

State-Space Control Fundamentals (00860721)

lecture no. 3

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Setup

Plant:

$$P : \begin{cases} \dot{x}(t) = Ax(t) + B_d d(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) + n(t) \end{cases}$$

where

d : disturbance effects of the environment, independent of our actions

u : control input we can choose it freely

y : measurement

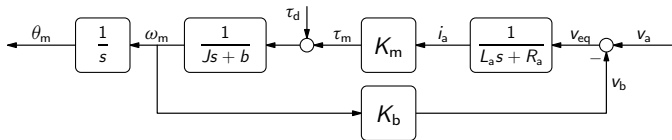
n : measurement noise sensor imperfections

and parameters $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $B_d \in \mathbb{R}^{n \times m_d}$, $C \in \mathbb{R}^{p \times n}$.

Special cases:

- if $y = x$, i.e. $C = I$ and $n = 0$, then the setup is called **state feedback**
- if C is not full column rank or $n \neq 0$, then **output feedback**
- if $B_d = B$, then d is known as a **load disturbance**
- if $B_d = BM_d$ for some M_d , then d is dubbed a **matched disturbance**

Example: DC motor



Realization (remember Lecture 2), with $x = (\theta_m, \omega_m, i_a)$:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -b/J & K_m/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix}}_{B_d} \tau_d + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix}}_B v_a$$

Would be

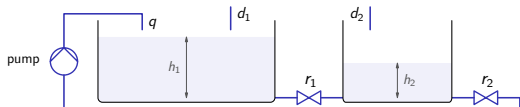
- state feedback if we measure θ_m , ω_m , and i_a perfectly
- output feedback otherwise (say, if only θ_m can be measured)

and

- τ_d is not a matched disturbance

would be matched with $M_d = R_a/K_m$ if $L_a = 0$, think why...

Example: two-tank system



Realization (remember Lecture 2), with $x = (h_1 - h_{eq1}, h_2 - h_{eq2})$:

$$\dot{x} = \underbrace{\frac{r_1 \sqrt{1 + \rho^2}}{2\rho \sqrt{h_{eq1}}} \begin{bmatrix} -1/a_1 & 1/a_1 \\ 1/a_2 & -(1 + \rho^2)/a_2 \end{bmatrix}}_A x + \underbrace{\begin{bmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{bmatrix}}_{B_d} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/a_1 \\ 0 \end{bmatrix}}_B u$$

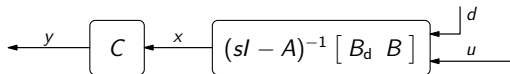
Would be

- state feedback if we measure both h_1 and h_2 perfectly
- output feedback otherwise (say, if we measure either h_1 or h_2)

and

- d_1 is matched (actually, load disturbance)
- d_2 is mismatched

Variants of control tasks



Regulator problem $\lim_{t \rightarrow \infty} x(t) = 0$ under arbitrary x_0 and $d = 0$

Tracking problem make y close to a known y_{ref} under $d = 0$

Disturbance attenuation reduce the effect of d on y

Remark 1: Because we conventionally work in terms of deviation variables (deviations from an equilibrium), the requirement $\lim_{t \rightarrow \infty} x(t) = 0$ normally means the convergence to that equilibrium.

Remark 2: Initial conditions play a role of the **test input**, much like the step reference does in the classical unity-feedback setup.

Outline

State feedback regulator problem

Controllability

Example: 2-mass system

State feedback: pole placement under $m = 1$

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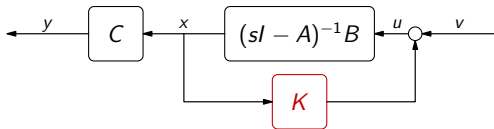
State feedback

The control law

$$u = Kx + v,$$

for a gain¹ $K \in \mathbb{R}^{m \times n}$ and exogenous signal v is dubbed the **state feedback**.

Ignoring disturbances for a while, the controlled system is

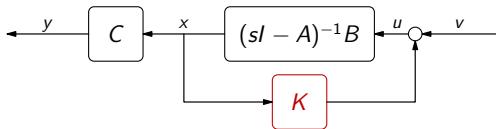


Note,

- in regulator problems $v = 0$
- nonzero v is used in tracking problems

¹This K is for Rudolf E. Kálmán.

State feedback: closed-loop dynamics



If $u = Kx + v$, then

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x}(t) = \overbrace{(A + BK)}^{:= A_K} x(t) + Bv(t) \\ y(t) = Cx(t) \end{cases}$$

so that

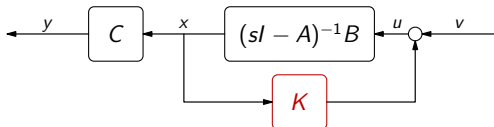
$$\underbrace{T_{xv}(s) = (sI - A_K)^{-1}B}$$

$$\Downarrow$$

$$\underbrace{T_{yv}(s) = C(sI - A_K)^{-1}B \quad \text{and} \quad T_{uv}(s) = I_m + K(sI_n - A_K)^{-1}B}$$

(because $y = Cx$ and $u = Kx + v$).

State feedback and system zeros



The Rosenbrock system matrix associated with T_{yv} is

$$\begin{bmatrix} A_K - sI & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A + BK - sI & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix} \underbrace{\begin{bmatrix} I & 0 \\ K & I \end{bmatrix}}_{\text{full-rank}}$$

Hence,

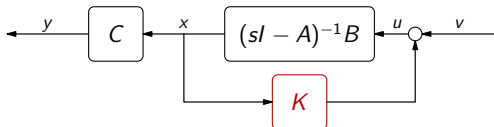
$$\text{rank} \begin{bmatrix} A + BK - sI & B \\ C & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix}$$

for all $s \in \mathbb{C}$, meaning that

- state feedback does **not move zeros**

(still, stable zeros may be canceled by closed-loop poles).

State feedback and closed-loop poles



With

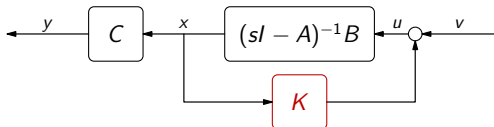
$$T_{yv}(s) = C(sI - A_K)^{-1}B \quad \text{and} \quad T_{uv}(s) = 1 + K(sI - A_K)^{-1}B$$

closed-loop poles are

- among the eigenvalues of $A_K = A + BK$.

Q: what freedom we have in assigning $\text{spec}(A + BK)$ by the choice of K ?

State feedback and closed-loop poles



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Special case: $m = 1$ and companion form

Assume that

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

If $K = [k_0 \ k_1 \ \cdots \ k_{n-1}]$, then

$$A_K = A + BK = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -(a_0 - k_0) & -(a_1 - k_1) & \cdots & -(a_{n-1} - k_{n-1}) \end{bmatrix}$$

is still a companion matrix and its characteristic polynomial

$$\chi_{A_K}(\lambda) = \lambda^n + (a_{n-1} - k_{n-1})\lambda^{n-1} + \cdots + (a_1 - k_1)\lambda + (a_0 - k_0).$$

Special case: companion form (contd)

Therefore, any desired closed-loop characteristic polynomial, say

$$\chi_{\text{cl}}(s) = s^n + \chi_{n-1}s^{n-1} + \cdots + \chi_1s + \chi_0$$

for some coefficients $\chi_i > 0$, can be assigned by

$$K = \begin{bmatrix} a_0 - \chi_0 & a_1 - \chi_1 & \cdots & a_{n-1} - \chi_{n-1} \end{bmatrix},$$

for which

$$A_K = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\chi_0 & -\chi_1 & \cdots & -\chi_{n-1} \end{bmatrix}$$

Q: under what condition this can be said about an arbitrary realization?

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Outline

State feedback regulator problem

Controllability

Example: 2-mass system

State feedback: pole placement under $m = 1$

Definition

Consider

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

This system, or the pair (A, B) , is said to be

- **controllable** if there are $t_1 > 0$ and $u : [0, t_1] \rightarrow \mathbb{R}^m$ that steers x from $x(0) = x_0$ to $x(t_1) = x_1$ for every pair $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$.

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Remark: Without loss of generality (w.l.o.g.) we can take $x_0 = 0$. Indeed,

$$x_1 = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds \iff \tilde{x}_1 := x_1 - e^{At_1}x_0 = \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds.$$

Therefore, a signal u steering x from $x(0) = x_0$ to $x(t_1) = x_1$ also steers it from $x(0) = 0$ to $x(t_1) = \tilde{x}_1$. Thus, if every $\tilde{x}_1$ is reachable from $x(0) = 0$, then every $x_1 = \tilde{x}_1 + e^{At_1}x_0$ is reachable from $x(0) = x_0$, and vice versa.

Remark: Alternatively, because

$$x_1 = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds \iff 0 = e^{At_1}(e^{-At_1}x_1 - x_0) + \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds,$$

we may take $x_1 = 0$ w.l.o.g. in the definition of controllability.

Criteria

Theorem 3.1

The following statements are equivalent:

1. (A, B) is controllable
2. $M_c := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times (nm)}$ has full row rank
3. $W_c(t) > 0$ for all $t > 0$, where $W_c(t) := \int_0^t e^{As} BB' e^{A's} ds \in \mathbb{R}^{n \times n}$
4. $\begin{bmatrix} A - \lambda I & B \end{bmatrix} \in \mathbb{C}^{n \times (n+m)}$ has full row rank $\forall \lambda \in \mathbb{C}$

If A is Hurwitz, then $W_c := \lim_{t \rightarrow \infty} W_c(t)$ exists and solves the Lyapunov

$$AW_c + W_c A' + BB' = 0$$

Terminology:

- matrix M_c is called the **controllability matrix**
- matrix W_c is known as the **controllability Gramian** sometimes $W_c(t)$
- 4. is known as the **PBH test** (Popov–Belevich–Hautus)

Controllability and similarity transformations

If $\tilde{A} = TAT^{-1}$ and $\tilde{B} = TB$, then (remember, $(TAT^{-1})^k = TA^kT^{-1}$)

$$\begin{aligned}\tilde{M}_c &= [\tilde{B} \quad \tilde{A}\tilde{B} \quad \dots \quad \tilde{A}^{n-1}\tilde{B}] = [TB \quad TAT^{-1}TB \quad \dots \quad TA^{n-1}T^{-1}TB] \\ &= T[B \quad AB \quad \dots \quad A^{n-1}B] = TM_c\end{aligned}$$

Likewise,

$$\tilde{W}_c(t) = \int_0^t e^{\tilde{A}s} \tilde{B} \tilde{B}' e^{\tilde{A}'s} ds = \int_0^t e^{TAT^{-1}s} TBB'T' e^{T^{-1}AT's} ds = TW_c(t)T'$$

Hence,

- controllability is not affected by similarity transformations.

If $m = 1$, then

$$T = \tilde{M}_c M_c^{-1}$$

is the transformation matrix connecting two similar controllable realizations.

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Gramian and control law steering x from x_0 to x_1

If

$$u(t) = u_*(t) := B'e^{A'(t_1-t)}[W_c(t_1)]^{-1}(x_1 - e^{At_1}x_0),$$

then

$$\begin{aligned} x(t) &= e^{At}x_0 + \int_0^t e^{A(t-s)}Bu_*(s)ds \\ &= e^{At}x_0 + \int_0^t e^{A(t-s)}BB'e^{A'(t_1-s)}[W_c(t_1)]^{-1}(x_1 - e^{At_1}x_0)ds \\ &= e^{At}x_0 + W_c(t)e^{A'(t_1-t)}[W_c(t_1)]^{-1}(x_1 - e^{At_1}x_0) \end{aligned}$$

Hence,

$$x(t_1) = e^{At_1}x_0 + W_c(t_1)e^{A'(t_1-t_1)}[W_c(t_1)]^{-1}(x_1 - e^{At_1}x_0) = x_1$$

In fact, u_* has

minimal energy, $E_u := \int_0^{t_1} \|u(t)\|^2 dt = \|W_c(t_1)\|^{-2} \|x_1 - e^{At_1}x_0\|^2$,
among all control laws steering x from $x(0) = x_0$ to $x(t_1) = x_1$.

Gramian and control law steering x from x_0 to x_1

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In fact, u_* has

- minimal energy, $E_u := \int_0^{t_1} \|u(t)\|^2 dt = \|[W_c(t_1)]^{-1/2}(x_1 - e^{At_1}x_0)\|^2$

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Minimum energy proof

If $u = u_\delta + u_*$, then, by linearity,

$$x(t_1) = \int_0^{t_1} e^{A(t_1-s)} B u_\delta(s) ds + \overbrace{e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1-s)} B u_*(s) ds}^{x_1}.$$

Hence, $x(t_1) = x_1$ iff u_δ satisfies

$$\int_0^{t_1} e^{A(t_1-s)} B u_\delta(s) ds = 0.$$

Now,

$$\begin{aligned} E_u &= \int_0^{t_1} \|u_\delta(t) + u_*(t)\|^2 dt = \int_0^{t_1} (u_\delta(t) + u_*(t))' (u_\delta(t) + u_*(t)) dt \\ &= \int_0^{t_1} \|u_\delta(t)\|^2 dt + 2 \int_0^{t_1} u_\delta'(t) u_*(t) dt + \int_0^{t_1} \|u_*(t)\|^2 dt \\ &= E_{u_\delta} + 2(x_1 - e^{At_1} x_0)' [W_c(t_1)]^{-1} \int_0^{t_1} e^{A(t_1-t)} B u_\delta(t) dt + E_{u_*} \\ &= E_{u_\delta} + E_{u_*} \end{aligned}$$

As $E_u \geq 0$ and $E_u = 0 \iff u = 0$, the minimum is attained by $u_\delta = 0$. $\square$

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$$= E_{u_\delta} + 2(x_1 - e^{A t_1} x_0)' [W_c(t_1)]^{-1} \int_0^{t_1} e^{A(t_1-s)} B u_\delta(s) ds + E_{u_*}$$

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Pythagoras!

As $E_{u_\delta} \geq 0$ and $E_{u_*} = 0 \iff u = 0$, the minimum is attained by $u_\delta = 0$. $\square$

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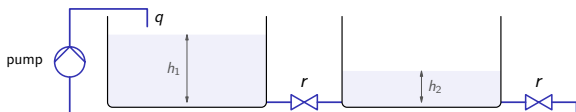
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Pythagoras!

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Minimum energy: example



If $a_1 = a_2 = 1$ and $r_1 = r_2 = r = 1$ (so $\rho = r_2/r_1 = 1$), then

$$- h_{eq1} = 0.5 \implies h_{eq2} = 0.5 h_{eq1} = 0.25 \text{ and } q_{eq} = \sqrt{0.5 h_{eq1}} = 0.5$$

Linearized dynamics

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad x(0) = x_0$$

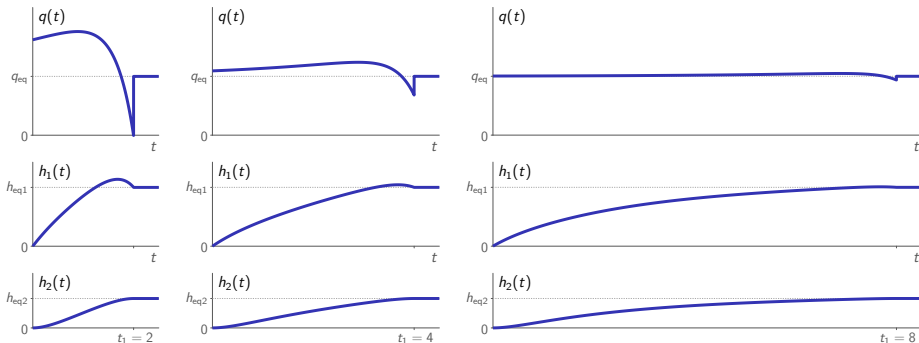
for $x = \begin{bmatrix} h_1 - h_{eq1} \\ h_2 - h_{eq1} \end{bmatrix} = \begin{bmatrix} h_1 - 0.5 \\ h_2 - 0.25 \end{bmatrix}$ and $u = q - q_{eq} = q - 0.5$. Choose

$$x_0 = - \begin{bmatrix} h_{eq1} \\ h_{eq2} \end{bmatrix} \quad \text{and} \quad x_1 = 0$$

(regulator problem), i.e. aim at $h_i(t_1) = h_{eq,i}$ starting at $h_i(0) = 0$, $i = 1, 2$.

Minimum energy: example (contd)

In terms of physical variables $q = u + q_{eq}$ and $h_i = x_i + h_{eqi}$,



Note,

- minimum energy $u \implies$ minimum energy $q - q_{eq}$
i.e. we effectively aim at the minimum energy deviation of q from its equilibrium q_{eq}
- if $t_1 < 2.006$, then $q(t_1) < 0$, which is not physical, beware

Calculating Gramian and minimum-energy trajectories

Let

$$\begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ 0 & \Gamma_{22}(t) \end{bmatrix} := \exp\left(\begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} t\right).$$

Thus, $e^{At} = \Gamma_{11}(t)$, $e^{-A't} = \Gamma_{22}(t)$, and, by Van Loan's formulae,

$$W_c(t) = \Gamma_{12}(t)\Gamma'_{11}(t) = \Gamma_{12}(t)\Gamma_{22}^{-1}(t).$$

Hence,

$$u_*(t) = B'\Gamma_{22}(t)\Gamma_{12}^{-1}(t_1)(x_1 - \Gamma_{11}(t_1)x_0)$$

and

$$x_*(t) = \Gamma_{11}(t)x_0 + \Gamma_{12}(t)\Gamma_{12}^{-1}(t_1)(x_1 - \Gamma_{11}(t_1)x_0).$$

Remark: Van Loan's formulae follow via

$$W_c(t)e^{-A't} = \int_0^t e^{As}BB'e^{A'(s-t)}ds = \int_0^t \underbrace{e^{A(t-s)}B}_{\text{impulse response}} \underbrace{B'e^{-A's}}_{\text{input}} ds.$$

which is the response of $(sI - A)^{-1}B$ to an input, which itself is the impulse response of $B'(sI - (-A'))^{-1}$. Overall, this is the impulse response of the series of these two systems.

Uncontrollable modes and stabilizability

If (A, B) is not controllable, then PBH test fails for some $\lambda_i \in \mathbb{C}$ such that

$$\text{rank} \begin{bmatrix} A - \lambda_i I & B \end{bmatrix} < n \implies \exists \tilde{\eta}_i \neq 0 \text{ such that } \tilde{\eta}_i' \begin{bmatrix} A - \lambda_i I & B \end{bmatrix} = 0.$$

Hence, PBH fails at $\lambda_i \in \mathbb{C}$ iff

$$\tilde{\eta}_i'(\lambda_i I - A) = 0 \quad \text{and} \quad \tilde{\eta}_i' B = 0,$$

meaning that $\lambda_i \in \text{spec}(A)$ and $\tilde{\eta}_i$ is the corresponding left eigenvector.

Eigenvalues of A at which PBH fails are called

– uncontrollable modes of (A, B) .

Pair (A, B) is said to be

– stabilizable if all its uncontrollable modes are stable,

i.e. in the open LHP $\{z \in \mathbb{C} \mid \text{Re } z < 0\}$.

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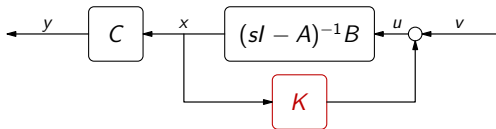
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Uncontrollable modes and state feedback



with (where $A_K = A + BK$)

$$T_{yv}(s) = C(sI - A_K)^{-1}B \quad \text{and} \quad T_{uv}(s) = 1 + K(sI - A_K)^{-1}B.$$

If λ is an uncontrollable mode of (A, B) and $\tilde{\eta}$ is its left eigenvector, then

$$\tilde{\eta}'_i A_K = \tilde{\eta}'_i A + \tilde{\eta}'_i B K = \tilde{\eta}'_i A = \lambda \tilde{\eta}'_i \iff \tilde{\eta}'_i (A_K - \lambda I) = 0.$$

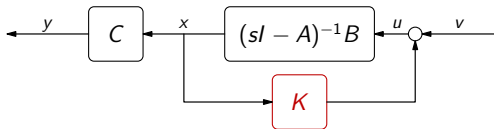
In other words,

- uncontrollable modes are eigenvalues of A_K for every K .

Hence,

- state feedback cannot affect uncontrollable modes.

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Outline

State feedback regulator problem

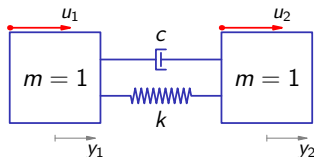
Controllability

Example: 2-mass system

State feedback: pole placement under $m = 1$

Setup

Consider the following 2-mass system:



with external forces u_1 and u_2 . It is described by the following equations:

$$\begin{bmatrix} \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

State-space model

Possible realization (non-unique):

$$\left\{ \begin{array}{l} \underbrace{\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_C \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix}}_{x(t)} \end{array} \right.$$

System has 4 modes (eigenvalues of A):

- $\lambda_1 = \lambda_2 = 0$ rigid body motion
- $\lambda_{3,4} = -c \pm \sqrt{c^2 - 2k}$ spring-damper dynamics

Controllability

If $u_1 = \alpha_1 u$ and $u_2 = \alpha_2 u$ for a force u and constants α_1 and α_2 , then

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} u(t)$$

Controllability matrix:

$$M_c = \begin{bmatrix} 0 & \alpha_1 & -c(\alpha_1 - \alpha_2) & -(2c^2 - k)(\alpha_1 - \alpha_2) \\ 0 & \alpha_2 & c(\alpha_1 - \alpha_2) & -(2c^2 - k)(\alpha_1 - \alpha_2) \\ \alpha_1 & -c(\alpha_1 - \alpha_2) & (2c^2 - k)(\alpha_1 - \alpha_2) & -4c(c^2 - k)(\alpha_1 - \alpha_2) \\ \alpha_2 & c(\alpha_1 - \alpha_2) & -(2c^2 - k)(\alpha_1 - \alpha_2) & 4c(c^2 - k)(\alpha_1 - \alpha_2) \end{bmatrix}$$

with $\det M_c = -k^2(\alpha_1^2 - \alpha_2^2)^2$. Thus, the system is

- uncontrollable for $\alpha_2 = \pm\alpha_1$.

Or what could it mean?

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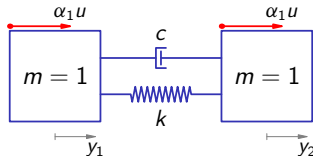
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Q: what could it mean?

Controllability with $\alpha_2 = \alpha_1$ ($u_1 = u_2$)

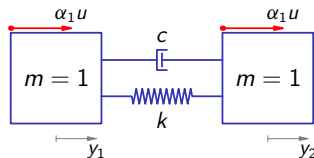


PBH test:

$$\text{rank} \begin{bmatrix} -\lambda & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 1 & 0 \\ -k & k & -c - \lambda & c & \alpha_1 \\ k & -k & c & -c - \lambda & \alpha_1 \end{bmatrix} \bigg|_{\lambda = -c \pm \sqrt{c^2 - 2k}} = 3,$$

(rank lost at **uncontrollable modes** of A). This agrees with our intuition that
 – if equal forces applied to each mass, oscillations not excited.

Controllability with $\alpha_2 = \alpha_1$ ($u_1 = u_2$)



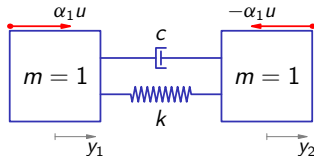
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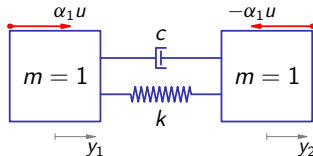


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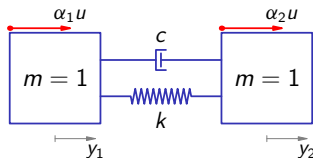
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Transfer functions for $u_1 = \alpha_1 u$ and $u_2 = \alpha_2 u$



Transfer function from u to y_1 :

$$P_1(s) = \frac{\alpha_1 s^2 + c(\alpha_1 + \alpha_2)s + k(\alpha_1 + \alpha_2)}{s^2(s^2 + 2cs + 2k)}$$

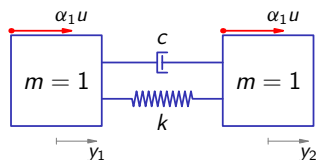
and transfer function from u to y_2 :

$$P_2(s) = \frac{\alpha_2 s^2 + c(\alpha_1 + \alpha_2)s + k(\alpha_1 + \alpha_2)}{s^2(s^2 + 2cs + 2k)}$$

(both obtained via $C(sI - A)^{-1}B$).

Transfer functions for $u_1 = \alpha_1 u$ and $u_2 = \alpha_2 u$ (contd)

$\alpha_2 = \alpha_1$:



then

$$P_1(s) = P_2(s) = \frac{\alpha_1}{s^2}.$$

$\alpha_2 = -\alpha_1$:

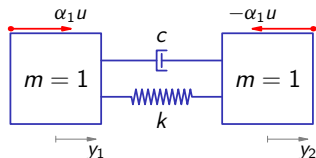
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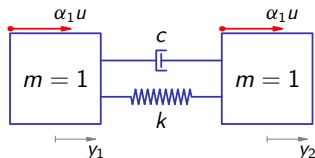
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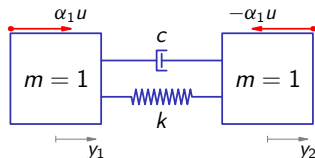
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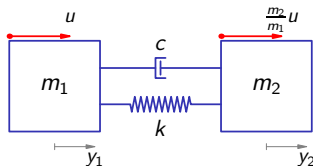
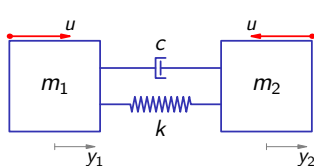
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In both cases we have **pole/zero cancellations** (of different modes though).

Extensions

- If masses are different, controllability is lost at



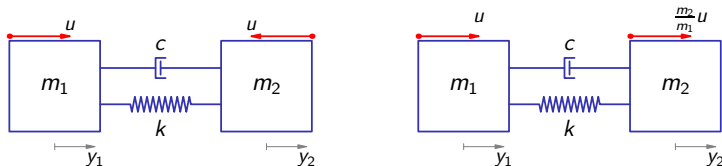
- If there are 3 masses, like in

controllability is lost at (try to explain)

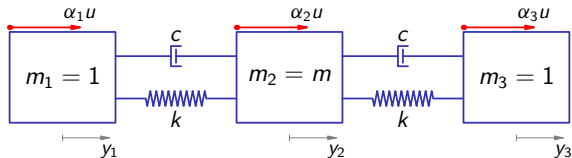
$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_1 = \alpha_3, \quad \text{and} \quad \alpha_1 + \alpha_3 = \frac{2}{m}\alpha_2$$

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Preliminary observations

We know by now that

- if (A, B) uncontrollable, then $\text{spec}(A_K)$ cannot be assigned arbitrarily
- if (A, B) is in the companion form, i.e.

$$(A, B) = (A_{\text{cf}}, B_{\text{cf}}) := \left(\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

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then $\text{spec}(A_K)$ can be assigned arbitrarily, with $K_i = a_i - \chi_i$

Remark: Another, brute-force, way to show the controllability of the companion form is

$$M_{\text{c,cf}} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}^{-1} \implies \det M_{\text{c,cf}} = (-1)^{\lfloor n/2 \rfloor}.$$

Pole placement for arbitrary realizations: logic

Given controllable (A, B) and a required $\chi_{cl}(s)$, all we need to do is to

- transform $(A, B) \rightarrow (A_{cf}, B_{cf}) = (TAT^{-1}, TB)$
- design K_{cf} such that $\text{spec}(A_{cf} + B_{cf}K_{cf})$ matches the roots $\chi_{cl}(s)$
- transform $K_{cf} \rightarrow K = K_{cf}T$ for the original realization
for this choice

$$A + BK = T^{-1}A_{cf}T + T^{-1}B_{cf}K_{cf}T = T^{-1}(A_{cf} + B_{cf}K_{cf})T$$

has the same spectrum as $A_{cf} + B_{cf}K_{cf}$

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An elegant algorithm to construct required K without explicitly calculating T is offered by **Ackermann's formula**.

Ackermann's formula: preliminaries

1. by Cayley–Hamilton, $A_{\text{cf}}^n = -a_{n-1}A_{\text{cf}}^{n-1} - \dots - a_1A_{\text{cf}} - a_0I$, so that

$$\begin{aligned}
 \chi_{\text{cl}}(A_{\text{cf}}) &= A_{\text{cf}}^n + \chi_{n-1}A_{\text{cf}}^{n-1} + \dots + \chi_1A_{\text{cf}} + \chi_0I \\
 &= -a_{n-1}A_{\text{cf}}^{n-1} - \dots - a_1A_{\text{cf}} - a_0I \\
 &\quad + \chi_{n-1}A_{\text{cf}}^{n-1} + \dots + \chi_1A_{\text{cf}} + \chi_0I \\
 &= (\chi_{n-1} - a_{n-1})A_{\text{cf}}^{n-1} + \dots + (\chi_1 - a_1)A_{\text{cf}} + (\chi_0 - a_0)I.
 \end{aligned}$$

2. if e_i is the i th standard basis in $\mathbb{R}^n$, then $\forall i \in \mathbb{Z}_{1,n-1}$,

$$e_i' A_{\text{cf}} = e_{i+1}' \quad \text{or, equivalently,} \quad e_i' A_{\text{cf}}' = e_{i+1}'.$$

3. if $A_{\text{cf}} = TAT^{-1}$ and $B_{\text{cf}} = TB$, then

$$M_{\text{cdf}} = TM_c \iff T = M_{\text{cdf}}M_c^{-1}.$$

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$$\begin{aligned}\chi_{\text{cl}}(A_{\text{cf}}) &= A_{\text{cf}}^n + \chi_{n-1}A_{\text{cf}}^{n-1} + \dots + \chi_1A_{\text{cf}} + \chi_0I \\ &= -a_{n-1}A_{\text{cf}}^{n-1} - \dots - a_1A_{\text{cf}} - a_0I \\ &\quad + \chi_{n-1}A_{\text{cf}}^{n-1} + \dots + \chi_1A_{\text{cf}} + \chi_0I \\ &= (\chi_{n-1} - a_{n-1})A_{\text{cf}}^{n-1} + \dots + (\chi_1 - a_1)A_{\text{cf}} + (\chi_0 - a_0)I.\end{aligned}$$

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$$e'_i A_{\text{cf}} = e'_{i+1} \quad \text{or, equivalently,} \quad e'_1 A_{\text{cf}}^i = e'_{i+1},$$

3. if $A_{\text{cf}} = TAT^{-1}$ and $B_{\text{cf}} = TB$, then

$$M_{\text{cd}} = TM_c \iff T = M_{\text{cd}}M_c^{-1}.$$

Ackermann's formula: preliminaries

1. by Cayley–Hamilton, $A_{\text{cf}}^n = -a_{n-1}A_{\text{cf}}^{n-1} - \dots - a_1A_{\text{cf}} - a_0I$, so that

$$\begin{aligned}\chi_{\text{cl}}(A_{\text{cf}}) &= A_{\text{cf}}^n + \chi_{n-1}A_{\text{cf}}^{n-1} + \dots + \chi_1A_{\text{cf}} + \chi_0I \\ &= -a_{n-1}A_{\text{cf}}^{n-1} - \dots - a_1A_{\text{cf}} - a_0I \\ &\quad + \chi_{n-1}A_{\text{cf}}^{n-1} + \dots + \chi_1A_{\text{cf}} + \chi_0I \\ &= (\chi_{n-1} - a_{n-1})A_{\text{cf}}^{n-1} + \dots + (\chi_1 - a_1)A_{\text{cf}} + (\chi_0 - a_0)I.\end{aligned}$$

2. if e_i is the i th standard basis in $\mathbb{R}^n$, then $\forall i \in \mathbb{Z}_{1..n-1}$,

$$e'_i A_{\text{cf}} = e'_{i+1} \quad \text{or, equivalently,} \quad e'_1 A_{\text{cf}}^i = e'_{i+1},$$

3. if $A_{\text{cf}} = TAT^{-1}$ and $B_{\text{cf}} = TB$, then

$$M_{\text{C,cf}} = TM_{\text{C}} \iff T = M_{\text{C,cf}}M_{\text{C}}^{-1}.$$

Ackermann's formula: preliminaries (contd)

4. Combining 1 and 2:

$$\begin{aligned}
 K_{\text{cf}} &= [a_0 - \chi_0 \quad a_1 - \chi_1 \quad \cdots \quad a_{n-1} - \chi_{n-1}] \\
 &= (a_0 - \chi_0)e'_1 + (a_1 - \chi_1)e'_2 + \cdots + (a_{n-1} - \chi_{n-1})e'_n \\
 &= (a_0 - \chi_0)e'_1 + (a_1 - \chi_1)e'_1 A_{\text{cl}} + \cdots + (a_{n-1} - \chi_{n-1})e'_1 A_{\text{cl}}^{n-1} \\
 &= e'_1((a_0 - \chi_0)I + (a_1 - \chi_1)A_{\text{cl}} + \cdots + (a_{n-1} - \chi_{n-1})A_{\text{cl}}^{n-1}) \\
 &= -e'_1 \chi_{\text{cl}}(A_{\text{cf}})
 \end{aligned}$$

5. By 2 and the fact that $B_{\text{cl}} = e_n$:

$$\begin{aligned}
 e'_1 M_{\text{cl}} &= e'_1 [B_{\text{cl}} \quad A_{\text{cl}} B_{\text{cl}} \quad \cdots \quad A_{\text{cl}}^{n-1} B_{\text{cl}}] \\
 &= [e'_1 B_{\text{cl}} \quad e'_1 A_{\text{cl}} B_{\text{cl}} \quad \cdots \quad e'_1 A_{\text{cl}}^{n-1} B_{\text{cl}}] \\
 &= [e'_1 e_n \quad e'_1 A_{\text{cl}} e_n \quad \cdots \quad e'_1 A_{\text{cl}}^{n-1} e_n] \\
 &= [0 \quad 0 \quad \cdots \quad 1] \\
 &= e'_n
 \end{aligned}$$

Ackermann's formula: preliminaries (contd)

4. Combining 1 and 2:

$$\begin{aligned}
 K_{\text{cf}} &= [a_0 - \chi_0 \quad a_1 - \chi_1 \quad \cdots \quad a_{n-1} - \chi_{n-1}] \\
 &= (a_0 - \chi_0)e'_1 + (a_1 - \chi_1)e'_2 + \cdots + (a_{n-1} - \chi_{n-1})e'_n \\
 &= (a_0 - \chi_0)e'_1 + (a_1 - \chi_1)e'_1 A_{\text{cl}} + \cdots + (a_{n-1} - \chi_{n-1})e'_1 A_{\text{cl}}^{n-1} \\
 &= e'_1((a_0 - \chi_0)I + (a_1 - \chi_1)A_{\text{cl}} + \cdots + (a_{n-1} - \chi_{n-1})A_{\text{cl}}^{n-1}) \\
 &= -e'_1 \chi_{\text{cl}}(A_{\text{cf}})
 \end{aligned}$$

5. By 2 and the fact that $B_{\text{cf}} = e_n$:

$$\begin{aligned}
 e'_1 M_{\text{c,cf}} &= e'_1 [B_{\text{cf}} \quad A_{\text{cf}} B_{\text{cf}} \quad \cdots \quad A_{\text{cf}}^{n-1} B_{\text{cf}}] \\
 &= [e'_1 B_{\text{cf}} \quad e'_2 B_{\text{cf}} \quad \cdots \quad e'_n B_{\text{cf}}] \\
 &= [e'_1 e_n \quad e'_2 e_n \quad \cdots \quad e'_n e_n] \\
 &= [0 \quad 0 \quad \cdots \quad 1] \\
 &= e'_n
 \end{aligned}$$

Ackermann's formula: derivation

We have:

$$K_{\text{cf}} = -e'_1 \chi_{\text{cl}}(A_{\text{cf}}) \quad \text{by item 4}$$

$$= -e'_1 \chi_{\text{cl}}(TAT^{-1}) = -e'_1 T \chi_{\text{cl}}(A) T^{-1}$$

$$= -e'_1 M_{\text{c,cf}} M_{\text{c}}^{-1} \chi_{\text{cl}}(A) T^{-1} \quad \text{by item 3}$$

$$= -e'_n M_{\text{c}}^{-1} \chi_{\text{cl}}(A) T^{-1} \quad \text{by item 5}$$

Now it is time to return to the original coordinates of the gain:

$$\begin{aligned} K &= K_{\text{cf}} T = -e'_n M_{\text{c}}^{-1} \chi_{\text{cl}}(A) T^{-1} T \\ &= -e'_n M_{\text{c}}^{-1} \chi_{\text{cl}}(A), \end{aligned}$$

voilà!

Ackermann's formula / Matlab: `acker(A,B,chi)`

Theorem 3.2 (Ackermann)

If $m = 1$, then the state-feedback gain assigning the eigenvalues of $A + BK$ to the roots of a given $\chi_{cl}(s)$ is

$$K = - \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_c^{-1} \chi_{cl}(A),$$

where

$$M_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

and

$$\chi_{cl}(A) = A^n + \chi_{n-1}A^{n-1} + \cdots + \chi_1A + \chi_0I.$$

Remark: The formula can be used also if (A, B) is not controllable. In that case $\chi_{cl}(s)$ must have all uncontrollable modes of (A, B) among its roots and M_c^{-1} needs to be replaced with its pseudo-inverse. The resulting $A + BK$ can be made Hurwitz iff (A, B) is stabilizable.

Initial condition vs. step responses

Classical control conventionally uses the step $\mathbb{1}$ as the test input signal. This is where our intuition works the best in

- connecting pole and zero locations to the time response

Because for every $t > 0$

$$\int_0^t e^{A_K(t-s)} A_K x_0 ds = - \int_0^t \frac{d}{ds} e^{A_K(t-s)} ds x_0 = \overbrace{e^{A_K t} x_0 - x_0}^{\text{initial condition response}}$$

we have

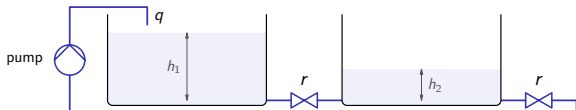
$$C e^{A_K t} x_0 \mathbb{1}(t) = \int_0^t C e^{A_K(t-s)} A_K x_0 \mathbb{1}(s) ds + \overbrace{C x_0 \mathbb{1}(t)}^{\text{constant shift}}$$

i.e. the initial condition response is a shifted, by $-T_{yx_0}(0)$, step response of

$$T_{yx_0}(s) = C(sI - A_K)^{-1} A_K x_0 = \left[\begin{array}{c|c} A_K & A_K x_0 \\ \hline C & 0 \end{array} \right].$$

The pole-zero map of $T_{yx_0}(s)$ is informative to understand $C e^{A_K t} x_0 \mathbb{1}(t)$.

Example: two-tank system



Again, let $a_1 = a_2 = 1$ and $r = 1$, for which

$$- h_{\text{eq}1} = 0.5 \implies h_{\text{eq}2} = 0.5h_{\text{eq}1} = 0.25 \text{ and } q_{\text{eq}} = \sqrt{0.5h_{\text{eq}1}} = 0.5$$

The state equation is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

The transfer functions of the systems $u \mapsto x_1$ and $u \mapsto x_2$ are

$$P_1(s) = \frac{s+2}{(s+2.618)(s+0.382)} \quad \text{and} \quad P_2(s) = \frac{1}{(s+2.618)(s+0.382)}$$

(derived via $P_i(s) = e_i'(sI - A)^{-1}B$, the poles are at $s = -(3 \pm \sqrt{5})/2$).

Example: state feedback (first option)

Let

$$\chi_{\text{cl}}(s) = s^2 + 2\zeta\omega_n s + \omega_n^2.$$

Because the controllability matrix

$$M_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad M_c^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

Ackermann's formula $K = - \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_c^{-1} \chi_{\text{cl}}(A)$ yields

$$\begin{aligned} K &= - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}^2 + 2\zeta\omega_n \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} + \omega_n^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 - 2\zeta\omega_n + \omega_n^2 & -3 + 2\zeta\omega_n \\ -3 + 2\zeta\omega_n & 5 - 4\zeta\omega_n + \omega_n^2 \end{bmatrix} \\ &= - \begin{bmatrix} -3 + 2\zeta\omega_n & 5 - 4\zeta\omega_n + \omega_n^2 \end{bmatrix} \end{aligned}$$

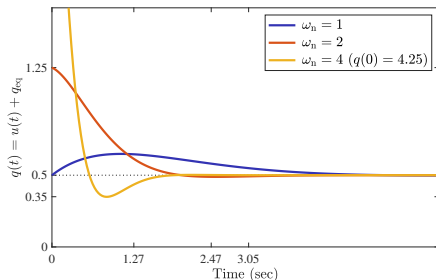
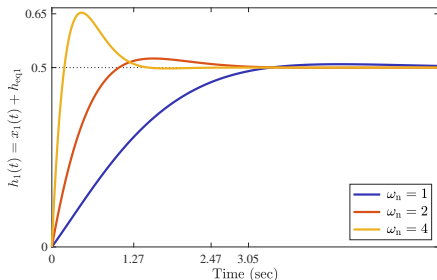
(the absolute values of both components of K eventually grow with ω_n).

Example: simulations

The control law

$$u(t) = Kx(t) \implies q(t) = Kh(t) + q_{eq} - Kh_{eq}.$$

With $\zeta = 0.8$, $\omega_n = \{1, 2, 5\}$, and $x_0 = -h_{eq} = -\begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$



under $K = -\begin{bmatrix} -1.4 & 2.8 \end{bmatrix}$, $K = -\begin{bmatrix} 0.2 & 2.6 \end{bmatrix}$, $K = -\begin{bmatrix} 3.4 & 8.2 \end{bmatrix}$. Thus,

- faster poles $\implies$ faster response & larger control effort
- faster poles $\implies$ large overshoot (why?)

Example: T_{yx_0}

If $x_{0,1} = 2x_{0,2}$, meaning we start from an equilibrium, then

$$\begin{aligned}(A + BK)x_0 &= \left(\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} K \right) \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_{0,2} \\ &= \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2k_1 + k_2 \\ 0 \end{bmatrix} \right) x_{0,2} \\ &= \begin{bmatrix} 2k_1 + k_2 - 1 \\ 0 \end{bmatrix} x_{0,2} = B(2k_1 + k_2 - 1)x_{0,2}\end{aligned}$$

so that (remember, state feedback shifts poles, but not zeros of T_{yv})

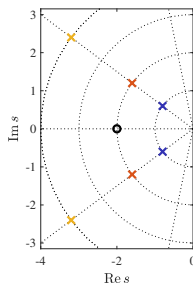
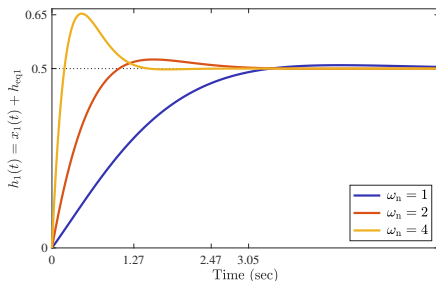
$$T_{x_1 x_0}(s) = (2k_1 + k_2 - 1)x_{0,2} T_{x_1 v}(s) = \frac{(s + 2)(2k_1 + k_2 - 1)0.5x_{0,1}}{\chi_{cl}(s)}$$

$$T_{x_2 x_0}(s) = (2k_1 + k_2 - 1)x_{0,2} T_{x_2 v}(s) = \frac{(2k_1 + k_2 - 1)x_{0,2}}{\chi_{cl}(s)}$$

In this design $\chi_{cl}(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ and $2k_1 + k_2 - 1 = -\omega_n^2$.

Example: simulations (contd)

The reason for growing overshoot with the increase of ω_n can be seen in the pole-zero map of $T_{x_1 x_0}(s)$:



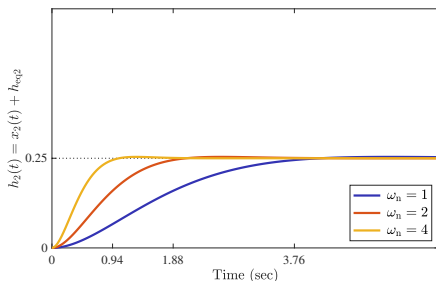
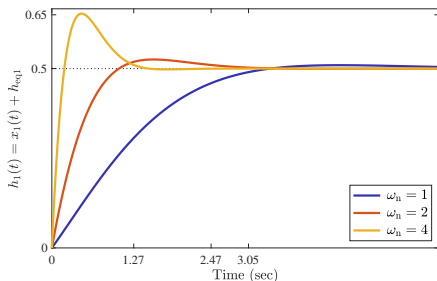
When the closed-loop poles move left, the zero at $s = -2$ of

$$T_{x_1 x_0}(s) = -\frac{(s+2)\omega_n^2 0.5x_{0,1}}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

becomes dominant.

Example: simulations (contd)

Compare responses of h_1 and h_2 :



where

$$T_{x_1 x_0}(s) = -\frac{(s+2)\omega_n^2 0.5x_{0,1}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \text{and} \quad T_{x_2 x_0}(s) = -\frac{\omega_n^2 x_{0,2}}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Example: state feedback (second option)

Let now choose (monic)

$$\chi_{\text{cl}}(s) = (s + 2)(s + 1/\tau).$$

Ackermann's formula $K = - \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_c^{-1} \chi_{\text{cl}}(A)$ yields

$$\begin{aligned} K &= - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} + 2I_2 \right) \left(\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} + \frac{1}{\tau} I_2 \right) \\ &= - \begin{bmatrix} 1/\tau - 1 & 1 \end{bmatrix}. \end{aligned}$$

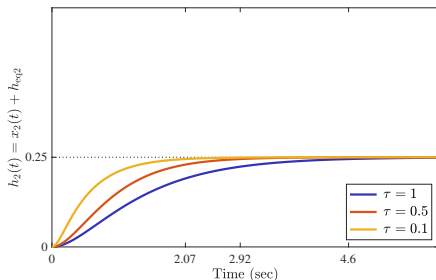
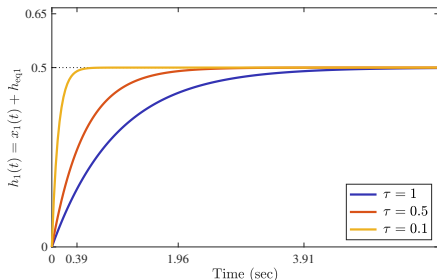
In this case $2k_1 + k_2 - 1 = -2/\tau$ and

$$T_{x_1 x_0}(s) = -\frac{x_{0,1}}{\tau s + 1} \quad \text{and} \quad T_{x_2 x_0}(s) = -\frac{x_{0,2}}{(0.5s + 1)(\tau s + 1)}$$

(remember, $x_{0,1} = 2x_{0,2}$), i.e. the zero of $P_1(s)$ at $s = -2$ is now canceled.

Example: simulations (contd)

Compare responses of h_1 and h_2 :



The settling time (to 2%) of h_1 is 3.91τ , as expected from a 1-order system. Yet the settling time of h_2 is never faster than 1.96, because

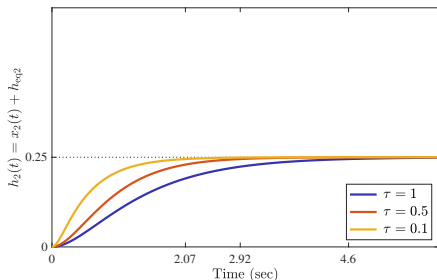
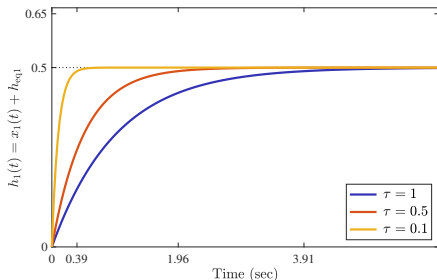
- as τ decreases, the pole of $T_{x_2x_0}(s)$ at $s = -2$ becomes dominant.

Conrad

there is more in closed-loop performance than pole locations.

Example: simulations (contd)

Compare responses of h_1 and h_2 :



The settling time (to 2%) of h_1 is 3.91τ , as expected from a 1-order system. Yet the settling time of h_2 is never faster than 1.96, because

- as τ decreases, the pole of $T_{x_2x_0}(s)$ at $s = -2$ becomes dominant.

Moral:

- there is more in closed-loop performance than pole locations.