

Nomenclature

$\mathbb{N}$	set of positive integers (natural numbers)
$\mathbb{Z}$	set of integers
$\mathbb{Z}_+$	set of nonnegative integers
$\mathbb{Z}_-$	set of non-positive integers, $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$
$\mathbb{Z}_{i_1..i_2}$	integer interval from i_1 to and including i_2 , i.e. $\mathbb{Z}_{i_1..i_2} := \{i \in \mathbb{Z} \mid i_1 \leq i \leq i_2\}$
$\mathbb{R}$	set of real numbers, $\mathbb{R} = (-\infty, \infty)$
$\mathbb{R}_+$	set of nonnegative real numbers, $\mathbb{R}_+ = [0, \infty)$
$\mathbb{R}_-$	set of non-positive real numbers, $\mathbb{R}_- = (\infty, 0]$
$j\mathbb{R}$	set of pure imaginary numbers
$\mathbb{C}$	set of complex numbers
$\operatorname{Re} z$	the real part of $z \in \mathbb{C}$
$\operatorname{Im} z$	the imaginary part of $z \in \mathbb{C}$
$\mathbb{C}_\alpha$	open right half-plane, to the right of $\alpha \in \mathbb{R}$, i.e. $\mathbb{C}_\alpha := \{z \in \mathbb{C} \mid \operatorname{Re} z > \alpha\}$
$\bar{\mathbb{C}}_\alpha$	closed right half-plane, to the right of $\alpha \in \mathbb{R}$, i.e. $\bar{\mathbb{C}}_\alpha := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \alpha\}$
$\mathbb{T}$	unit circle, $\mathbb{T} := \{z \in \mathbb{C} \mid z = 1\}$
$\mathbb{D}$	interior of $\mathbb{T}$ (open unit disk), $\mathbb{D} := \{z \in \mathbb{C} \mid z < 1\}$
$\bar{\mathbb{D}}$	closed unit disk, $\bar{\mathbb{D}} := \{z \in \mathbb{C} \mid z \leq 1\} = \mathbb{D} \cup \mathbb{T}$
e_i	the i th standard basis in $\mathbb{R}^n$, i.e. $e_1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 := \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$
I_n	$n \times n$ identity matrix (just I if the dimension is irrelevant)
$0_{n \times m}$	$n \times m$ zero matrix (just 0 if the dimensions are irrelevant)
a_{ij}	the ij th element of a matrix $A \in \mathbb{R}^{p \times m}$
A'	transpose of a matrix $A \in \mathbb{R}^{n \times m}$ / complex-conjugate transpose of a matrix $A \in \mathbb{C}^{n \times m}$
$\lambda_i(A)$	i th eigenvalue of a matrix $A \in \mathbb{R}^{n \times n}$
$\operatorname{spec}(A)$	spectrum of a matrix $A \in \mathbb{R}^{n \times n}$, i.e. the set of all its eigenvalues
$\rho(A)$	spectral radius of a matrix $A \in \mathbb{R}^{n \times n}$, $\rho(A) := \max\{ \lambda_1(A) , \dots, \lambda_n(A) \}$
$\operatorname{tr}(A)$	trace of a matrix $A \in \mathbb{R}^{n \times n}$, $\operatorname{tr}(A) := \sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i(A)$
$A > 0$	positive definite matrix (i.e. such that $x'Ax > 0$ for all $x \neq 0$)
$A < 0$	negative definite matrix (i.e. such that $x'Ax < 0$ for all $x \neq 0$)

$A \geq 0$	positive semi-definite matrix (i.e. such that $x'Ax \geq 0$ for all x)
$A \leq 0$	negative semi-definite matrix (i.e. such that $x'Ax \leq 0$ for all x)
$\lambda_{\max}(A)$	maximal eigenvalue of $A = A'$
$\lambda_{\min}(A)$	minimal eigenvalue of $A = A'$
$\ A\ $	spectral norm of a matrix $A \in \mathbb{R}^{n \times m}$, $\ A\ ^2 = \rho(A'A) = \rho(AA')$
$\ A\ _F$	Frobenius norm of a matrix $A \in \mathbb{R}^{n \times m}$, $\ A\ _F^2 = \sum_{i=1}^n \sum_{j=1}^m a_{ij} ^2 = \text{tr}(A'A) = \text{tr}(AA')$
$\mathfrak{F}\{x\}$	Fourier transform of $x : \mathbb{R} \rightarrow \mathbb{R}^n$
$\mathfrak{L}\{x\}$	Laplace transform of $x : \mathbb{R} \rightarrow \mathbb{R}^n$
$\mathfrak{Z}\{x\}$	the Z transform of $x : \mathbb{Z} \rightarrow \mathbb{R}^n$
$\mathbb{1}$	unit step, $\mathbb{1}(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$ and $\mathbb{1}[t] = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$
δ	Dirac delta in continuous time or unit pulse, $\delta[t] = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$, in discrete time
$\exp_a$	exponential function, $\exp_a(t) = e^{at}$
g	impulse response of an LTI system G
$G(s)$	transfer function of an LTI system G , $G(s) = (\mathfrak{L}\{g\})(s)$
(A, B, C, D)	state-space realization of a finite-dimensional LTI system G
$\left[\begin{array}{c c} A & B \\ \hline C & D \end{array} \right]$	transfer function in terms of realization (A, B, C, D) , $\left[\begin{array}{c c} A & B \\ \hline C & D \end{array} \right] = D + C(sI - A)^{-1}B$

Classical Control in a Nutshell

THIS NOTE revises some basic notions and ideas of classical systems and control methods for LTI (linear time-invariant) systems. It is not aimed at presenting a comprehensive overview, but rather at introducing key notions used throughout the course. The main emphasis is placed on the plant inversion ideas and frequency-domain analysis and design philosophy.

1 Signals and systems

In numerous situations we may be concerned with quantities, whose values change as some *independent variables*, like time, evolve. Two such examples are presented in Fig. 1. The audio waveform of a short word

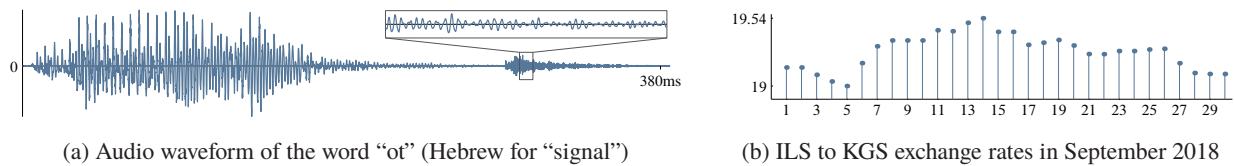


Fig. 1: Examples of signals

is depicted in Fig. 1(a). This very waveform makes the voice recognizable (and reproducible, as a matter of fact). The independent variable here is the analog time, a subset of $\mathbb{R}$. Fig. 1(b) shows the exchange rates of **Israeli new shekel** to **Kyrgyzstani som** during one month. In this case, the independent variable is the discrete time, a subset of $\mathbb{Z}$. These quantities can be described as functions of independent variables, say $f(t)$ if $t \in \mathbb{R}$ or $f[t]$ if $t \in \mathbb{Z}$. When associated with some physical process, such functions are called *signals*. Thus, signals are functions conveying information about the behavior of some phenomenon.

Remark 1. The independent variable need be neither time nor scalar. For example, we may be interested in the steady-state temperature profile of a heated body as a function of the distance from the heating element or, in image processing, in the RGB values of pixels as a function of their two spatial coordinates. Nonetheless, time is by far the most prevalent independent variable in control applications, so we frequently interchange these terms in the sequel. ∇

Signals are not “in a vacuum” (although they do exist in a vacuum, thanks Per-Olof). They are related via laws of nature, economics, society, and so on. In some cases these relations, and their applicability scopes, are well understood. For example, we know that applied forces and the mass position are coupled via **Newton’s law**; that the electromotive force around a closed path depends on the magnetic flux enclosed by it (by **Faraday’s law**); that the angles of incidence and refraction of light passing through a boundary between two different isotropic media are related via **Snell’s law**; et cetera. In other cases, mostly in life sciences, humanities, sociology, economics, and the like, relations between signals may be empirical and vague. For example, it may be accepted that unemployment is linked with losses in a country’s production (by **Okun’s law**); the utility of social networks can scale exponentially with the size of the network (by

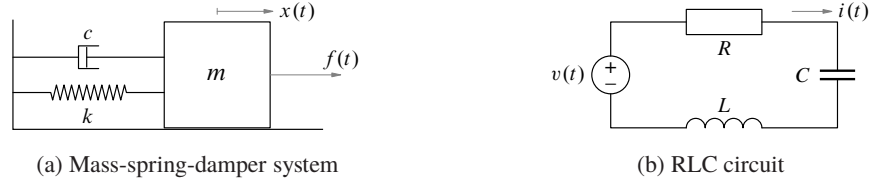
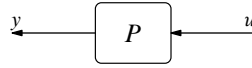


Fig. 2: Two simple systems

Reed's law); the metabolic rate of animals is related to their mass (by **Kleiber's law**); passion is inversely proportional to the amount of real information available (by **Benford's law of controversy**); the bitterness of discussions in academia is in inverse proportion to the importance of issues at stake (by **Sayre's law**); et cetera. But even then the very fact that signals of interest are interrelated is beyond doubt.

By *systems* we may then understand constraints imposed on interdependent signals. Consider e.g. a mechanical system comprising a mass point with mass m connected to a fixed base via a spring with spring constant k and a damper (**dashpot**) with damping coefficient c , see Fig. 2(a). It can be seen as a constraint on mutual relation between the force f and the mass position x . If the system is in its equilibrium and we change the force, then the position can be uniquely determined via **Newton's law**. It is worth emphasizing that it also works the other way around. Namely, if somehow we can change the mass position, it applies a unique force (think of a **door closer**, which works this way). Thus, two involved signals, f and x , are constrained by properties of this system. Another example of a system is the RLC electrical circuit in Fig. 2(b), with a resistor having resistance R , a solenoid having inductance L , and a capacitor having capacitance C . Such a system constraints mutual relations between the voltage v and the current i , which can be derived by Kirchhoff's voltage law.

Although all involved signals are “born equal,” two groups of signals associated with a given system are often distinguished in control applications. One group is assumed to be an action (input) and another one is then a reaction (output), see the block-diagram in Fig. 3, which represents a system P with an input

Fig. 3: Block-diagram of an I/O system $P : u \mapsto y$

u and an output y . This relation is conventionally written as $y = Pu$, where P should be understood as an operator acting on u . This philosophy is referred to as the *input / output approach*¹ and the corresponding systems as I/O systems. It treats systems as signal processors, in a sense that systems are understood as mappings “ $\mapsto$ ” from input signals to output signals, like $P : u \mapsto y$ in Fig. 3, where this relation is reflected in the arrows added to the signal lines. This way of thinking is more restrictive, because the separation between inputs and outputs might impose artificial causality relations. Nevertheless, it is widely acceptable and is sufficiently rich in many situations.

A system $G : u \mapsto y$ is said to be SISO (single-input, single-output) if both its input and output signals are scalar-valued, normally, taking values in $\mathbb{R}$. Otherwise, it is MIMO (multiple-input, multiple-output). It is called *linear* if it satisfies the property of superposition, i.e. if

$$G(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 (G u_1) + \alpha_2 (G u_2) \quad \text{for all admissible signals } u_1, u_2 \text{ and all scalars } \alpha_1, \alpha_2.$$

A system is referred to as *time-invariant* (or shift-invariant) if any constant time shift of its input results in the same time shift of its output. Linear time-invariant systems are often abbreviated to LTI.

¹A more general viewpoint, with which we started, is known as the behavioral approach of Jan C. Willems [3].

2 Models

Let us return to the systems in Fig. 2 and be more concrete about their behavior. When considered as a mapping $f \mapsto x$, the mechanical system in Fig. 2(a) can be described by the differential equation

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t). \quad (1)$$

This relation follows from **Newton's** and **Hooke's** laws and by assuming that the dashpot produces a resistive force proportional to the velocity of its end. The electrical circuit in Fig. 2(b) can be viewed as a mapping $v \mapsto q$, where q is the charge across the capacitor, with $i = \dot{q}$. This mapping satisfies

$$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = v(t) \quad (2)$$

and follows by **Kirchhoff's**, **Faraday's**, **Ohm's**, and **Gauss's** laws. It is readily seen that (2) is essentially the same as (1), modulo the replacements $m \rightarrow L$, $c \rightarrow R$, and $k \rightarrow 1/C$. It may then be convenient to present both (1) and (2) as an abstract mapping $u \mapsto y$ described by the non-physical second-order differential equation

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2y(t) = k_{st}\omega_n^2u(t), \quad (3)$$

where the parameters ω_n , ζ , and k_{st} represent the natural frequency, the damping factor, and the static gain, respectively. Their values have one to one correspondence with physical parameters of the respective systems and determine qualitative and quantitative system properties, like oscillations / decay profiles, speed of transients, steady-state response to various harmonic signals, et cetera. Thus, properties of mechanical and electrical systems in Fig. 2 can be studied in a uniform fashion via their abstract models.

This is a momentous observation. Once we abstract from the physical nature of signals and systems, be they mechanical, electrical, biological, social, etc, we can analyze them from unified points of view by universal theories. This is a key idea behind the notion of *mathematical model*, which is a description of systems in a mathematical language. In control applications we most frequently encounter mathematical language based on differential or difference equations. Yet there are many other ways to describe systems, like logical models used in computer sciences, heuristic models used in psychology, and so on. Model-based way of thinking is ubiquitous in engineering applications, even in methods claiming to be model-free (perhaps the only difference is that some model-based approaches are explicit and others are implicit).

An important aspect of mathematical models, which should always be remembered, is that they are *never perfect*. Any mathematical model is merely a (more or less accurate) approximation of the actual physical process, capturing its properties only to a certain degree. For example, in modeling the mechanical system in Fig. 2(a) we assumed that the mass movements are perfectly one-dimensional, the dashpot and the the spring are linear, massless, and do not change their properties when heated up, that there is no friction at the mass base and with the air, and so on and so forth. Likewise, the model (2) of the RLC circuit in Fig. 2(b) is true only under unrealistic assumptions that the resistor, solenoid, and capacitor are linear and their properties do not depend on working conditions, that the whole circuit is perfectly isolated from its environment, both electrically and thermally, and suchlike, not to mention that the picture becomes substantially more complicated on the atomic level. Still, models (1) and (2) are useful if deviations from linear regimes and changes in components properties are relatively small. These considerations are true in general as well. It may be safe to claim that there are no linear systems in nature and that no physical system can be exhaustively described by a finite number of ordinary differential / difference equations or are time invariant. Yet we do use rather simple LTI models in many situations. In general, the required accuracy and model granularity level depend on applications and a model, sufficiently accurate in one application, might not be adequate in another. **Occam's razor** may be a good principle for deciding on the type of model in every situation. Finally, note that with a certain abuse of terminology, we frequently say “system” meaning its model, this is a common practice.

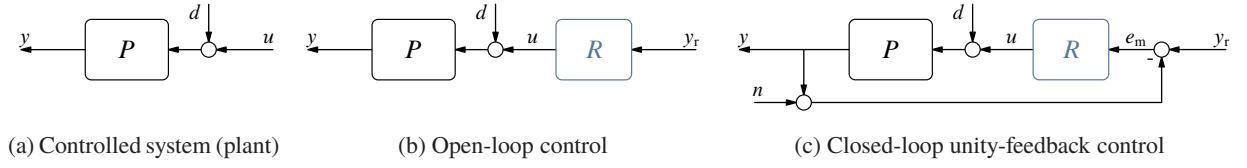


Fig. 4: Basic control setups

3 A prototype control problem

Control is about affecting systems so that they behave in a desired manner. Systems can be affected in various ways, up to rebuilding their physical structure. Control theory normally addresses situations, when systems are fixed and affected via some of their input signals, naturally called *control inputs*, that can be freely shaped by a controller.

A heavily simplified—yet sufficiently representative as a starting point—control setup can be described in terms of the system presented in Fig. 4(a). Here the studied LTI system P , dubbed the *plant*, is affected via its control input u to generate a desired output y , aka the *controlled output*, at the end. The signal d , which also has an effect on y , is called the *disturbance* (sometimes, the load disturbance). Its role is to account for impacts of exogenous phenomena, like impacts of the environment, which are uncertain or too complicated to model. The disturbance signal is often supposed to be unmeasurable (although in some situations it can be) and uncorrelated with control goals. Throughout this and the next section we assume that all involved signals are scalar valued, so that P is SISO.

Requirements to the controlled output may be conveniently expressed in terms of a *reference signal* y_r . The rationale behind its choice varies from task to task. In some problems, called *tracking problems*, this is a physical signal, which should be tracked by the plant output. Its generation might then be an independent task, like recovering y_r from on-line measurements of a target, e.g. in **video tracking** problems. In other situations, like in *set-point regulation*, the reference signal is calculated analytically from knowing the initial and required steady-state positions of y . Although this is not frequently highlighted, the reference signal in such situations should reflect not only requirements to the behavior of y , but also limitations of the plant, see the discussion in §A.2. In any case, given a reference signal y_r , a *basic control problem* can be formulated as selecting the control signal u rendering (at least, approaching)

$$y = y_r \quad (4)$$

despite the presence of unmeasured exogenous signals and modeling uncertainties.

There are essentially two control configurations for meeting this requirement. An *open-loop control* scheme is shown in Fig. 4(b). In this case the controller (regulator) R generates the control signal u as a function of the reference signal y_r only (it may also depend on d if the latter is measurable). In *closed-loop control*, whose simplest unity-feedback configuration is depicted in Fig. 4(c), the controller R generates the control input using both the reference signal and measurements of the controlled signal y (or signals correlated with y , if it is not directly measurable). Measurements of y require separate sensors, which are never perfect. These imperfections can be accounted for by introducing an additive measurement noise signal, like n in Fig. 4(c). In both these configurations, the designed element is the controller R .

4 Ultimate control methodology: plant inversion and its limitations

There are many ways to think of designing R . Arguably, they all have the idea of plant inversion under the hood. It is conceptually simple and intuitive, although (practically) never implementable literally, which is why control engineering is such a nontrivial discipline.

The idea can be grasped by considering the idealized version of the system in Fig. 4(a) under $d = 0$ and a perfectly known P . Rewriting (4) as $y_r = Pu$ yields the following control signal meeting this requirement:

$$u = P^{-1}y_r, \quad (5)$$

where P^{-1} should be understood as the system producing the unity system when placed in series with P (its transfer function is the reciprocal of the transfer function of the plant, $1/P(s)$). The control law (5) is called the plant inversion.

4.1 Open-loop plant inversion

The control law (5) appears to be a perfect fit for the open-loop system in Fig. 4(b). It corresponds to the choice

$$R(s) = \frac{1}{P_0(s)}, \quad (6)$$

where P_0 is our model of the plant P . However, the use of this controller has its pitfalls, some of which severely limit its applicability scope.

The first of these pitfalls is associated with the notion of *stability*, which we have not discussed yet. Loosely speaking, an I/O system is said to be stable if its output remains bounded for all bounded inputs. It is then well known that a finite-dimensional LTI system is stable iff its (rational) transfer function is proper, i.e. the degree of its numerator does not exceed that of its denominator, and has no poles in the closed right half-plane. Consider now the relation between all exogenous and internal signals in the system in Fig. 4(b)

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} PR & P \\ R & 0 \end{bmatrix} \begin{bmatrix} y_r \\ d \end{bmatrix}. \quad (7)$$

We obviously need the boundedness of both y and u . We also cannot ignore the effects of either y_r or d (even if d is insignificant, its effect on unstable systems might be destructive). We thus must guarantee that all three nonzero systems in (7) are stable. This is known as the *internal stability* of interconnected systems. Internal stability is equivalent to the stability of both P and R . If this is the case, then the third system, PR , is always stable. Consequently, the open-loop setup in Fig. 4(b) cannot be used if the plant is unstable, no matter what control algorithm is considered. Another outcome of the internal stability requirement is that controller (6) cannot be used if $P_0(s)$ is nonminimum-phase or strictly proper. Indeed, the former property would add unstable poles to the controller and the latter would render $R(s)$ non-proper.

Remark 2 (properness). Non-proper controllers may be accepted in some situations, viz. if they always act on analytically known reference signals with sufficiently many bounded derivatives, see the discussion in §A.3. Still, properness might be a safe property to impose as a standard requirement. ∇

There is a workaround for problems with nonminimum-phase or strictly proper plant models. Instead of (6) we may use

$$R(s) = \frac{T_{\text{ref}}(s)}{P_0(s)} \quad (8)$$

for some stable system T_{ref} . Technically, the only constraint on T_{ref} , apart from its own stability, is that the resulting controller is stable. This boils down to the following two conditions:

1. $T_{\text{ref}}(s)$ must have all nonminimum-phase zeros of $P_0(s)$ as its own zeros (multiplicities counted),
2. the pole excess of $T_{\text{ref}}(s)$ must be greater than or equal to the pole excess of $P_0(s)$.

While the second of these conditions is effectively non-restrictive, the first one might be, depending on the location of unstable zeros of $P_0(s)$.

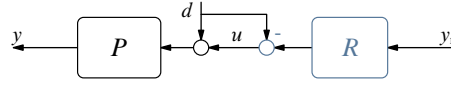


Fig. 5: Open-loop control with disturbance feedforward

Conceptually, the introduction of T_{ref} renders the controlled output (still assuming that $P_0 = P$)

$$y = T_{\text{ref}} y_r. \quad (9)$$

In other words, we may only expect to match y_r processed by T_{ref} . This motivates the term *reference model* for this system, it may be thought of as representing a “best” pragmatic response of the controlled system to y_r . Unless $T_{\text{ref}} = 1$, (9) is no longer the same as our original goal (4). But the latter can be met only if the plant has a bi-proper transfer function (i.e. has infinite bandwidth), which is not realistic. It thus does make sense to switch to a more pragmatic (9). This is especially so if we know that y_r is not arbitrary, but rather belongs to a more narrow class of signals. For example, we often need to track band-limited signals only. In such situations it may be reasonable to require the frequency response $T_{\text{ref}}(j\omega) \approx 1$ only within the frequency range of the spectrum of y_r . This normally produces a low-pass T_{ref} and does not impose any restrictive constraints on the pole excess of $T_{\text{ref}}(s)$.

Now it is time to address the effect of other idealizations. Specifically, consider the behavior of the controlled output under a nonzero disturbance d and not perfectly known plant, with $P_0 \neq P$. Define $e := T_{\text{ref}} y_r - y$ as the error signal representing the deviation from required behavior. It is readily seen that (8) renders

$$e = (1 - PP_0^{-1})T_{\text{ref}} y_r - Pd =: e_{y_r} - e_d. \quad (10)$$

Because y_r and d are independent, we cannot cause e_{y_r} and e_d to cancel each other or alleviate their effect. Thus, these two terms should be considered individually.

An important observation is that we are left with no tools to affect the mismatch terms in (10). This would also be true for any other choice of R in Fig. 4(b). Equation (10) would then remain unchanged modulo the replacement of T_{ref} with the model $P_0 R$ of the attained system $y_r \mapsto y$. The effect of modeling uncertainty, e_{y_r} , is entirely determined by the relative modeling error system, $1 - PP_0^{-1}$. Hence, open-loop control can only be effective in frequency ranges where this error is small. The effect of the exogenous disturbance, e_d , is also independent of the controller choice. In fact, a controller acting in open loop can affect the disturbance response only if it can measure d or at least a signal correlated with it.

Remark 3 (disturbance feedforward). If d is measurable, the control configuration in Fig. 5 can be used instead, with the very same rationale behind the design of R , just now with $y_r = Pd + Pu$ to satisfy (4). For this scheme we obviously have $e_d = 0$. Even if d is not measured perfectly, such measurements can still be useful to reduce the impact of d on the controlled variables, at least to some extent. ∇

Finally, it should be mentioned that the two conditions on p. 5 are not the only limiting factors in the choice of T_{ref} . Another limitation, a soft one, is the control effort. Actuating resources are always limited, so we would be interested in attaining control goals with “affordable” u . It may be convenient to normalize control signals, so that $|u| = 1$ is the borderline between “high” and “low” control efforts. The condition $|R(j\omega)| \gg 1$ may be then viewed as an indication of “affordable” control. In addition, we may be interested in attaining $|R(j\omega)| \ll 1$ at high frequencies, to avoid amplifying parasitic components in y_r . These considerations, viewed in the context of (8), can be translated to the rule of thumb that *the bandwidth of T_{ref} should not considerably exceed that of P* . Control problems tend to become more complicated if this suggestion is not followed.

4.2 Closed-loop plant inversion

Relationships between (4) and the setup in Fig. 4(c) are less evident than those between (4) and the setup in Fig. 4(b). Yet they exist in an unexpected and fascinating way.

To start with, consider relations between all exogenous and internal signals for the system in Fig. 4(c). It is not hard to see that

$$\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} T & T_d & -T \\ T_c & -T & -T_c \\ S & -T_d & T \end{bmatrix} \begin{bmatrix} y_r \\ d \\ n \end{bmatrix}, \quad (11)$$

where $e := y_r - y = e_m + n$ and

$$\begin{bmatrix} T(s) & T_c(s) \\ T_d(s) & S(s) \end{bmatrix} := \frac{1}{1 + P(s)R(s)} \begin{bmatrix} P(s)R(s) & R(s) \\ P(s) & 1 \end{bmatrix} = \frac{1}{1 + P(s)R(s)} \begin{bmatrix} R(s) \\ 1 \end{bmatrix} \begin{bmatrix} P(s) & 1 \end{bmatrix} \quad (12)$$

are four fundamental closed-loop transfer functions, termed the *sensitivity* (S), *complementary sensitivity* (as $T = 1 - S$), *control sensitivity* ($T_c = TP^{-1}$), and *disturbance sensitivity* ($T_d = PS$) transfer functions. They are also known as the *Gang of Four* (the term coined by **Karl Johan Åström** back in 2000).

In line with its definition for open-loop systems, the *internal stability* of the system in Fig. 4(c) is defined as the stability of all possible closed-loop systems, i.e. the four systems whose transfer functions are defined in (12). The internal stability requirement rules out unstable pole / zero cancellations between $P(s)$ and $R(s)$. Indeed, if unstable poles of $P(s)$ are canceled by the controller, then they are still poles of $T_d(s)$. Likewise, if unstable zeros of $P(s)$ are canceled by $R(s)$, these zeros show up as poles of $T_c(s)$. Internal stability also rules out non-proper $R(s)$'s, because then $T_c(s)$ is not proper either. But if there are no unstable pole / zero cancellations between $P(s)$ and $R(s)$ and the latter transfer function is proper, internal stability is equivalent to the stability of either one of the four transfer functions in (12).

Consider now the control signal. Assuming for the moment that measurements of y are perfect, i.e. that $n = 0$, we have:

$$u = T_c y_r - T d = \frac{R}{1 + PR} y_r - \frac{PR}{1 + PR} d = \frac{1}{1/R + P} y_r - \frac{P}{1/R + P} d. \quad (13)$$

It is worth emphasizing that the mechanical inversion of the plant by the controller, as in (6), is futile here. Indeed, even if admissible, the choice $R = P^{-1}$ would result in $u = 0.5P^{-1}y_r - 0.5d$ and then in $y = 0.5y_r + 0.5P^{-1}d$, which makes little sense. A proper “plant inversion” choice can be seen by noticing that the terms $1/R$ on the right-hand side of (13) become less significant as the controller gain grows. In the limit (ignore mathematical meaning of this for now), as $R \rightarrow \infty$, the control signal

$$u \rightarrow u_\infty := P^{-1}y_r - d \quad (14)$$

and the controlled output meets (4) despite the presence of d . This u_∞ is the very control signal generated by the open-loop control system with disturbance feedforward in Fig. 5 under the plant-inversion choice of the controller, as in (5). But unlike its open-loop version, the closed-loop plant inversion requires neither the knowledge of the plant model nor direct measurements of d (although measuring y can be thought of as an indirect measurement of the disturbance). This sounds like a miracle and should naturally arouse suspicions of being too simple to be true.

These suspicions would be well founded. The situation in the closed-loop case is even more complicated than that in the open-loop case. Stability is again a primary concern. It can be shown, e.g. by **root-locus** asymptotes arguments, that closed-loop systems can be stable under controllers with uniformly high gains only for a handful of plants. Namely, these are minimum-phase systems with a pole excess of

at most two. And even for these plants the use of such controllers is not practical. First, no measurements are perfect. If the measurement noise is taken into account, the control signal in (14) turns

$$u_\infty = P^{-1}(y_r - n) - d, \quad (15)$$

resulting in $y = y_r - n$. Tracking measurement noise, especially its high-frequency components, is not what we need. Moreover, P^{-1} is typically a high-pass system, so fast components of n would be amplified by (15), producing large high-frequency oscillations in the control signal, which is not healthy either. Second, uniformly high-gain feedback frequently produces lightly damped modes in closed-loop systems. This, in turn, yields poor transients, with fast but slowly decaying oscillations. Third, feedback systems with high loop gains at high frequencies are normally quite sensitive to modeling uncertainty, loop delays, digital implementation with sampling and roundoff errors, et cetera, i.e. they are not *robust*. This lack of robustness is deemed unacceptable in applications.

Still, the underlying idea behind (14)—that a high loop gain effectively inverts the plant and measures disturbances—is insightful and can be recognized as the guiding principle behind many control design methods. For example, adding an integral action to the controller renders the loop gain infinite at the zero frequency. This is why controllers with integral action can guarantee zero steady-state errors under reference signals converging to a constant value despite the presence of constant disturbances and without the need to know the static gain of the plant. The loop-shaping method, which manipulates the loop gain ideas explicitly, is discussed in more details below.

4.3 Loop shaping

Consider again the relation (11). Our goal is to render e “small” and u “not too large,” in whatever senses. Similarly to treating the open-loop problem in §4.1, we assume that the exogenous signals are not correlated and cannot be used to cancel the effect of each other. The requirements above might then be loosely expressed as requirements to reduce gains of S , T_d , and T , while keeping gains of T_c and T not too large. The problem is that these requirements are intrinsically conflicting. For instance, it is readily seen that $S + T = 1$. Hence, S and T cannot be made small simultaneously.

An elegant way to circumvent this obstacle can be seen via treating the system in Fig. 4(c) in the frequency domain. The underlying assumptions are that (i) spectra of y_r and n are well separated and (ii) only low-frequency components of d are harmful and should be compensated by the controller. Indeed, it is the case in many applications that y_r is relatively slow, sensors are less reliable at high frequencies, and P is low-pass, so high-frequency components of d do not affect y anyway. Adopting these assumptions, we may reformulate the requirements as rendering $|S(j\omega)|$ and $|T_d(j\omega)|$ small at low frequencies, $|T(j\omega)|$ small at high frequencies, and keeping $|T_c(j\omega)|$ and $|T(j\omega)|$ not too large at all frequencies. From the transient performance point of view, it is beneficial to avoid high resonance peaks in $|T(j\omega)|$ (smoother transients) and to have a sufficiently high bandwidth of T (faster transients). At the same time, aiming at too high bandwidth of T , especially that exceeding the bandwidth of P , would result in high peaks in $|T_c(j\omega)|$ (cf. the discussion at the end of §4.1) and poor robustness. This set of requirements is rather simplified, but still provides a flavor of the ideas behind frequency-domain design methods and complexity of goals involved. The complexity is manifested in both the number of transfer functions involved and the fact that they all are nonlinear functions of the design parameter, R .

A workaround is to translate requirements on closed-loop transfer functions to those on the loop transfer function

$$L(s) = P(s)R(s),$$

which is a linear function of $R(s)$. It is readily seen that $S = (1 + L)^{-1}$ and $T = L(1 + L)^{-1}$. Hence, if the loop gain is high, i.e. if $|L(j\omega)| \gg 1$, we have small $|S(j\omega)|$ and, in many cases, $|T_d(j\omega)|$. If the loop gain is

low, i.e. if $|L(j\omega)| \ll 1$, we have a small $|T(j\omega)|$. These relations are intuitive. Indeed, at frequencies where the loop gain is high, we effectively invert the plant and have good tracking and disturbance attenuation. But this comes at a price of an increased sensitivity to measurement noise. At frequencies where the loop gain is low we effectively disconnect the sensor with its noise, thus rendering measurement noise irrelevant. Less intuitive is that the other requirements can also be expressed in terms of $L(j\omega)$. This is true for the peaks of $|T(j\omega)|$ ($L(j\omega)$ should be sufficiently far from the critical point $(-1, 0)$ on the Nyquist complex plane to avoid them), its bandwidth (connected with the crossover frequency ω_c , at which $|L(j\omega_c)| = 1$), and even the closed-loop stability (the **Nyquist** criterion).

A representative loop-shaping cascade design involves the following steps.

1. Pick the required crossover frequency ω_c around which the whole procedure is carried out. Too high ω_c would normally complicate the design and might render the closed-loop bandwidth too high, while an overly low ω_c would produce a too slow closed-loop system. Arguably, this is the most important choice in the whole procedure. The right choice is normally reached via trial-and-error iterations.
2. Add a low-pass filter, whose bandwidth exceeds ω_c , to provide a required high-frequency roll-off.
3. Add a static gain to render the actual crossover frequency equal to the chosen ω_c .
4. If the closed-loop system is unstable in this stage (check via the Nyquist criterion), or if stability margins are insufficient, add phase lead element(s). A properly parametrized lead does not alter ω_c .
5. If the low-frequency gain is not sufficiently high, add a phase lag element (includes a PI as a special case). A properly parametrized lag almost does not alter ω_c .
6. Simulate the resulted closed-loop system. If some of its properties are unsatisfactory, return to Step 1, change ω_c , and repeat all steps. Sometimes, it might be possible to return to some later steps, keeping ω_c intact and tuning loop gains, roll-off, stability margins, etc.

In some situations one may use skewed notch filters as a subtler lead element. It is more localized, yet comes at a lower price. Loop-shaping procedure for systems with lightly damped modes might require different tools. For instance, a phase lag might be required there to increase the distance from the critical point, which might not appear intuitive at first sight. An example of the use of loop-shaping ideas can be found in §A.4.

Although loop-shaping design guidelines are to a large extent ad hoc, there are analytic results shedding light on their potential limitations. One of them is **Bode's** gain-phase relation, which quantifies the fact that the magnitude and the phase of the loop frequency response $L(j\omega)$ cannot be manipulated independent of each other. For example, if $L(s)$ is stable, its phase at every frequency ω_0 must satisfy

$$\arg L(j\omega_0) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d \ln |L(j\omega_0 e^v)|}{dv} \ln \left(\coth \frac{|v|}{2} \right) dv - 2 \sum_{i=1}^{n_{\text{mpz}}} \arctan \frac{\omega_0 + \text{Im } z_i}{\text{Re } z_i}, \quad (16)$$

where $v := \ln(\omega/\omega_0)$, $n_{\text{mpz}} \in \mathbb{Z}_+$ is the number of nonminimum-phase zeros of $L(s)$, and z_i , with $\text{Re } z_i \geq 0$, are these zeros. The function

$$\ln \left(\coth \frac{|v|}{2} \right) = \frac{1}{v} \quad \text{for } v > 0, \quad \text{and} \quad \ln \left(\coth \frac{|v|}{2} \right) = -\frac{1}{v} \quad \text{for } v < 0,$$

may be thought of as an approximate Dirac delta. The first term on the right-hand side of (16) indicates then that the phase depends on the slope of the Bode magnitude plot,

$$\frac{d \ln |L(j\omega_0 e^v)|}{dv} = \frac{d \ln |L(j\omega)|}{d \ln \omega} = \frac{d \log |L(j\omega)|}{d \log \omega},$$

around $\omega = \omega_0$. The faster we need to reduce $|L(j\omega)|$, the more phase lag we have to tolerate. Because the Nyquist stability criterion normally requires the phase lag of $L(j\omega)$ to be limited around the crossover

frequency, the first term on the right-hand side of (16) effectively necessitates the crossover region to be sufficiently wide, so that the negative slope of the gain is not too steep. The second terms on the right-hand side of (16) imply that every nonminimum-phase zero aggravates this situation, especially at frequencies exceeding $|z_i|$. This effectively imposes limitations on the attainable crossover frequency for nonminimum-phase systems.

Another classical quantitative relation is **Bode's** sensitivity integral. If the loop transfer function $L(s)$ is real-rational and strictly proper, then the following relation holds whenever $S(s)$ is stable:

$$\int_0^\infty \ln|S(j\omega)| d\omega = \pi \left(\sum_{i=1}^{n_{\text{rhp}}} \text{Re } p_i - \frac{1}{2} \lim_{s \rightarrow \infty} sL(s) \right), \quad (17)$$

where $n_{\text{rhp}} \in \mathbb{Z}_+$ is the number of poles of $L(s)$ in the open right half-plane $\mathbb{C}_0$ and p_i , with $\text{Re } p_i > 0$, are these poles. Relation (17) is known [4] as *generalized² Bode's sensitivity integral*. The integrand on the left-hand side of (17) is negative if $|S(j\omega)| < 1$ and nonnegative otherwise. Thus, (17) effectively says that any feedback design with a strictly proper $sL(s)$ is an *art of tradeoffs*, where a reduction of $|S(j\omega)|$ at some frequencies inevitably leads to its increase at others. This phenomenon is known as the *waterbed effect*. Bode's sensitivity integral itself does not indicate that $|S(j\omega)|$ necessarily grows outside the frequency range where the $|S(j\omega)|$ is pushed down as this range widens. But this kind of results can be derived if some additional information on the decay of $|L(j\omega)|$ at high frequencies is available.

Appendix SISO loop-shaping case study: set-point control of a DC motor

To illustrate the ideas discussed in this chapter, consider the problem of controlling the shaft angle θ of a armature-controlled DC motor connected to a rigid mechanical load.

A.1 Model

A **basic property** of armature-controller DC motors is that the generated torque, τ_m , is proportional to the armature current, i , i.e.

$$\tau_m(t) = K_m i(t),$$

where K_m is called the torque constant of the motor. The torque is then applied to the load (which includes the rotor itself), whose angular velocity $\omega = \dot{\theta}$ is assumed to satisfy

$$J \frac{d}{dt} \omega(t) + f \omega(t) = \tau_m(t) + \tau_e(t),$$

where J and f are the moment of inertia and the viscous friction coefficient of the load, respectively, and τ_e is an external torque representing possible interactions with the environment. The armature electric circuit satisfies

$$L_a \frac{d}{dt} i(t) + R_a i(t) = v(t) - v_b(t),$$

where L_a and R_a are the inductance and the resistance of the motor armature, respectively, v is the external voltage supplied to the armature and v_b is the back emf (electromotive force), which by **Lenz's law** is proportional to the rotor angular velocity, i.e.

$$v_b(t) = K_b \omega(t),$$

where K_b is the back emf constant (or speed constant) of the motor. Normally, it **satisfies** $K_b = K_m$, if expressed in compatible units. Finally, we assume that the armature voltage is generated by an amplifier as

$$v(t) = K_a u(t),$$

²Bode's original formula is for a stable $L(s)$ with a pole excess of at least 2, for which the right-hand side of (17) is zero.

Parameter	K_a	K_m [N m/A]	R_a [Ω]	L_a [H]	J [kg m ²]	f [N m s/rad]	$\tau_{\max}$ [N m]
Value	12	0.126	2.08	0.000264	0.008	0.005	0.235

Table .1: Numerical values of motor and load parameters

where K_a is its gain and u is a normalized control signal, which is assumed to be limited as $|u(t)| \leq 1$ for all t . Numerical values of all parameters above, which will be used throughout this example, are presented in Table .1.

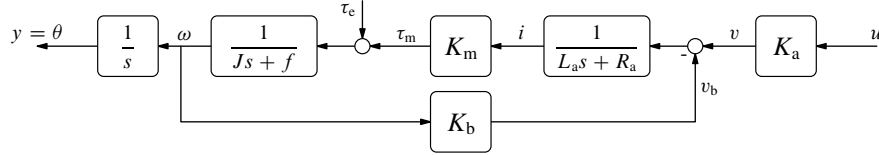


Fig. 6: Armature-controlled DC motor connected to a rigid mechanical load

The equations above can be combined in the block-diagram in Fig. 6, with the control input u and the controlled output $y = \theta$. This corresponds to the system in Fig. 4(a), with $P = P_L$ under

$$P_L(s) := \frac{K_m K_a}{s((L_a s + R_a)(J s + f) + K_m K_b)} \quad \text{and} \quad D(s) = \frac{L_a s + R_a}{K_m K_a} T_e(s). \quad (18)$$

This plant P_L is unstable, because of a pole at the origin. The disturbance might be unbounded in its high frequency range because τ_e passes through a system with a non-proper transfer function. Yet this should not be a problem as those high-frequency components are filtered out by the plant, which has a pole excess of two and natural low-pass properties. In many situations the time constant of the electrical part is substantially smaller than that of the mechanical load, i.e. $L_a/R_a \ll J/f$. In this case we may take $L = 0$ without substantial loss of accuracy. This yields the simplified transfer function of the plant

$$P_0(s) = \frac{K_m K_a}{s(R_a J s + (R_a f + K_b K_m))}, \quad (19)$$

which is still unstable. In what follows, this P_0 will be used as the plant model in controller design because of its simplicity.

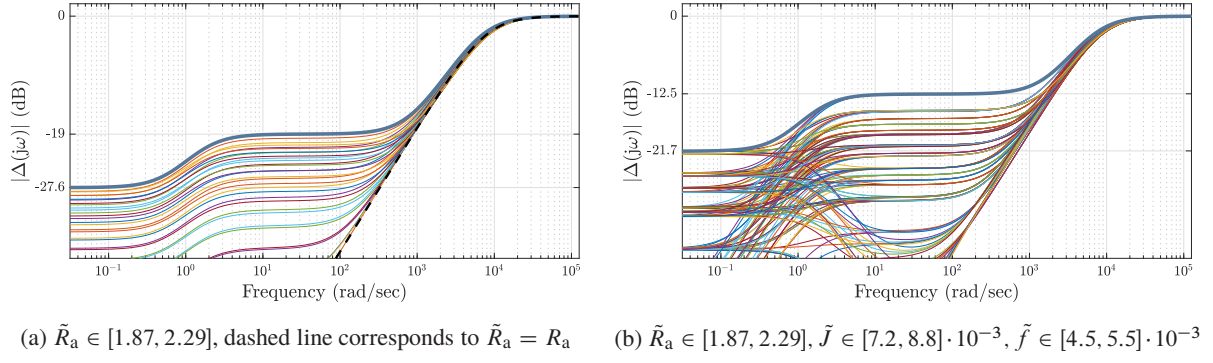
In any case, both P_L and P_0 above are approximations of a real, possibly nonlinear, DC motor P . Moreover, physical parameters of the motor might change in course of its operation. For example, the resistance increases as the armature circuit heats up and the load changes frequently. On top of this, there are complex high-frequency (like flexible structural modes of the load) and nonlinear (like dry friction) phenomena, which are not taken into account in deriving (18). All this implies that we have to account for modeling errors. Such errors are convenient to express in terms of *multiplicative modeling uncertainty*

$$\Delta := 1 - P P_0^{-1}, \quad (20)$$

where P_0 is a *nominal plant*, that in (19) for some chosen values of its parameters, cf. (10). If $P = P_L$ from (18) under $L_a \neq 0$ and known other parameters, then the relative modeling error is LTI with the transfer function

$$\Delta(s) = \frac{L_a s(J s + f)}{(L_a s + R_a)(J s + f) + K_m K_b}. \quad (21)$$

This is a stable high-pass system with the zero DC gain, $\Delta(0) = 0$, and a monotonically increasing magnitude of its frequency-response, approaching 1 at high frequencies (the magnitude of its frequency response

Fig. 7: Modeling errors Δ for various uncertainty sources

is shown by the black dashed line in Fig. 7(a)). If $P = P_L$ under a mismatched resistance, say $\tilde{R}_a \neq R_a$, then

$$\Delta(s) = \frac{(L_a s + \tilde{R}_a - R_a)(J s + f)}{(L_a s + \tilde{R}_a)(J s + f) + K_m K_b}, \quad (22)$$

which is also stable, but now $\Delta(0) = (\tilde{R}_a - R_a)f / (\tilde{R}_a f + K_m K_b) \neq 0$, indicating that modeling error exists also at low frequencies. Thin lines in Fig. 7(a) represent $|\Delta(j\omega)|$ for various values of $\tilde{R}_a \in [1.87, 2.29]$, which is about 10% deviation from the nominal $\tilde{R}_a = R_a$.

If the value of $\tilde{R}_a$ is not known a priori, or drifts during the motor operation, then Δ is not a fixed given LTI system, but rather a family of systems. There are many approaches to handle such situations. Conceptually, it may be the simplest to treat modeling mismatch as an arbitrary stable system, whose norm (size) is bounded by corresponding bounds on the actual Δ . Such a treatment significantly simplifies the analysis, rendering it suitable for small-gain arguments [2, Sec. III.2]. However, it introduces *conservatism*, in the sense that the class of modeling errors is enlarged and a negative result for the whole norm-bounded class does not necessarily imply that for the original class of modeling errors. Still, simplicity is a great asset and below we analyze the system via such an upper bound. Obtaining it is relatively simple for constant $\tilde{R}_a$'s, in which case it is merely the frequency-wise upper bound on $|\Delta(j\omega)|$ over all admissible $\tilde{R}_a$, see the **thick line** in Fig. 7(a). This approach readily extends to modeling errors arising from uncertainty in other constant parameters, see for example Fig. 7(b), which represents combined uncertainty in R_a , J , and f , each is again about 10% deviation from nominal values. Obtaining an upper-bound Δ in the time-varying or nonlinear case might be more involved.

A.2 Reference signal

The control goal here is to rotate the motor shaft to a specific destination angle θ_d as fast as possible and stay there. To simplify the formulae and without loss of generality we may assume that the initial steady state is at $y(0) = 0$ and that $\theta_d > 0$. This kind of problems arise in numerous applications and are dubbed the *set-point regulation*.

In a constraint-free world, this kind of problems would prompt a step reference signal, like

$$y_r(t) = \theta_d \mathbb{1}(t) = \begin{cases} 0 & t < 0 \\ \theta_d & t \geq 0 \end{cases},$$

which is obviously the fastest move to any θ_d . Yet this signal would not be realistic. No motor can generate an infinite torque or admit an infinite input voltage required for a shaft jump. Of course, we may still use the step reference, or its filtered version, essentially as a declaration of intent. This will not render y matching

y_r closely in its initial stage anyway. But this will put the control system into saturation modes if θ_d is “too large” and might cause unnecessary, and definitely unwanted, spikes / oscillations in y and u .

A far better approach is to incorporate practical constraints into the choice of y_r . The rationale behind this is twofold. First, it is always healthier to demand from a system something that it can supply than something unachievable anyway. This is a good recipe to avoid complications, like saturation and overloads, which might then be way harder to correct by a controller. Second, the fastest trajectory in a constrained world is almost certainly a nonlinear function of the destination point and system parameters. It is then always beneficial, analysis-wise, to pull nonlinear parts of a control algorithm outside feedback loops, i.e. to y_r .

Apparently, the most restrictive constraint for our problem is that on the torque generated by the motor (equivalently, on the current in its armature circuit). In many cases, at least if f is not too small, the voltage constraint is inactive while the current constraint is satisfied. A possible choice of the reference trajectory then is to rotate the shaft to its target position in minimum time t_f under given load dynamics and the constraint that $|\tau_m(t)| \leq \tau_{\max}$ for all t and a given $\tau_{\max} > 0$ dependent on the concrete motor. This trajectory can be derived by minimum-time optimization techniques [1, Ch. 7] for the second-order $1/(Js^2 + fs)$. The optimal torque should be of a bang-bang type, with exactly one switch point t_s in $(0, t_f)$, because both poles of the plant are real. The Laplace transform of the resulting y_r , which is a nonlinear function of θ_d and $\tau_{\max}$, is

$$Y_r(s) = \frac{1}{s(Js + f)} \frac{1 - 2e^{-t_s s} + e^{-t_f s}}{s} \tau_{\max}, \quad (23)$$

where the time instances

$$t_s = \frac{f\theta_d}{\tau_{\max}} + \frac{J}{f} \ln\left(1 + \sqrt{1 - e^{-f^2\theta_d/(J\tau_{\max})}}\right) \quad \text{and} \quad t_f = \frac{f\theta_d}{\tau_{\max}} + 2\frac{J}{f} \ln\left(1 + \sqrt{1 - e^{-f^2\theta_d/(J\tau_{\max})}}\right)$$

are derived from the conditions

$$\lim_{s \rightarrow 0} sY_r(s) = \theta_d \quad \text{and} \quad \lim_{J \rightarrow -f} (Js + f)Y_r(s) = 0 \quad (24)$$

(require $f > 0$, which is naturally assumed to hold true). This reference signal has the form

$$y_r(t) = \begin{cases} 0 & 0 \leq t < t_s \\ \theta_d & t_s \leq t < t_f \\ \theta_d & t \geq t_f \end{cases} \quad \left(\text{attained by the bang-bang } \tau_m(t) = \begin{cases} \tau_{\max} & 0 \leq t < t_s \\ -\tau_{\max} & t_s \leq t < t_f \\ 0 & t \geq t_f \end{cases} \right). \quad (25)$$

The numerical values here and in what follows are taken from Table .1 and with $\theta_d = 6\pi$ [rad], resulting in $t_s = 1.017$ [s] and $t_f = 1.634$ [s].

It should be mentioned that in some applications there might be additional constraints, like limited load velocity, acceleration, or jerk (e.g. if the load is the cabin of an elevator), accounting for which would result in more complicated reference profiles. Still, handling such constraints on the level of y_r is way easier than on the level of a controller.

A.3 Open-loop control

First, consider the design of an open-loop controller in the configuration presented in Fig. 4(b). Let us use the plain plant inversion strategy (6) here, namely select

$$R(s) = \frac{1}{P_0(s)} = \frac{R_a J}{K_m K_a} s^2 + \frac{R_a f + K_m K_b}{K_m K_a} s$$

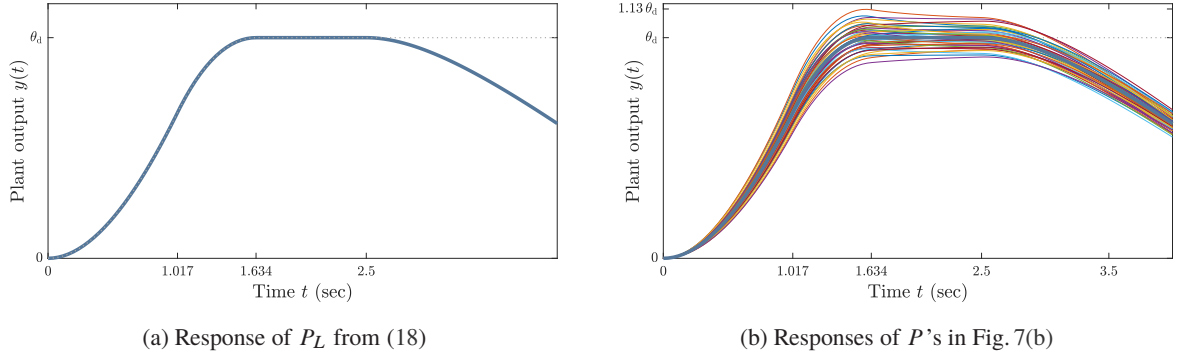


Fig. 8: Open-loop controlled command responses y , the disturbance τ_e satisfies $\tau_e(t) = 0.11(t - 2.5)$

Although this transfer function is non-proper, and hence unstable, it acts only on y_r known analytically. We just need to calculate two first derivatives of y_r above and verify that they are bounded. This is indeed the case and the control trajectory corresponding to the optimal y_r is

$$\begin{aligned}
 u(t) &= \frac{R_a J}{K_m K_a} \ddot{y}_r(t) + \frac{R_a f + K_m K_b}{K_m K_a} \dot{y}_r(t) \\
 &= \underbrace{\begin{array}{c} \text{trapezoidal profile} \\ \text{from } 0 \text{ to } t_f \end{array}}_{u_{\text{opt}}(t)} + \underbrace{\begin{array}{c} \text{triangular profile} \\ \text{from } 0 \text{ to } t_f \end{array}}_{u_{\text{opt}}(t)} = \begin{array}{c} \text{trapezoidal profile} \\ \text{from } 0 \text{ to } t_f \end{array} \quad (26)
 \end{aligned}$$

Note that the peak value of the input signal, 0.56, is still far from its maximal allowable value of 1. In fact, even when θ_d grows, the peak voltage never exceeds $(K_b/f + R_a/K_m)\tau_{\max} = 0.818K_a < K_a$, which justifies ignoring the voltage constraint in the choice of y_r in our case.

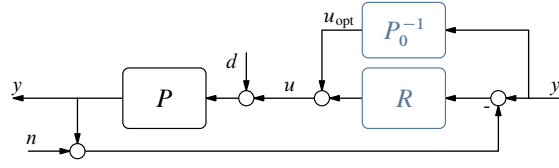
As discussed in §4.1, the controller above attains (4) only under the condition that $P = P_0$, which is not the case. If we consider P as in (18), still with $d = 0$, the deviation from the ideal response in the Laplace domain, according to (10) and (21), is

$$E(s) = Y_r(s) - Y(s) = \Delta(s)Y_r(s) = \frac{L_a}{(L_a s + R_a)(Js + f) + K_m K_b} \frac{1 - 2e^{-t_s s} + e^{-t_f s}}{s} \tau_{\max}.$$

This corresponds to an exponentially decaying e , which is proportional to L . If L is very small, as assumed in justifying the use of P_0 , the error is also insignificant, as can be seen from the **thick line** in Fig. 8(a) for $t \in [0, 2.5]$, which is virtually indistinguishable from the ideal response shown by the dotted gray line. Moreover, the steady-state value is still θ_d for every L , as $\lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = 0$. This is because L does not affect properties of $P(s)$ at low frequencies.

The resistance and load friction do affect low-frequency properties of P , so the open-loop control is less successful if either of them changes. Thin lines in Fig. 8(b) for $t \in [0, 2.5]$ represent responses of the system to the control signal (26) for plants from the family depicted in Fig. 7(b). It is readily seen that those up to 10% deviations from nominal values of R_a , J , and f produce rather diverse responses, with steady-state errors up to 8% of the value of θ_d . The responses are also visually different from the expected response represented by the **thick line** in transient phases, demonstrating that open-loop control has no ability to cope with modeling mismatches.

Expectably, the shaft angle θ starts to diverge if a constant external torque τ_e , even a very small one, is applied to the system as shown in Fig. 6. After all, the plant is unstable. This is seen from the plots in Fig. 8, which represent, after $t = 2.5$ [sec], responses to a step torque of -0.1 [N m] applied at that time instance. This external torque causes the angle to diverge for every P , decaying linearly after short transients. This is a typical problem with controlling unstable processes in open loop, where no mechanism for compensating the effect of unmeasurable disturbances exist.

Fig. 9: 2DOF closed-loop control aiming at $y = y_r$ under $P = P_0$ and $d = n = 0$

A.4 Closed-loop control

Because the transfer function of the plant in (18) has a pole excess of 3, no proper $R(s)$ with uniformly high gain will stabilize it. We thus cannot use the arguments of the beginning of §4.2. Rather, consider designing R via loop-shaping techniques. The design is simplified by the use of the 2-degrees-of-freedom (2DOF) controller architecture, a simplified version of which is depicted in Fig. 9. This scheme assumes that the signal $(P_0^{-1})y_r$ can be implemented, which is indeed the case for our choice of the reference signal in (25) as is evident from (26).

It can be verified that the tracking error signal $e = y_r - y$ in this case is of the form

$$e = S(\Delta y_r - P d) + T n =: e_{y_r} - e_d + e_n, \quad (27)$$

where $S = (1 + PR)^{-1}$ is the *actual* sensitivity function, $T = 1 - S$ is the complementary sensitivity function, the modeling error Δ is as defined in (20), and the components in the right-hand side are counterparts of what we had in the open-loop case in (10). An important property of this closed-loop relation is that in the nominal case, with $P = P_0$ and $\Delta = 0$, the effect of the reference signal on the tracking error $e_{y_r} = 0$, exactly as in the open-loop control case, regardless the choice of R (as long as it stabilizes the system, of course). This is an advantage of the 2DOF architecture over the unity-feedback one in Fig. 4(c). The latter, by (11), has $e_{y_r} = S y_r$, which can be zero only in the infeasible uniformly high-gain case. Moreover, the choice of R does not need to take nominal tracking considerations into account, which simplifies the design procedure.

By the logic of the discussion on p. 6, consider each of the components in the right-hand side of (27) separately.

e_{y_r} : The only source of problems here is the modeling error. Unlike the open-loop case, where $e_{y_r} = \Delta y_r$, the closed-loop sensitivity S is also a factor now. We thus can affect e_{y_r} via a choice of the feedback controller R . A natural approach to reduce e_{y_r} is to decrease $|S(j\omega)|$ at dominant frequencies of possible Δy_r . To estimate this dominant part, observe that by (23)

$$|Y_r(j\omega)| = \frac{|1 - 2e^{-j\omega t_s} + e^{-j\omega t_f}| \tau_{\max}}{\omega^2 \sqrt{f^2 + J^2 \omega^2}} \leq \frac{4\tau_{\max}}{\omega^2 \sqrt{f^2 + J^2 \omega^2}}, \quad \forall \omega \in \mathbb{R} \quad (28)$$

which follows by the triangle inequality and can be arbitrarily tight at every ω under some θ_d . Remarkably, the bound above is independent of θ_d , so that requirements on S can be formulated independently of the destination point. For example, for uncertainty as in Fig. 7(b) we have that an upper bound on the frequency response of ΔY_r is monotonically decreasing function of ω such that $|\Delta(j\omega)Y_r(j\omega)| \leq 0.1$ for all $\omega > 6.44$. In any case, we should aim at increasing the frequency range in which the sensitivity magnitude is small. This is, in turn, equivalent to increasing the target crossover as much as possible while preserving stability and reasonable stability margins under every possible incarnation of P . It is also worth emphasizing that $S(0) = 0$ under any stabilizing R , because of the integral action in the plant. Hence, $\lim_{t \rightarrow \infty} e_{y_r}(t) = 0$ for every y_r converging to a constant, despite modeling uncertainty.

e_d : The change with respect to the open-loop $e_d = P d$ is again the addition of the sensitivity function. Hence, reducing $|S(j\omega)|$ at low frequencies reduces the effect of the load disturbance. But because

the plant itself has an unbounded static gain, zero steady-state error requires an integral action in the controller as well. Thus, a policy to reduce e_d would be similar to that for reducing e_{y_r} , with the additional requirement to have an integral action in the controller.

e_n : This component of the error signal is an artefact of closing the feedback loop, it does not exist in open-loop control. In many situations, the spectrum of the measurement noise is dominated by high frequencies. This implies upper bounds on the choice of the crossover frequency and requirements on the high-frequency roll-off of the controller. Although the effect of n is not studied below, some specifications are inspired by its presence.

With these considerations in mind, consider the design of the feedback part of the controller, R , for the system in Fig. 9 with the following specifications in terms of the nominal plant P_0 from (19):

1. an integral action in $R(s)$,
2. a high-frequency roll-off of at least 1 for $R(s)$,
3. a modulus margin³ of above 0.5, i.e. $|S(j\omega)| < 2$ for all ω ,
4. the control sensitivity magnitude $|T_c(j\omega)| < 1$ for all ω ,
5. as large crossover frequency ω_c of the nominal loop $P_0 R$ as possible under the requirements above.

Attaining the first requirement is trivial. The second requirement is motivated by the need to render the magnitude of the complementary and control sensitivity frequency responses small at high frequencies to reduce the effect of the measurement noise. Its fulfillment is also technically simple, we just need to end up with a strictly proper $R(s)$. The third one requires the nominal loop to be sufficiently far from the critical point, which will be pursued via adding enough phase margin by the lead component. The specification on $|T_c(j\omega)|$ aims at avoiding high control effort and is the main source of conflict with the crossover requirement in the last item above. So the design is a sequence of iterations, in which we increase or decrease the target crossover frequency and see whether the resulting control sensitivity is strictly contractive.

Although specifications are formulated for the *design model* P_0 , we need to take into account modeling uncertainty. To this end, design steps will be verified for a family of plants, those under the conditions of Fig. 7(b), both during checking frequency-domain properties of the designed loops and in time-domain simulations. The Nichols plots of those plants are depicted by thin solid lines in Fig. 10(a). Hollow dots represent two selected frequencies, both of them will be eventually our crossover choices. It can be seen that the frequency responses from the studied class are relatively dense, not far from the nominal one shown by the **thick line** in Fig. 10. As such, nominal margins and crossover are expected to be close to those for all these plants.

Technically, the controller design is done in the following steps. First, the first two requirements above can be guaranteed by augmenting a strictly proper integrator of the form $R_{\text{int}}(s) = k/s$ to the plant and then designing a proper remaining part of the controller. In doing so, we choose the gain k so that the crossover frequency of the resulted augmented plant $P_0 R_{\text{int}}$ is exactly the chosen ω_c . This yields

$$R_{\text{int}}(s) = \frac{\omega_c / |P_0(j\omega_c)|}{s}.$$

But the integrator in this controller adds additional 90° of phase lag, leading to loops like that shown by the **dashed line** in Fig. 10(a) for all crossover candidates in $[1, 20]$, which can safely be considered the required range with the uncertain Δy_r negligible above $\omega = 6.44$ [rad/sec] and the plant bandwidth below $\omega = 12.54$ [rad/sec]. By the Nyquist arguments, we now need then at least 75° of phase lead at $\omega = \omega_c$ to render the closed-loop system stable. In fact we need more to have a reasonable phase margin. This implies that a single lead controller, whose phase lead is below 90° , would not be enough. A general⁴ second-order

³The modulus margin μ_m is the shortest Euclidean distance of the frequency response from the critical point on the Nyquist plane.

⁴A yet more general form would take different damping factors in the numerator and denominator, resulting in *skew notches*.

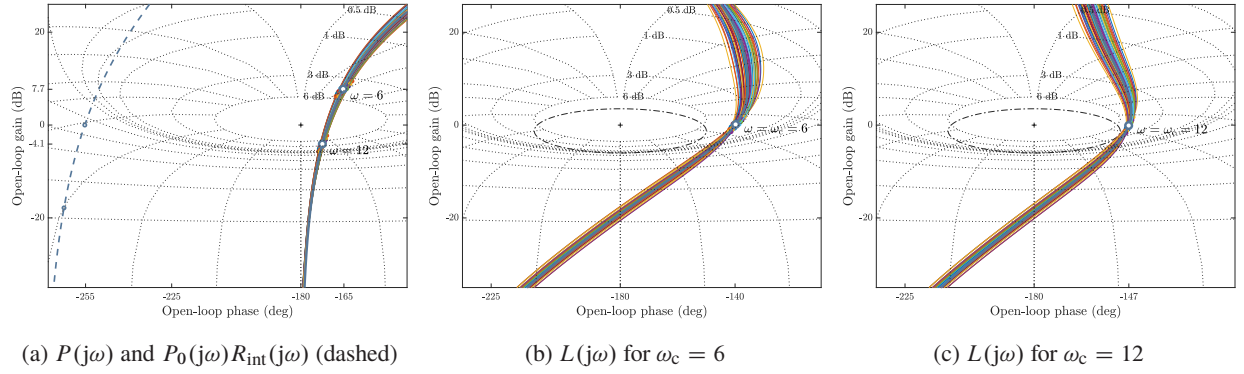


Fig. 10: Nichols charts of the plant and designed loops (dash-dotted line is the 6 dB inverse M -circle)

lead tuned for a given crossover frequency ω_c can be written in the form

$$R_{\text{lead}}(s) = \frac{\alpha s^2 + 2\zeta\sqrt{\alpha}\omega_c s + \omega_c^2}{s^2 + 2\zeta\sqrt{\alpha}\omega_c s + \alpha\omega_c^2} \quad \text{for some } \alpha > 1 \text{ and } \zeta \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]. \quad (29)$$

Its maximal phase lead, always at $\omega = \omega_c$, is $180 - 2 \arctan(2\zeta\sqrt{\alpha}/(\alpha - 1))$ [deg]. It grows as ζ decreases and as α increases. An increase of α also implies a decrease of the static gain of R_{lead} and an increase of its high-frequency gain. In other words, α may be thought of as the cost of phase lead in this controller. A small damping factor ζ implies that the phase lead is concentrated in a more narrow frequency band. In our system, because of modeling uncertainty, we would prefer to keep this frequency band wide enough.

The procedure described above is technically simple, although the need to tune three parameters, ω_c , α , and ζ , could make it time consuming. The design may be simplified by limiting the consideration to damping factors slightly above one and aiming at a phase margin of about 40° . After some trial and error process, the choice $\omega_c = 6$ [rad/sec] was found suitable, accompanied by $\alpha = 20$ and $\zeta = 3/\sqrt{5} \approx 1.342$. After rounding coefficients, the overall controller

$$R(s) = R_{\text{int}}(s)R_{\text{lead}}(s) = \frac{50(s+3)(s+0.6)}{s(s+60)(s+12)}.$$

The frequency responses of the resulting loops are shown in Fig. 10(b), where the nominal loop is represented by the **thick line** and loops for other possible plants from the considered set are depicted by thin lines. We can see that the closed-loop system is always stable, with the phase margin about 40° for every plant from the set, which is reasonably large. Checking the closed-loop control sensitivity magnitudes separately, we could see that the bound $|T_c(j\omega)| < 1$ holds for all frequencies, as required. Likewise, plotting the sensitivity functions for all plants from the studied class would reveal that they are all high-pass filters with the cutoff frequencies⁵ above $\omega = 3.2$ [rad/sec] and $|S(j\omega)| < 1.767 < 2$. In fact, the latter could be seen already from the *inverse M-circles* on the open-loop Nichols chart. The dash-dotted line in Fig. 10(b) represents one such inverse circle for the 6 dB level, which evidently does not touch loop frequency responses. The closed-loop sensitivity cutoff bound suggests that the effect of modeling uncertainty on the command response is suppressed at dominant frequencies of y_r . This suppression shows up clearly in the closed-loop command responses in Fig. 11(a), where the **thick line** again represents the nominal case and thin lines represent all others, all in the time interval $[0, 2.5]$. These responses are considerably closer to the nominal response than in the open-loop case, which demonstrates clearly advantages of feedback in reducing effects of modeling uncertainty.

⁵By the *cutoff frequency* of a high-pass $F(s)$ we understand the largest ω_{cutoff} such that $|F(j\omega)| \leq -3$ dB, $\forall |\omega| \leq \omega_{\text{cutoff}}$.

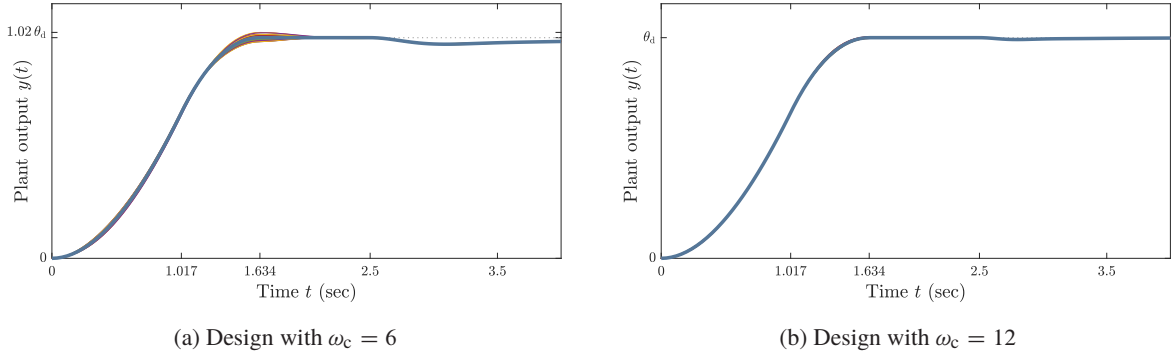


Fig. 11: Closed-loop controlled command responses y , disturbance τ_e satisfies $\tau_e(t) = 0.1\mathbb{1}(t - 2.5)$

Another advantage of feedback can be seen in responses to a step torque disturbance applied at $t = 2.5$. Comparing closed-loop responses in Fig. 11(a) with those in Fig. 8(b) under open-loop control, we can see that the effect of τ_e on the former is substantially smaller. Moreover, the presence of an integral action in R guarantees that y converges to y_r in steady state, which is a qualitative difference from the diverging open-loop results.

At the end, note that if the requirement on the control sensitivity is relaxed, then even tighter closed-loop results can be obtained. For example, let us double the crossover frequency, to $\omega_c = 12$ [rad/sec]. Keeping the design logic and the parameters α and ζ in (29) untouched, we end up with the loop in Fig. 10(c) for the controller

$$R(s) = \frac{380(s + 6)(s + 1.2)}{s(s + 120)(s + 24)},$$

for which $\max_{\omega} |T_c(j\omega)| = 3.5$. Although stability margins of this design are smaller than those for $\omega_c = 6$, we still have that $|S(j\omega)| < 2$ for all frequencies, which is seen via the inverse M -circle, depicted by the dash-dotted line in Fig. 10(c). As a result of the increase in ω_c we have a wider suppression region of the sensitivity functions, whose cutoff frequencies are now above $\omega = 6.1$ [rad/sec]. And then the time responses of all plants from the family in Fig. 7(b) are nearly indistinguishable, see Fig. 11(b). The sensitivity to torque disturbances is also lower than in the less aggressive design in Fig. 11(a).

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