

State-Space Control Fundamentals (00860721)

lecture no. 2

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Linear systems and their representations

A continuous-time system is a mapping, denoted $G : u \mapsto y$, between input and output signals, $u : \mathbb{R} \rightarrow \mathbb{R}^m$ and $y : \mathbb{R} \rightarrow \mathbb{R}^p$. We write

$$y = Gu \quad \text{or} \quad \begin{array}{c} \xleftarrow{y} \boxed{G} \xleftarrow{u} \end{array}$$

(not a multiplication). A system is

- linear if $G(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 G u_1 + \alpha_2 G u_2$ superposition
- time invariant if any time shift in $u \implies$ the same time shift in y

Linear time-invariant (LTI) system G can be described via

convolution integral $y = g * u \iff y(t) = \int_{-\infty}^{\infty} g(t-s)u(s)ds$, where g is its impulse response

transfer function $G(s) = (\mathcal{L}\{g\})(s)$, so $Y(s) = G(s)U(s)$ product

ODEs / PDEs / FDEs e.g. $m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t)$

Outline

State-space models

Transfer functions and state-space realization (mostly SISO)

Initial conditions

System interconnections in terms of state-space realizations

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State-space models

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State-space representation: continuous-time case

Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, the set of equations

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

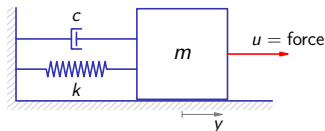
is known as the state-space representation of a continuous-time $G : u \mapsto y$. Here

- the (internal) signal $x : \mathbb{R} \rightarrow \mathbb{R}^n$ is called the state vector of G
- the upper equation (differential) is called the **state equation**
- the lower one (algebraic) is called the **output equation**

The quadruple (A, B, C, D) is dubbed a **state-space realization** of G .

Remark: We may be mostly familiar with SISO systems, i.e. the case of $m = p = 1$. Still, most results of the course apply to MIMO systems seamlessly. As such, we shall try to keep results general whenever possible w/o entering dangerous corners (for that take *Control of Multi-Input Multi-Output Systems*, 00860289).

Example: spring-mass-damper system 1



Dynamics,

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t),$$

can be rewritten as

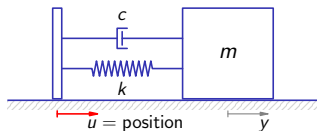
$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

where

$$x(t) := \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

is the **state vector**, having clear physical meaning.

Example: spring-mass-damper system 2



Dynamics,

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = c\dot{u}(t) + ku(t),$$

can be rewritten as

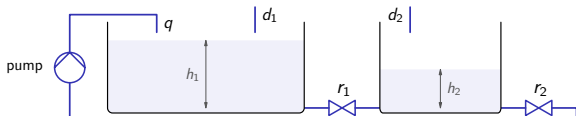
$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} k/m & c/m \end{bmatrix} x(t) \end{cases}$$

where

$$x(t) := \begin{bmatrix} v(t) \\ \dot{v}(t) \end{bmatrix} \quad \text{for } \ddot{v}(t) = u(t) - y(t)$$

is the **state vector**, having no clear physical meaning.

Example: two-tank system



Here:

- q is the inflow rate (control input) volume per time unit
- h_1 and h_2 are liquid levels (heights)
- r_1 and r_2 are the resistances of the valves
- a_1 and a_2 are the crossing areas of tanks
- d_1 and d_2 are parasitic inflows (disturbances) volume per time unit

Dynamics (volume balance and Torricelli's law), provided $h_1 \geq h_2$:

$$\begin{bmatrix} \dot{a}_1 h_1(t) \\ \dot{a}_2 h_2(t) \end{bmatrix} = \begin{bmatrix} q(t) - r_1 \sqrt{h_1(t) - h_2(t)} + d_1(t) \\ r_1 \sqrt{h_1(t) - h_2(t)} - r_2 \sqrt{h_2(t)} + d_2(t) \end{bmatrix}$$

Example: two-tank system (contd)

Equilibria under $d_1 = d_2 = 0$ satisfy

$$q_{\text{eq}} = r_1 \sqrt{h_{\text{eq1}} - h_{\text{eq2}}} \quad \text{and} \quad r_1 \sqrt{h_{\text{eq1}} - h_{\text{eq2}}} = r_2 \sqrt{h_{\text{eq2}}}$$

so

$$h_{\text{eq2}} = \frac{h_{\text{eq1}}}{1 + \rho^2} < h_{\text{eq1}} \quad \text{and} \quad q_{\text{eq}} = \frac{r_2 \sqrt{h_{\text{eq1}}}}{\sqrt{1 + \rho^2}}$$

for any $h_{\text{eq1}} \geq 0$, where $\rho := r_2/r_1$. With the Jacobian matrix

$$\frac{\partial f(h, q, d)}{\partial h} = \frac{r_1}{2} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}^{-1} \begin{bmatrix} -1/\sqrt{h_1 - h_2} & 1/\sqrt{h_1 - h_2} \\ 1/\sqrt{h_1 - h_2} & -1/\sqrt{h_1 - h_2} - \rho/\sqrt{h_2} \end{bmatrix},$$

linearized state equation is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \frac{r_1 \sqrt{1 + \rho^2}}{2\rho \sqrt{h_{\text{eq1}}}} \begin{bmatrix} -1/a_1 & 1/a_1 \\ 1/a_2 & -(1 + \rho^2)/a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/a_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{bmatrix} \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix}$$

where $x_i = h_i - h_{\text{eq},i}$ ($i \in \{1, 2\}$) and $u = q - q_{\text{eq}}$ are deviation variables.

Solution to the state equation: (educated) guess

Consider the function

$$x(t) = \int_{-\infty}^t e^{A(t-s)} Bu(s) ds$$

Using the Leibniz integral rule

$$\begin{aligned} \frac{d}{dt} \int_{a(t)}^{b(t)} f(s, t) ds \\ = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(s, t) ds + \frac{db(t)}{dt} f(b(t), t) - \frac{da(t)}{dt} f(a(t), t) \end{aligned}$$

and the relation $\frac{\partial}{\partial t} e^{A(t-s)} = A e^{A(t-s)}$, we have that

$$\dot{x}(t) = \int_{-\infty}^t A e^{A(t-s)} Bu(s) ds + e^{A(t-t)} Bu(t) = Ax(t) + Bu(t)$$

This is exactly the state equation.

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This is exactly the state equation.

Solution to the state equation

Thus, the solution of

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is

$$y(t) = \int_{-\infty}^t Ce^{A(t-s)} Bu(s) ds + Du(t).$$

As a matter of fact, this $y = g * u$ under

$$g(t) = D\delta(t) + Ce^{At}B1(t),$$

which implies that

- the system G is LTI
- g above is its impulse response in terms of that state-space realization
- the system G is causal (since g has support in $\mathbb{R}_+$)

Solution to the state equation and impulse response

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As a matter of fact, this $y = g * u$ under

$$g(t) = D\delta(t) + Ce^{At}B\mathbb{1}(t),$$

which implies that

- the system G is LTI
- g above is its **impulse response** in terms of that state-space realization
- the system G is causal

g has support in $\mathbb{R}_+$

Similar realizations

Let $T \in \mathbb{R}^{n \times n}$ be nonsingular. Define $\tilde{x} := Tx$, where x is a state of an LTI system G . Its derivative

$$\dot{\tilde{x}}(t) = T\dot{x}(t) = T(Ax(t) + Bu(t)) = TAT^{-1}\tilde{x}(t) + TBu(t)$$

Because $y(t) = Cx(t) + Du(t) = CT^{-1}\tilde{x}(t) + Du(t)$, we have

$$G : \begin{cases} \dot{\tilde{x}}(t) = TAT^{-1}\tilde{x}(t) + TBu(t) & =: \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) = CT^{-1}\tilde{x}(t) + Du(t) & =: \tilde{C}\tilde{x}(t) + \tilde{D}u(t) \end{cases}$$

and conclude that $\tilde{x}$ is also a state vector of G , with the realization

$$(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) := (TAT^{-1}, TB, CT^{-1}, D).$$

The realizations (A, B, C, D) and $(TAT^{-1}, TB, CT^{-1}, D)$ are called **similar**.

Similar realizations (contd)

Not surprisingly,

$$\begin{aligned}\tilde{g}(t) &= \tilde{D}\delta(t) + \tilde{C}e^{\tilde{A}t}\tilde{B}\mathbb{1}(t) \\ &= D\delta(t) + CT^{-1}e^{TAT^{-1}t}TB\mathbb{1}(t) \\ &= D\delta(t) + CT^{-1}Te^{At}T^{-1}TB\mathbb{1}(t) \\ &= D\delta(t) + Ce^{At}B\mathbb{1}(t) \\ &= g(t)\end{aligned}$$

meaning that

- similarity transformation does not affect I/O relations, it only change the internal variable.

Remark. Similarity transformations is a powerful tool to make realizations more transparent, i.e. convenient for particular purposes. In other words, “**All realizations are equal, but some realizations are more equal than others**” (one of the commandments of control engineering). We shall see repeatedly that during the course.

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Transfer functions and state-space realization (mostly SISO)

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From state space to transfer function

By the differentiation property of the Laplace transform,

$$\left. \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array} \right\} \xrightarrow{\text{Laplace}} \begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

The state equation yields

$$(sI - A)X(s) = BU(s) \implies X(s) = (sI - A)^{-1}BU(s)$$

well defined in $\{s \in \mathbb{C} \mid \operatorname{Re} s > \max_{\lambda \in \operatorname{spec}(A)} \operatorname{Re} \lambda\}$, and the output equation yields

$$Y(s) = (D + C(sI - A)^{-1}B)U(s).$$

Hence, the transfer function of this system is

$$G(s) = D + C(sI - A)^{-1}B = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

It is indeed the Laplace transform of g for which $g(t) = D\delta(t) + Ce^{At}B1(t)$.

From state space to transfer function

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It is indeed the Laplace transform of g for which $g(t) = D\delta(t) + Ce^{At}B\mathbb{1}(t)$.

From state space to transfer function (contd)

The transfer function

$$\begin{aligned} G(s) &= D + C(sI - A)^{-1}B \\ &= D + C \left(\frac{1}{\det(sI - A)} \operatorname{adj}(sI - A) \right) B \\ &= \frac{1}{\det(sI - A)} (D \det(sI - A) + C \operatorname{adj}(sI - A)B), \end{aligned}$$

where $\operatorname{adj}(sI - A)$ is the adjugate matrix having *polynomial* entries. Hence, both the numerator and the denominator of $G(s)$ are polynomials. Thus,

- $G(s)$ is **rational** (i.e. a ratio between two finite-order polynomials)
- **poles** of $G(s)$ are among¹ the eigenvalues of A , i.e. in **$\operatorname{spec}(A)$**

¹It's a bit more complicated, we shall study the precise relation later on in the course.

Transfer function via realization: properness in SISO case

Let $p = m = 1$. Because

$$\lim_{|s| \rightarrow \infty} C(sI - A)^{-1}B = \lim_{|s| \rightarrow \infty} \frac{1}{s} C(I - \frac{1}{s}A)^{-1}B = \lim_{\sigma \rightarrow 0} \sigma C(I - \sigma A)^{-1}B = 0$$

we have that

$$\lim_{|s| \rightarrow \infty} (D + C(sI - A)^{-1}B) = D.$$

Compare

$$\lim_{|s| \rightarrow \infty} \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = b_n.$$

Thus, the transfer function

$D + C(sI - A)^{-1}B$ is strictly proper iff $D = 0$ (or bi-proper iff $D \neq 0$).

The parameter D is dubbed the **feed-through** term of the realization.

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Pole excess and high-frequency gain: SISO case

Let

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

be strictly proper (i.e. $n > m$). Its pole excess, $n - m$, can be interpreted as

- the minimum $k \in \mathbb{N}$ such that $s^k G(s)$ is bi-proper.

Now, because

$$s(sl - A)^{-1} = I + A(sl - A)^{-1} = I + (sl - A)^{-1}A,$$

we have that

$$\begin{aligned} sC(sl - A)^{-1}B &= CB + CA(sl - A)^{-1}B = CB + C(sl - A)^{-1}AB \\ s^2C(sl - A)^{-1}B &= sCB + CAB + CA^2(sl - A)^{-1}B \\ &\vdots \\ s^kC(sl - A)^{-1}B &= \sum_{i=1}^{k-1} s^i CA^{k-1-i}B + CA^{k-1}B + CA^k(sl - A)^{-1}B \end{aligned}$$

Pole excess and high-frequency gain: SISO case (contd)

Thus, given $G(s) = D + C(sI - A)^{-1}B$ for² $D = 0$, its

- pole excess is the minimum $k \in \mathbb{N}$ such that $CA^{k-1}B \neq 0$.

The quantities $CA^{k-1}B$, $k \in \mathbb{N}$, are known as **Markov parameters** of $G(s)$.

The high-frequency gain of $G(s)$,

$$b_m = \lim_{|s| \rightarrow \infty} s^{n-m} G(s).$$

In state space,

$$s^{n-m} C(sI - A)^{-1} B = CA^{n-m-1} B + CA^{n-m}(sI - A)^{-1} B.$$

Hence, the high-frequency gain of $G(s) = C(sI - A)^{-1} B$

$$b_m = CA^{n-m-1} B,$$

where n is the dimension of A and $n - m$ is the pole excess of $G(s)$.

²If $D \neq 0$, then the pole excess is always zero.

Pole excess and high-frequency gain: SISO case (contd)

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- pole excess is the minimum $k \in \mathbb{N}$ such that $CA^{k-1}B \neq 0$.

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where n is the dimension of A and $n - m$ is the pole excess of $G(s)$.

²If $D \neq 0$, then the pole excess is always zero.

Zeros of transfer functions via state space: SISO case

If $G(s) = D + C(sI - A)^{-1}B$, then³

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} A - sI & 0 \\ C & G(s) \end{bmatrix} \overbrace{\begin{bmatrix} I & -(sI - A)^{-1}B \\ 0 & I \end{bmatrix}}^{\text{full-rank}}$$

for all $s \notin \text{spec}(A)$. Hence, if $s \notin \text{spec}(A)$, then

$$\text{rank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \text{rank} \begin{bmatrix} A - sI & 0 \\ C & G(s) \end{bmatrix} = n + \text{rank } G(s).$$

Hence, in the SISO case

— **zeros** of $G(s)$ are points $s \in \mathbb{C}$ where $\text{rank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n$
(i.e. where the RSM loses its rank).

Remark: RSM is defined the same way for MIMO systems, with the same definition of zeros (invariant zeros of realizations). Yet MIMO system might have poles and zeros at the same location w/o canceling them, so arguments are more delicate.

³This matrix is called the Rosenbrock system matrix (RSM) of G .

From transfer functions to state space via PFE: SISO case

If $G(s)$ is proper and its poles are simple, its partial fraction expansion

$$G(s) = d + \sum_{i=1}^n \frac{k_i}{s - a_i} = d + [c_1 \cdots c_n] \begin{bmatrix} \frac{1}{s - a_1} & 0 \\ \vdots & \vdots \\ 0 & \frac{1}{s - a_n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$= d + [c_1 \cdots c_n] \left(sI_n - \begin{bmatrix} a_1 & 0 \\ \vdots & \vdots \\ 0 & a_n \end{bmatrix} \right)^{-1} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

for any b_i and c_i such that $k_i = c_i b_i$. Hence, a possible realization may be

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{cc|c} a_1 & 0 & b_1 \\ \vdots & \vdots & \vdots \\ 0 & a_n & b_n \\ \hline c_1 & \cdots & c_n & d \end{array} \right]$$

and diagonalizing "A" $\iff$ the partial fraction expansion of $G(s)$.

From transfer functions to state space via PFE: SISO case

If $G(s)$ is proper and its poles are simple, its partial fraction expansion

$$\begin{aligned}
 G(s) &= d + \sum_{i=1}^n \frac{k_i}{s - a_i} = d + [c_1 \ \cdots \ c_n] \begin{bmatrix} \frac{1}{s - a_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{s - a_n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \\
 &= d + [c_1 \ \cdots \ c_n] \left(sI_n - \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} \right)^{-1} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}
 \end{aligned}$$

for any b_i and c_i such that $k_i = c_i b_i$. Hence, a possible realization may be

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{ccc|c} a_1 & & 0 & b_1 \\ & \ddots & & \\ 0 & & a_n & b_n \\ \hline c_1 & \cdots & c_n & d \end{array} \right]$$

and diagonalizing “A” $\iff$ the partial fraction expansion of $G(s)$.

Canonical realizations: strictly proper SISO case

If $G(s) = \frac{b_{n-1}s^{n-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$, then

— companion form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{c|c} A_{\text{cf}} & B_{\text{cf}} \\ \hline C_{\text{cf}} & D_{\text{cf}} \end{array} \right] := \left[\begin{array}{cccc|c} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -a_0 & -a_1 & \dots & -a_{n-1} & 1 \\ \hline b_0 & b_1 & \dots & b_{n-1} & 0 \end{array} \right]$$

— observer form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{c|c} A_{\text{of}} & B_{\text{of}} \\ \hline C_{\text{of}} & D_{\text{of}} \end{array} \right] := \left[\begin{array}{cccc|c} -a_{n-1} & 1 & \dots & 0 & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & 0 & \dots & 1 & b_1 \\ -a_0 & 0 & \dots & 0 & b_0 \\ \hline 1 & 0 & \dots & 0 & 0 \end{array} \right]$$

Canonical realizations: bi-proper SISO case

Let

$$G(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

for $b_n \neq 0$. The trick is to rewrite it as

$$\begin{aligned} & \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_0} \\ &= \frac{b_n (s^n + a_{n-1} s^{n-1} + \dots + a_0) + (b_{n-1} - b_n a_{n-1}) s^{n-1} + \dots}{s^n + a_{n-1} s^{n-1} + \dots + a_0} \\ &= b_n + \underbrace{\frac{(b_{n-1} - b_n a_{n-1}) s^{n-1} + \dots + (b_1 - b_n a_1) s + (b_0 - b_n a_0)}{s^n + a_{n-1} s^{n-1} + \dots + a_0}}_{\tilde{G}(s)} \end{aligned}$$

Hence, canonical realizations of $G(s)$ are those of the (strictly proper) $\tilde{G}(s)$ complemented by $D = b_n$.

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System interconnections in terms of state-space realizations

State as history accumulator

Given t_0 , let us take another look at the state solution at $t_0 + t > t_0$

$$\begin{aligned} x(t_0 + t) &= \int_{-\infty}^{t_0+t} e^{A(t_0+t-s)} Bu(s) ds \\ &= \int_{-\infty}^{t_0} e^{A(t_0+t-s)} Bu(s) ds + \int_{t_0}^{t_0+t} e^{A(t_0+t-s)} Bu(s) ds \end{aligned}$$

and because $e^{A(t_0+t-s)} = e^{At} e^{A(t_0-s)}$ and e^{At} does not depend on s ,

$$= e^{At} \int_{-\infty}^{t_0} e^{A(t_0-s)} Bu(s) ds + \int_{t_0}^{t_0+t} e^{A(t_0+t-s)} Bu(s) ds$$

$x(t_0)$
 $u(s) \text{ for } s > t_0$

Therefore, the knowledge of $x(t_0)$ and $u(t)$ for $t > t_0$ is enough to compute $x(t)$ for all $t > t_0$. In other words,

$x(t)$ at a given t accumulates the effect of the input history up to t .

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Starting point

The reasoning above implies that

- systems can be analyzed from any time point $t = t_0$ if $x(t_0)$ is known.

In the time-invariant case this starting point can be always chosen as $t = 0$. Consequently, $\mathbb{R}_+$ is taken as the domain of all involved signals and systems are considered in the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with an **initial condition** $x_0 \in \mathbb{R}^n$. In other words,

- systems may be treated as mappings $(x_0, u) \mapsto y$ operating on $\mathbb{R}_+$ in the state space.

Remark: It shall be emphasized that “operating on $\mathbb{R}_+$ ” is not the same as “having support in $\mathbb{R}_+$ ” assumed when systems are studied from pure I/O perspectives. Unless $x_0 = 0$, the presence of initial conditions implies that involved signals acted in $\mathbb{R}_-$ as well. The notion of the state just enables us to ignore their particular forms. All we need to know is the state at $t = 0$.

System dimension

Since *dynamical* systems are those with *memory*, the property of a state to hold the whole history suggests that

- **state dimension** may be regarded as a **measure of the complexity** of corresponding dynamics. Systems admitting a finite vector as their state are then called **finite dimensional**.

We know that

- an LTI system with a state realization has a proper & rational transfer function
- an LTI system with a proper & rational transfer function always admits a state-space realization (state-space realization)

Hence, an LTI system T (or a transfer function $T(s)$) is

- finite dimensional iff its transfer function is rational.

Systems with irrational transfer functions, like the delay element with e^{-sT} , are infinite dimensional.

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- an LTI system with a state realization has a proper & rational transfer function
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Hence, an LTI system (with a proper transfer function) is

- **finite dimensional** iff its transfer function is **rational**.

Systems with **irrational** transfer functions, like the delay element with $e^{-\tau s}$, are **infinite dimensional**.

Solution with initial conditions

If

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

then

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds$$

(we already saw that) and

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)$$

As a matter of fact,

- effect of initial conditions on the state is $e^{At}x_0$
- effect of $u = \delta$ on the state is $e^{At}B$ if $x_0 = 0$ and $t > 0$

Thus, response to nonzero initial condition and the Dirac delta at the input are closely related, although the former is richer (x_0 is arbitrary, B is fixed).

Initial conditions and similarity transformations

Nothing really special, still the transformation $x \rightarrow \tilde{x} = Tx$. Just remember that in the new coordinates

$$\begin{cases} \dot{\tilde{x}}(t) = (TAT^{-1})\tilde{x}(t) + (TB)u(t), & \tilde{x}(0) = T\mathbf{x}_0 \\ y(t) = (CT^{-1})\tilde{x}(t) + Du(t) \end{cases}$$

i.e. initial conditions are also affected by T .

Systems with initial conditions in the Laplace domain

In

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

we

- no longer assume that the state x has support in $\mathbb{R}_+$
- do not know how x behaved on $\mathbb{R}_-$

As such,

- it is natural to switch from the bilateral to unilateral Laplace analysis.

The main consequence of that is the differentiation rule, which is now

$$y(t) = \dot{x}(t) \implies Y(s) = sX(s) - x_0$$

Thus, the state equation reads

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

Outline

State-space models

Transfer functions and state-space realization (mostly SISO)

Initial conditions

System interconnections in terms of state-space realizations

Problem

We know how to carry out parallel, cascade, and feedback interconnections in terms of transfer functions $G_1(s)$ and $G_2(s)$. Indeed:

- parallel: $G_1(s) + G_2(s)$
- series: $G_2(s)G_1(s)$
- feedback: $(1 \mp G_1(s)G_2(s))^{-1}G_1(s)$

The question is: how this can be done in state space?

We assume that

$$G_1: \begin{cases} \dot{x}_1(t) = A_1x_1(t) + B_1u_1(t) \\ y_1(t) = C_1x_1(t) + D_1u_1(t) \end{cases} \quad \text{and} \quad G_2: \begin{cases} \dot{x}_2(t) = A_2x_2(t) + B_2u_2(t) \\ y_2(t) = C_2x_2(t) + D_2u_2(t) \end{cases}$$

A universal way of interconnecting systems via their realizations is to

- determine what inputs / outputs to equate
- unite state vectors

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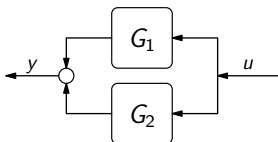
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A universal way of interconnecting systems via their realizations is to

- determine what inputs / outputs to equate
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Parallel interconnection



This corresponds to

- $u_1 = u_2 = u$
- $y = y_1 + y_2$

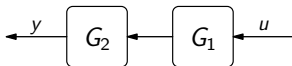
Then

$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + (D_1 + D_2)u(t) \end{cases}$$

We already know (spectrum of block-diagonal matrices) that

- poles of $G(s)$ are in $\text{spec}(A_1) \cup \text{spec}(A_2)$

Series / cascade interconnection



This corresponds to

- $u_1 = u$
- $u_2 = y_1$
- $y = y_2$

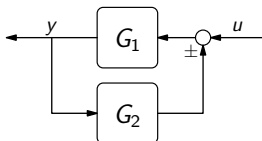
Then

$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + D_2 D_1 u(t) \end{cases}$$

We already know (spectrum of block-triangular matrices) that

- poles of $G(s)$ are in $\text{spec}(A_1) \cup \text{spec}(A_2)$

Feedback interconnection



This corresponds to

- $u_1 = u \pm y_2$
- $u_2 = y_1$
- $y = y_1$

Then, assuming that $D_1 = 0$ for simplicity (enough for our purposes),

$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 \pm B_1 D_2 C_1 & \pm B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

Inversion

Let

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with $p = m$ and $\det D \neq 0$. Its inverse is the system mapping $y \mapsto u$. Then

$$u(t) = -D^{-1}Cx(t) + D^{-1}y(t)$$

and

$$\dot{x}(t) = Ax(t) + B(-D^{-1}Cx(t) + D^{-1}y(t))$$

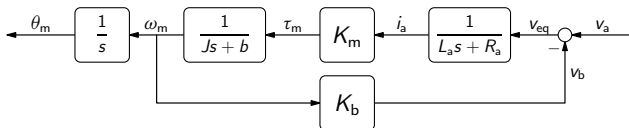
Therefore,

$$G^{-1} : \begin{cases} \dot{x}(t) = (A - BD^{-1}C)x(t) + BD^{-1}y(t) \\ u(t) = -D^{-1}Cx(t) + D^{-1}y(t) \end{cases}$$

and **zeros of $G(s)$** (= poles of $G^{-1}(s)$) are⁴ in **$\text{spec}(A - BD^{-1}C)$** .

⁴Try to draw the same conclusion via the Rosenbrock system matrix.

Exercise: DC motor



$G : v_a \mapsto \theta_m$, where v_a is input voltage and θ_m is shaft angle. Subsystems:

electric circuit: $\dot{i}_a = -(R_a/L_a)i_a + (1/L_a)v_{eq}$ 1-order, state i

motor: $\tau_m = K_m i_a$ static

load: $\dot{\omega}_m = -(b/J)\omega_m + (1/J)\tau_m$ 1-order, state ω_m

back emf: $v_b = K_b \omega_m$ static

voltage balance: $v_{eq} = v_a - v_b$ static

velocity $\rightarrow$ angle: $\dot{\theta}_m = \omega_m$ 1-order, state θ_m

Hint: start with defining the physical state $x = \begin{bmatrix} \theta_m \\ \omega_m \\ i_a \end{bmatrix} \dots$