_1 = a_2 = 1$, $r = 1$, $h_{\text{eq}} = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$, and $q_{\text{eq}} = 0.5$,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

where $x_i = h_i - h_{\text{eq},i}$ and $u = q - q_{\text{eq}}$.

Our **goal** is to

- regulate x from $x(0)$ to $\lim_{t \rightarrow \infty} x(t) = 0$ in a desired manner (regulator problem), which is effectively the set-point tracking of h_{eq} .

Example 1: LQR formulations

We may consider several objectives (quantities to penalize in the cost):

1. level deviation from its steady state, $\gamma_i := \int_0^\infty \|x_i(t)\|^2 dt$, for $i = 1, 2$
2. level smoothness, $\gamma_{s,i} := \int_0^\infty \|\dot{x}_i(t)\|^2 dt$, for $i = 1, 2$
3. control effort, $\gamma_u := \int_0^\infty \|u(t)\|^2 dt$
4. level asynchronization, $\gamma_{12} := \int_0^\infty \|x_1(t) - 2x_2(t)\|^2 dt$

These goals are conflicting, e.g. faster response needs higher control effort, so the design is

- the art of tradeoffs,

whose essence is to seek for a right blend of a number of them via weights.

Example 1: building Q , R , and S

First, express each penalty via LQR quadratic forms. To this end, note that

$$x_i = \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \dot{x}_i = C_i \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

where $C_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $C_2 := \begin{bmatrix} 0 & 1 \end{bmatrix}$. Hence (mind that $\|v\|^2 = v'v$),

$$\gamma_i = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} C_i' C_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

$$\gamma_{s,i} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} A' C_i' C_i A & A' C_i' C_i B \\ B' C_i' C_i A & B' C_i' C_i B \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

$$\gamma_u = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

and

$$\gamma_{12} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} (C_1 - 2C_2)'(C_1 - 2C_2) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

Example 1: building Q , R , and S

First, express each penalty via LQR quadratic forms. To this end, note that

$$x_i = \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \dot{x}_i = C_i \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

where $C_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $C_2 := \begin{bmatrix} 0 & 1 \end{bmatrix}$. Hence (mind that $\|v\|^2 = v'v$),

$$\gamma_i = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} C_i' C_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

$$\gamma_{s,i} = \int_0^\infty \begin{bmatrix} x(t) & u(t) \end{bmatrix} \begin{bmatrix} A' C_i' C_i A & A' C_i' C_i B \\ B' C_i' C_i A & B' C_i' C_i B \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

$$\gamma_u = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

and

$$\gamma_{12} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} (C_1 - 2C_2)'(C_1 - 2C_2) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

Example 1: building Q , R , and S

First, express each penalty via LQR quadratic forms. To this end, note that

$$x_i = \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \dot{x}_i = C_i \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

where $C_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $C_2 := \begin{bmatrix} 0 & 1 \end{bmatrix}$. Hence (mind that $\|v\|^2 = v'v$),

$$\begin{aligned} \gamma_i &= \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} C_i' C_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \\ \gamma_{s,i} &= \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} A' C_i' C_i A & A' C_i' C_i B \\ B' C_i' C_i A & B' C_i' C_i B \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \\ \gamma_u &= \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \end{aligned}$$

and

$$\gamma_{12} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} (C_1 - 2C_2)'(C_1 - 2C_2) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

Example 1: building Q , R , and S

First, express each penalty via LQR quadratic forms. To this end, note that

$$x_i = \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \dot{x}_i = C_i \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

where $C_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $C_2 := \begin{bmatrix} 0 & 1 \end{bmatrix}$. Hence (mind that $\|v\|^2 = v'v$),

$$\begin{aligned} \gamma_i &= \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} C_i' C_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \\ \gamma_{s,i} &= \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} A' C_i' C_i A & A' C_i' C_i B \\ B' C_i' C_i A & B' C_i' C_i B \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \\ \gamma_u &= \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \end{aligned}$$

and

$$\gamma_{12} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} (C_1 - 2C_1\gamma(C_1 - 2C_2)) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

Example 1: building Q , R , and S

First, express each penalty via LQR quadratic forms. To this end, note that

$$x_i = \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \dot{x}_i = C_i \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

where $C_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $C_2 := \begin{bmatrix} 0 & 1 \end{bmatrix}$. Hence (mind that $\|v\|^2 = v'v$),

$$\gamma_i = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} C_i' C_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

$$\gamma_{s,i} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} A' C_i' C_i A & A' C_i' C_i B \\ B' C_i' C_i A & B' C_i' C_i B \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

$$\gamma_u = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

and

$$\gamma_{12} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} (C_1 - 2C_2)'(C_1 - 2C_2) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

Example 1: building Q , R , and S (contd)

Now we could easily mix them. For example, for every weight $r > 0$

$$\gamma_i + r\gamma_u = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \underbrace{\begin{bmatrix} C_i' C_i & 0 \\ 0 & r \end{bmatrix}}_{\geq 0} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

which is a standard LQR cost.

Likewise, for every weight $\lambda \in (0, 1)$

$$\lambda\gamma_i + (1-\lambda)\gamma_{u,i} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

where

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} \lambda C_i' C_i + (1-\lambda) A' C_i' C_i A & (1-\lambda) A' C_i' C_i B \\ (1-\lambda) B' C_i' C_i A & (1-\lambda) B' C_i' C_i B \end{bmatrix} \geq 0$$

(show why ≥ 0). Note that $R > 0$ iff $C_i B \neq 0$, true for $i = 1$, not for $i = 2$.

Example 1: building Q , R , and S (contd)

Now we could easily mix them. For example, for every weight $r > 0$

$$\gamma_i + r\gamma_u = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \underbrace{\begin{bmatrix} C_i' C_i & 0 \\ 0 & r \end{bmatrix}}_{\geq 0} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

which is a standard LQR cost.

Likewise, for every weight $\lambda \in (0, 1)$

$$\lambda\gamma_i + (1 - \lambda)\gamma_{s,i} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

where

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} \lambda C_i' C_i + (1 - \lambda) A' C_i' C_i A & (1 - \lambda) A' C_i' C_i B \\ (1 - \lambda) B' C_i' C_i A & (1 - \lambda) B' C_i' C_i B \end{bmatrix} \geq 0$$

(show why ≥ 0). Note that $R > 0$ iff $C_i B \neq 0$, true for $i = 1$, not for $i = 2$.

Example 2: minimum-energy stabilization

Let

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

and assume that we need to find a

- stabilizing u having minimal energy, $\gamma_u = \int_0^\infty \|u(t)\|^2 dt$.

This is an LQR problem, with

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \geq 0.$$

If

- A is Hurwitz, the trivial solution is $u = 0$,
- A is not Hurwitz, the solution is nontrivial. Can you guess what is it?

Example 2: minimum-energy stabilization

Let

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

and assume that we need to find a

- stabilizing u having minimal energy, $\gamma_u = \int_0^\infty \|u(t)\|^2 dt$.

This is an LQR problem, with

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \geq 0.$$

If

- A is Hurwitz, the trivial solution is $u = 0$
- A is not Hurwitz, the solution is nontrivial. Can you guess what is it?

Example 2: minimum-energy stabilization

Let

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

and assume that we need to find a

- stabilizing u having minimal energy, $\gamma_u = \int_0^\infty \|u(t)\|^2 dt$.

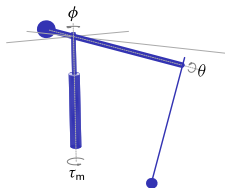
This is an LQR problem, with

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \geq 0.$$

If

- A is Hurwitz, the trivial solution is $u = 0$,
- A is not Hurwitz, the solution is nontrivial. Can you guess what is it?

Exercise: Furuta pendulum damping



whose model is 4-order, with the state vector

$$x = \begin{bmatrix} \phi \\ \dot{\phi} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

Simulate with

- $Q = \text{diag}\{q_1, q_2, q_3, q_4\}$, for $q_1 > 0$ and $q_i \geq 0$ for $i \in \mathbb{Z}_{2..4}$
- $R = 1$

Outline

Linear-Quadratic Regulator (LQR) problem: formulation

Technical background: continuous-time algebraic Riccati equation

CARE / Matlab: `icare(A,B,Q,R,S)` or `icare(A,B,Q,R)`

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $Q = Q' \in \mathbb{R}^{n \times n}$, $R = R' \in \mathbb{R}^{m \times m}$, and $S \in \mathbb{R}^{n \times m}$ be given. The matrix equation

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

is called the **continuous-time algebraic Riccati equation** (CARE). Its solution $\bar{X} \in \mathbb{R}^{n \times n}$ is said to be stabilizing if the matrix

$$A_K := A - BR^{-1}(S' + B'\bar{X})$$

is Hurwitz, i.e. having eigenvalues in $\{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$. Meanwhile we are interested in CAREs for

$$R > 0 \quad \text{and} \quad \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$$

(by Theorem 1.4.2, the latter condition is equivalent to $Q - SR^{-1}S' \geq 0$).

Example

Let $A = a$, $B = b$, $Q = q \geq 0$, $R = r > 0$, and $S = 0$. The CARE reads

$$2a\bar{X} + q - \frac{b^2}{r}\bar{X}^2 = 0.$$

1. If $b = 0$, then the CARE becomes $2a\bar{X} + q = 0$ and $A_K = a$.

→ if $a \geq 0$, then no stabilizing solution exists

→ if $a < 0$, then $\bar{X} = -q/(2a) \geq 0$ is the stabilizing solution

2. If $b \neq 0$, then the CARE above is quadratic, solvable by

$$\bar{X} = \frac{ar \pm \sqrt{(a^2 r + b^2 q)r}}{b^2} \Rightarrow A_K = a - \frac{b^2}{r}\bar{X} = \mp \sqrt{a^2 + b^2 \frac{q}{r}}$$

and it is Hurwitz for "+" in $\bar{X}$, unless $a = q = 0$. Thus, the stabilizing

$$\bar{X} = \frac{ar + \sqrt{(a^2 r + b^2 q)r}}{b^2} > 0.$$

Example

Let $A = a$, $B = b$, $Q = q \geq 0$, $R = r > 0$, and $S = 0$. The CARE reads

$$2a\bar{X} + q - \frac{b^2}{r}\bar{X}^2 = 0.$$

1. If $b = 0$, then the CARE becomes $2a\bar{X} + q = 0$ and $A_K = a$.
 - if $a \geq 0$, then no stabilizing solution exists
 - if $a < 0$, then $\bar{X} = -q/(2a) \geq 0$ is the stabilizing solution

2. If $b \neq 0$, then the CARE above is quadratic, solvable by

$$\bar{X} = \frac{ar \pm \sqrt{(a^2 r + b^2 q)r}}{b^2} \Rightarrow A_K = a - \frac{b^2}{r}\bar{X} = \mp \sqrt{a^2 + \frac{b^2 q}{r}}$$

and it is Hurwitz for “+” in $\bar{X}$, unless $a = q = 0$. Thus, the stabilizing

$$\bar{X} = \frac{ar + \sqrt{(a^2 r + b^2 q)r}}{b^2} > 0.$$

Example

Let $A = a$, $B = b$, $Q = q \geq 0$, $R = r > 0$, and $S = 0$. The CARE reads

$$2a\bar{X} + q - \frac{b^2}{r}\bar{X}^2 = 0.$$

1. If $b = 0$, then the CARE becomes $2a\bar{X} + q = 0$ and $A_K = a$.
 - if $a \geq 0$, then no stabilizing solution exists
 - if $a < 0$, then $\bar{X} = -q/(2a) \geq 0$ is the stabilizing solution
2. If $b \neq 0$, then the CARE above is quadratic, solvable by

$$\bar{X} = \frac{ar \pm \sqrt{(a^2r + b^2q)r}}{b^2} \implies A_K = a - \frac{b^2}{r}\bar{X} = \mp \sqrt{a^2 + b^2 \frac{q}{r}}$$

and it is Hurwitz for “+” in $\bar{X}$, unless $a = q = 0$. Thus, the stabilizing

$$\bar{X} = \frac{ar + \sqrt{(a^2r + b^2q)r}}{b^2} > 0.$$

Alternative forms

It can be shown that

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

$$\Leftrightarrow$$

$$(A - BR^{-1}S')'\bar{X} + \bar{X}(A - BR^{-1}S') + Q - SR^{-1}S' - \bar{X}BR^{-1}B'\bar{X} = 0$$

$$\Leftrightarrow$$

$$(A + BK)'\bar{X} + \bar{X}(A + BK) + \begin{bmatrix} I & K' \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} = 0$$

where $K := -R^{-1}(S' + B'\bar{X})$.

By the second form,

$$- S = 0 \text{ w.l.o.g.} \quad \text{via } A \rightarrow A - BR^{-1}S' \text{ and } Q \rightarrow Q - SR^{-1}S' \geq 0$$

By the third (Lyapunov) form,

$$- A + BK \text{ is Hurwitz} \implies \bar{X} = \bar{X}' \geq 0$$

Uniqueness

Lemma 4.1

Stabilizing solution $\bar{X} = \bar{X}'$ to the CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

is unique whenever exists.

Proof (for $S = 0$): Suppose $\bar{X}_1$ and $\bar{X}_2$ are stabilizing solutions, such that $A - BR^{-1}B'\bar{X}_1$ and $A - BR^{-1}B'\bar{X}_2$ are Hurwitz. Denoting $\Delta := \bar{X}_1 - \bar{X}_2$,

$$\begin{aligned} 0 &= A'\bar{X}_1 + \bar{X}_1A + Q - \bar{X}_1BR^{-1}B'\bar{X}_1 - A'\bar{X}_2 - \bar{X}_2A - Q + \bar{X}_2BR^{-1}B'\bar{X}_2 \\ &= A'\Delta + \Delta A - \bar{X}_1BR^{-1}B'\bar{X}_1 + \bar{X}_2BR^{-1}B'\bar{X}_2 \pm \bar{X}_2BR^{-1}B'\bar{X}_1 \\ &= A'\Delta + \Delta A - \Delta BR^{-1}B'\bar{X}_1 - \bar{X}_2BR^{-1}B'\Delta \\ &= (A - BR^{-1}B'\bar{X}_2)'\Delta + \Delta(A - BR^{-1}B'\bar{X}_1). \end{aligned}$$

This is a **Sylvester equation**, whose solution is unique (both $A - BR^{-1}B'\bar{X}_2$ and $A - BR^{-1}B'\bar{X}_1$ are Hurwitz). Trivially, this $\Delta = 0 \implies \bar{X}_1 = \bar{X}_2$.

Uniqueness

Lemma 4.1

Stabilizing solution $\bar{X} = \bar{X}'$ to the CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

is unique whenever exists.

Proof (for $S = 0$): Suppose $\bar{X}_1$ and $\bar{X}_2$ are stabilizing solutions, such that $A - BR^{-1}B'\bar{X}_1$ and $A - BR^{-1}B'\bar{X}_2$ are Hurwitz. Denoting $\Delta := \bar{X}_1 - \bar{X}_2$,

$$\begin{aligned} 0 &= A'\bar{X}_1 + \bar{X}_1A + Q - \bar{X}_1BR^{-1}B'\bar{X}_1 - A'\bar{X}_2 - \bar{X}_2A - Q + \bar{X}_2BR^{-1}B'\bar{X}_2 \\ &= A'\Delta + \Delta A - \bar{X}_1BR^{-1}B'\bar{X}_1 + \bar{X}_2BR^{-1}B'\bar{X}_2 \pm \bar{X}_2BR^{-1}B'\bar{X}_1 \\ &= A'\Delta + \Delta A - \Delta BR^{-1}B'\bar{X}_1 - \bar{X}_2BR^{-1}B'\Delta \\ &= (A - BR^{-1}B'\bar{X}_2)'\Delta + \Delta(A - BR^{-1}B'\bar{X}_1). \end{aligned}$$

This is a **Sylvester equation**, whose solution is unique (both $A - BR^{-1}B'\bar{X}_2$ and $A - BR^{-1}B'\bar{X}_1$ are Hurwitz). Trivially, this $\Delta = 0 \implies \bar{X}_1 = \bar{X}_2$. $\square$

Hamiltonian matrices

A matrix $H \in \mathbb{R}^{2n \times 2n}$ is said to be **Hamiltonian** if

$$\underbrace{\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}}_J H = \left(\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} H \right)'.$$

Because $J' = -J$,

$$JH = (JH)' = H'J' = -H'J \implies JHJ^{-1} = -H',$$

i.e. H and $-H'$ are similar. Hence, $\text{spec}(H) = \text{spec}(-H') = \text{spec}(-H)$ and $-\lambda \in \text{spec}(H) \iff \lambda \in \text{spec}(H)$.

In particular, there is $k \in \mathbb{Z}_{0,n}$ such that H has

- $2k$ eigenvalues on the imaginary axis $\{x \in \mathbb{C} \mid \text{Re } x = 0\}$
- $n - k$ eigenvalues in the OLHP $\{x \in \mathbb{C} \mid \text{Re } x < 0\}$
- $n - k$ eigenvalues in the ORHP $\{x \in \mathbb{C} \mid \text{Re } x > 0\}$

Hamiltonian matrices

A matrix $H \in \mathbb{R}^{2n \times 2n}$ is said to be **Hamiltonian** if

$$\underbrace{\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}}_J H = \left(\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} H \right)'.$$

Because $J' = -J$,

$$JH = (JH)' = H'J' = -H'J \implies JHJ^{-1} = -H',$$

i.e. H and $-H'$ are similar. Hence, $\text{spec}(H) = \text{spec}(-H') = \text{spec}(-H)$ and

$$- \lambda \in \text{spec}(H) \iff -\lambda \in \text{spec}(H).$$

In particular, there is $k \in \mathbb{Z}_{0..n}$ such that H has

- $2k$ eigenvalues on the imaginary axis $\{z \in \mathbb{C} \mid \text{Re } z = 0\}$
- $n - k$ eigenvalues in the OLHP $\{z \in \mathbb{C} \mid \text{Re } z < 0\}$
- $n - k$ eigenvalues in the ORHP $\{z \in \mathbb{C} \mid \text{Re } z > 0\}$

Hamiltonian matrix associated with CARE

CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

can be rewritten as (remember, $A_K = A - BR^{-1}(S' + B'\bar{X})$)

$$\underbrace{\begin{bmatrix} A - BR^{-1}S' & -BR^{-1}B' \\ -Q + SR^{-1}S' & -(A - BR^{-1}S')' \end{bmatrix}}_{H_{\text{ric}}} \begin{bmatrix} I \\ \bar{X} \end{bmatrix} = \begin{bmatrix} I \\ \bar{X} \end{bmatrix} A_K,$$

Because

$$JH_{\text{ric}} = \begin{bmatrix} Q - SR^{-1}S' & (A - BR^{-1}S')' \\ A - BR^{-1}S' & -BR^{-1}B' \end{bmatrix}$$

is symmetric, H_{ric} is Hamiltonian. Moreover, $\text{spec}(A_K) \subset \text{spec}(H_{\text{ric}})$, since

$$A_K \eta = \lambda \eta \implies H_{\text{ric}} \begin{bmatrix} \eta \\ \bar{X} \eta \end{bmatrix} = \lambda \begin{bmatrix} \eta \\ \bar{X} \eta \end{bmatrix}$$

and $\eta \neq 0 \implies \begin{bmatrix} \eta \\ \bar{X} \eta \end{bmatrix} \neq 0$.

Hamiltonian matrix associated with CARE

CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

can be rewritten as (remember, $A_K = A - BR^{-1}(S' + B'\bar{X})$)

$$\underbrace{\begin{bmatrix} A - BR^{-1}S' & -BR^{-1}B' \\ -Q + SR^{-1}S' & -(A - BR^{-1}S')' \end{bmatrix}}_{H_{\text{ric}}} \begin{bmatrix} I \\ \bar{X} \end{bmatrix} = \begin{bmatrix} I \\ \bar{X} \end{bmatrix} A_K,$$

Because

$$JH_{\text{ric}} = \begin{bmatrix} Q - SR^{-1}S' & (A - BR^{-1}S')' \\ A - BR^{-1}S' & -BR^{-1}B' \end{bmatrix}$$

is symmetric, H_{ric} is Hamiltonian. Moreover, $\text{spec}(A_K) \subset \text{spec}(H_{\text{ric}})$, since

$$A_K\eta = \lambda\eta \implies H_{\text{ric}} \begin{bmatrix} \eta \\ \bar{X}\eta \end{bmatrix} = \lambda \begin{bmatrix} \eta \\ \bar{X}\eta \end{bmatrix}$$

and $\eta \neq 0 \iff \begin{bmatrix} \eta \\ \bar{X}\eta \end{bmatrix} \neq 0$.

Solving CARE via the Hamiltonian matrix

Let

$$T = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}^{-1}$$

for some $U_{ij} \in \mathbb{R}^{n \times n}$ be nonsingular and such that

$$TH_{\text{ric}}T^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

for some $A_{ij} \in \mathbb{R}^{n \times n}$ (clearly, $\text{spec}(H_{\text{ric}}) = \text{spec}(A_{11}) \cup \text{spec}(A_{22})$). Hence,

$$H_{\text{ric}}T^{-1} = T^{-1} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \implies H_{\text{ric}} \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} A_{11}$$

If $\det U_{11} \neq 0$, then

$$H_{\text{ric}} \begin{bmatrix} I \\ U_{21}U_{11}^{-1} \end{bmatrix} = \begin{bmatrix} I \\ U_{21}U_{11}^{-1} \end{bmatrix} U_{11}A_{11}U_{11}^{-1}$$

so that $\bar{X} = U_{21}U_{11}^{-1}$ and $A_K = U_{11}A_{11}U_{11}^{-1}$, with $\text{spec}(A_K) = \text{spec}(A_{11})$.

Solving CARE via the Hamiltonian matrix (contd)

The decomposition

$$TH_{\text{ric}}T^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

is not unique, any partition of $\text{spec}(H_{\text{ric}})$ into two n -tuples leads to some T . This is why CAREs may have many solutions. The

- stabilizing solution requires A_{11} to be Hurwitz,

Together with the symmetry of $\text{spec}(H_{\text{ric}})$ around the imaginary axis, this is another way to see that the stabilizing solution is unique.

- To solve CARE, ensure that H_{ric} has no pure imaginary eigenvalues
- partition $\text{spec}(H_{\text{ric}})$ into stable and anti-stable parts
- the Schur decomposition does that with a unitary T , i.e. in a numerically stable way
- check if U_{11} is nonsingular

→ [MATLAB code](#)

Solving CARE via the Hamiltonian matrix (contd)

The decomposition

$$TH_{\text{ric}}T^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

is not unique, any partition of $\text{spec}(H_{\text{ric}})$ into two n -tuples leads to some T . This is why CAREs may have many solutions. The

- stabilizing solution requires A_{11} to be Hurwitz,

Together with the symmetry of $\text{spec}(H_{\text{ric}})$ around the imaginary axis, this is another way to see that the stabilizing solution is unique. To solve CARE,

- ensure that H_{ric} has no pure imaginary eigenvalues
- partition $\text{spec}(H_{\text{ric}})$ into stable and anti-stable parts

the *Schur decomposition* does that with a unitary T , i.e. in a numerically stable way

- check if U_{11} is nonsingular

Remark. If $\bar{X}$ solves the CARE, then

$$\begin{bmatrix} I & 0 \\ -\bar{X} & I \end{bmatrix} \begin{bmatrix} A - BR^{-1}S' & -BR^{-1}B' \\ -Q + SR^{-1}S' & -(A - BR^{-1}S')' \end{bmatrix} \begin{bmatrix} I & 0 \\ \bar{X} & I \end{bmatrix} = \begin{bmatrix} A_K & -BR^{-1}B' \\ 0 & -A'_K \end{bmatrix},$$

which is a way to construct T .

Solvability in terms of H_{ric}

Lemma 4.2

Let $R > 0$ and $Q - SR^{-1}S' \geq 0$. The CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

has a stabilizing solution iff (A, B) is stabilizable and $\text{spec}(H_{\text{ric}}) \cap j\mathbb{R} = \emptyset$.

Proof (only if): If (A, B) is not stabilizable, then there exists no K making $A + BK$ Hurwitz and, hence, no stabilizing $K = -R^{-1}(S' + B'\bar{X})$ either. If H_{ric} has pure imaginary eigenvalues, then H_{ric} has less than n eigenvalues in the OLHP and there is no Hurwitz $A_K \in \mathbb{R}^{n \times n}$ with $\text{spec}(A_K) \subset \text{spec}(H_{\text{ric}})$.

The proof of sufficiency is quite technical, especially in showing that in

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}^{-1} H_{\text{ric}} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad \text{with Hurwitz } A_{11},$$

the stabilizability of (A, B) necessarily implies that $\det U_{11} \neq 0$. □

Example (contd)

If $A = a$, $B = b$, $Q = q \geq 0$, $R = r > 0$, and $S = 0$, then

$$H_{\text{ric}} = \begin{bmatrix} a & -b^2/r \\ -q & -a \end{bmatrix} \implies \text{spec}(H_{\text{ric}}) = \left\{ \pm \sqrt{a^2 + b^2 \frac{q}{r}} \right\}.$$

Hence,

- if $a = 0 \wedge bq = 0$, then $\text{spec}(H_{\text{ric}}) \in j\mathbb{R}$ and there is no stabilizing $\bar{X}$
- otherwise, $\text{spec}(H_{\text{ric}}) \cap j\mathbb{R} = \emptyset$, but if $(a \geq 0) \wedge (b = 0)$, then (a, b) is not stabilizable and there is no stabilizing $\bar{X}$

if $b = 0$ and $a > 0$, then

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \overbrace{\begin{bmatrix} a & 0 \\ -q & -a \end{bmatrix}}^{H_{\text{ric}}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -a & -q \\ 0 & a \end{bmatrix}$$

and $U_{11} = 0$ indeed; if $b = 0$ and $a < 0$, then (remember, A_{11} should be Hurwitz)

$$\begin{bmatrix} -2a & -q \\ q & -1 \end{bmatrix}^{-1} \begin{bmatrix} a & 0 \\ -q & -a \end{bmatrix} \begin{bmatrix} -2a & -q \\ q & -1 \end{bmatrix} = \begin{bmatrix} a & -q \\ 0 & -a \end{bmatrix}$$

and $U_{11} = -2a \neq 0$

Hamiltonian matrix associated with CARE (contd)

Lemma 4.3

If (A, B) stabilizable, then $j\omega \in \text{spec}(H_{\text{ric}})$ iff $\text{rank} \begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix} < n + m$.

Proof (for $S = 0$): $j\omega \in \text{spec}(H_{\text{ric}})$ iff

$$\begin{bmatrix} A - j\omega I & -BR^{-1}B' \\ -Q & -A' - j\omega I \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0 \iff \begin{cases} (A - j\omega I)\eta_1 = BR^{-1}B'\eta_2 \\ \eta_2'(A - j\omega I) = -\eta_1'Q \end{cases}$$

Hence, $\eta_2'(A - j\omega I)\eta_1 \geq 0$ and $\eta_2'(A - j\omega I)\eta_1 \leq 0$,
 $\eta_2'(A - j\omega I)\eta_1 = \eta_2'BR^{-1}B'\eta_2$ and $\eta_2'(A - j\omega I)\eta_1 = -\eta_1'Q\eta_1$,
 whence $\eta_1'Q = 0$ and $\eta_2'B = 0$. Hence, $\eta_2' \begin{bmatrix} A - j\omega I & B \end{bmatrix} = 0 \implies \eta_2 = 0$
 (by stabilizability). Therefore,

$$j\omega \in \text{spec}(H_{\text{ric}}) \iff \begin{bmatrix} A - j\omega I \\ -Q \end{bmatrix} \eta_1 = 0 \iff \text{rank} \begin{bmatrix} A - j\omega I \\ Q \end{bmatrix} < n,$$

which is equivalent to the rank condition under $S = 0$ and $R > 0$. □

Hamiltonian matrix associated with CARE (contd)

Lemma 4.3

If (A, B) stabilizable, then $j\omega \in \text{spec}(H_{\text{ric}})$ iff $\text{rank} \begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix} < n + m$.

Proof (for $S = 0$): $j\omega \in \text{spec}(H_{\text{ric}})$ iff

$$\begin{bmatrix} A - j\omega I & -BR^{-1}B' \\ -Q & -A' - j\omega I \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0 \iff \begin{cases} (A - j\omega I)\eta_1 = BR^{-1}B'\eta_2 \\ \eta_2'(A - j\omega I) = -\eta_1'Q \end{cases}$$

Hence,

$$\eta_2'(A - j\omega I)\eta_1 = \overbrace{\eta_2'BR^{-1}B'\eta_2}^{\geq 0} \quad \text{and} \quad \eta_2'(A - j\omega I)\eta_1 = \overbrace{-\eta_1'Q\eta_1}^{\leq 0},$$

whence $\eta_1'Q = 0$ and $\eta_2'B = 0$. Hence, $\eta_2' \begin{bmatrix} A - j\omega I & B \end{bmatrix} = 0 \implies \eta_2 = 0$ (by stabilizability). Therefore,

$$j\omega \in \text{spec}(H_{\text{ric}}) \iff \begin{bmatrix} A - j\omega I \\ -Q \end{bmatrix} \eta_1 = 0 \iff \text{rank} \begin{bmatrix} A - j\omega I \\ Q \end{bmatrix} < n,$$

which is equivalent to the rank condition under $S = 0$ and $R > 0$. $\square$

Hamiltonian matrix associated with CARE (contd)

Lemma 4.3

If (A, B) stabilizable, then $j\omega \in \text{spec}(H_{\text{ric}})$ iff $\text{rank} \begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix} < n + m$.

Proof (for $S = 0$): $j\omega \in \text{spec}(H_{\text{ric}})$ iff

$$\begin{bmatrix} A - j\omega I & -BR^{-1}B' \\ -Q & -A' - j\omega I \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0 \iff \begin{cases} (A - j\omega I)\eta_1 = BR^{-1}B'\eta_2 \\ \eta_2'(A - j\omega I) = -\eta_1'Q \end{cases}$$

Hence,

$$\eta_2'(A - j\omega I)\eta_1 = \overbrace{\eta_2'BR^{-1}B'\eta_2}^{\geq 0} \quad \text{and} \quad \eta_2'(A - j\omega I)\eta_1 = \overbrace{-\eta_1'Q\eta_1}^{\leq 0},$$

whence $\eta_1'Q = 0$ and $\eta_2'B = 0$. Hence, $\eta_2' \begin{bmatrix} A - j\omega I & B \end{bmatrix} = 0 \implies \eta_2 = 0$ (by stabilizability). Therefore,

$$j\omega \in \text{spec}(H_{\text{ric}}) \iff \begin{bmatrix} A - j\omega I \\ -Q \end{bmatrix} \eta_1 = 0 \iff \text{rank} \begin{bmatrix} A - j\omega I \\ Q \end{bmatrix} < n,$$

which is equivalent to the rank condition under $S = 0$ and $R > 0$. $\square$

Hamiltonian matrix associated with CARE (contd)

Lemma 4.3

If (A, B) stabilizable, then $j\omega \in \text{spec}(H_{\text{ric}})$ iff $\text{rank} \begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix} < n + m$.

Proof (for $S = 0$): $j\omega \in \text{spec}(H_{\text{ric}})$ iff

$$\begin{bmatrix} A - j\omega I & -BR^{-1}B' \\ -Q & -A' - j\omega I \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0 \iff \begin{cases} (A - j\omega I)\eta_1 = BR^{-1}B'\eta_2 \\ \eta_2'(A - j\omega I) = -\eta_1'Q \end{cases}$$

Hence,

$$\eta_2'(A - j\omega I)\eta_1 = \overbrace{\eta_2'BR^{-1}B'\eta_2}^{\geq 0} \quad \text{and} \quad \eta_2'(A - j\omega I)\eta_1 = \overbrace{-\eta_1'Q\eta_1}^{\leq 0},$$

whence $\eta_1'Q = 0$ and $\eta_2'B = 0$. Hence, $\eta_2' \begin{bmatrix} A - j\omega I & B \end{bmatrix} = 0 \implies \eta_2 = 0$ (by stabilizability). Therefore,

$$j\omega \in \text{spec}(H_{\text{ric}}) \iff \begin{bmatrix} A - j\omega I \\ -Q \end{bmatrix} \eta_1 = 0 \iff \text{rank} \begin{bmatrix} A - j\omega I \\ Q \end{bmatrix} < n,$$

which is equivalent to the rank condition under $S = 0$ and $R > 0$. □

Solvability

Theorem 4.4

Let $R > 0$ and $Q - SR^{-1}S' \geq 0$. The CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

has a stabilizing solution $\bar{X} \geq 0$ iff

1. (A, B) is stabilizable,
2. $\begin{bmatrix} A - \lambda I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank for all $\lambda \in \{z \in \mathbb{C} \mid \operatorname{Re} z = 0\}$.

Moreover, $\bar{X} > 0$ iff the second condition is replaced with

- 2'. $\begin{bmatrix} A - \lambda I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank for all $\lambda \in \{z \in \mathbb{C} \mid \operatorname{Re} z \leq 0\}$.

The rank condition

Because $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$, there are $C_z \in \mathbb{R}^{p_z \times n}$ and $D_{zu} \in \mathbb{R}^{p_z \times m}$ such that

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} C'_z \\ D'_{zu} \end{bmatrix} \begin{bmatrix} C_z & D_{zu} \end{bmatrix}$$

for $p_z = \text{rank} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$ (Lecture 1, Slide 29). For example, $\gamma_1 + r\gamma_u$ has

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} C'_1 C_1 & 0 \\ 0 & r \end{bmatrix} = \begin{bmatrix} C'_1 & 0 \\ 0 & \sqrt{r} \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & \sqrt{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \sqrt{r} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \sqrt{r} \end{bmatrix}$$

If $M = N'N$, then $Mx = 0 \iff N'Nx = 0 \iff x'N'Nx = 0 \iff Nx = 0$.

Hence, the rank condition reads

$$2. \begin{bmatrix} A - \lambda I & B \\ C_z & D_{zu} \end{bmatrix} \text{ has full column rank for all } \lambda \in \{z \in \mathbb{C} \mid \text{Re } z = 0\}$$

and implies that the transfer function

$$-D_{zu} + C_z(sI - A)^{-1}B \text{ has no pure imaginary zeros.}$$

The rank condition

Because $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$, there are $C_z \in \mathbb{R}^{p_z \times n}$ and $D_{zu} \in \mathbb{R}^{p_z \times m}$ such that

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} C'_z \\ D'_{zu} \end{bmatrix} \begin{bmatrix} C_z & D_{zu} \end{bmatrix}$$

for $p_z = \text{rank} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$ (Lecture 1, Slide 29). For example, $\gamma_1 + r\gamma_u$ has

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} C'_1 C_1 & 0 \\ 0 & r \end{bmatrix} = \begin{bmatrix} C'_1 & 0 \\ 0 & \sqrt{r} \end{bmatrix} \begin{bmatrix} C_1 & 0 \\ 0 & \sqrt{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \sqrt{r} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \sqrt{r} \end{bmatrix}$$

If $M = N'N$, then $Mx = 0 \iff N'Nx = 0 \iff x'N'Nx = 0 \iff Nx = 0$.

Hence, the rank condition reads

2. $\begin{bmatrix} A - \lambda I & B \\ C_z & D_{zu} \end{bmatrix}$ has full column rank for all $\lambda \in \{z \in \mathbb{C} \mid \text{Re } z = 0\}$

and implies that the transfer function

— $D_{zu} + C_z(sI - A)^{-1}B$ has no pure imaginary zeros.