



STATE-SPACE CONTROL FUNDAMENTALS (00860721)

HOMEWORK 2

(submission deadline: 23/6/2026, 20:00; do make an effort to be concise, clear, and accurate)

Question 1 (30pt). Consider a DC motor $u \mapsto y$ having the transfer function $P(s) = 1/(s(s+1))$ with measured y and $\dot{y}$. Formulate LQR problems, i.e. form the parameters A , B , Q , R , and S , for the following cost functions (here $\lambda \in [0, 1]$ and $r > 0$).

1. $\mathcal{J} = \lambda \int_0^\infty \|y(t)\|^2 dt + (1 - \lambda) \int_0^\infty \|\dot{y}(t)\|^2 dt + r \int_0^\infty \|u(t)\|^2 dt$
2. $\mathcal{J} = \lambda \int_0^\infty \|y(t)\|^2 dt + (1 - \lambda) \int_0^\infty \|\ddot{y}(t)\|^2 dt + r \int_0^\infty \|u(t)\|^2 dt$
3. $\mathcal{J} = \lambda \int_0^\infty \|y(t)\|^2 dt + (1 - \lambda) \int_0^\infty \|\ddot{y}(t)\|^2 dt + r \int_0^\infty \|u(t)\|^2 dt$

Question 2 (70pt). A linearized model of an inverted Furuta pendulum, like that presented in Fig. 1, is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & J_b + mr^2 & 0 & -mrl \\ 0 & 0 & 1 & 0 \\ 0 & -mrl & 0 & J_p + ml^2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -f & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & gml & -\zeta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tau_m, \quad (1)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} := \begin{bmatrix} \phi \\ \dot{\phi} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \text{arm angle} \\ \text{arm angular velocity} \\ \text{pendulum angle} \\ \text{pendulum angular velocity} \end{bmatrix}$$

(the angles are in radians) and

- J_b and f are the total moment of inertia (around its pivot point) and viscous friction of the rotating base (rod + arm);
- r is the distance from the pendulum rotation plane to the arm pivot point;
- m is the pendulum mass;
- l is the distance from the pendulum pivot point to its centroid;

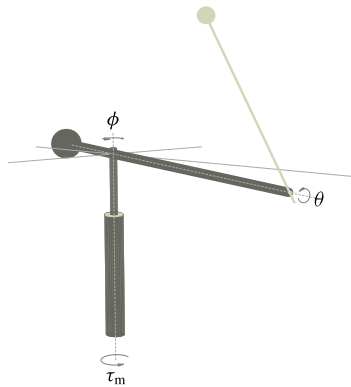


Fig. 1: Inverted Furuta pendulum.

Parameter	l	r	m	J_p	ζ	J_b	f	g	R	K_m	N_g	K_a
Value (in SI)	0.3226	0.3	0.04	0.0003	0.0005	0.01	0.01	9.81	2.08	0.0243	4.5	12

Table 1: Parameters of the system

- J_p and ζ are the moment of inertia around its centroid and damping coefficient of the pendulum, respectively;
- g is the standard gravity.

The input torque τ_m is generated by a DC motor having its torque and back emf constants K_m and $K_b = K_m$, respectively, a resistance R , *negligible inductance*, and a gearbox with the ratio N_g (so that the angular velocity of the motor shaft equals $N_g \dot{\phi}$ and $\tau_m = N_g \tau_0$, where τ_0 is the torque generated by the rotor). The armature voltage of the motor is generated by an amplifier, modeled as a static gain K_a , so the control input is the input u to the amplifier.

1. Prove that if $J_b > 0$, $J_p > 0$, and $m > 0$, then

$$E := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & J_b + mr^2 & 0 & -mrl \\ 0 & 0 & 1 & 0 \\ 0 & -mrl & 0 & J_p + ml^2 \end{bmatrix} > 0.$$

2. Derive, in a symbolic form, the parameters A and B of the state equation of the system $u \mapsto x$ (no need to resolve all relations there, it is enough to present the realization matrices as products of known matrices, and there is no need to explicitly invert E).

As a debugging hint, the poles of the system for the parameters in Table 1 should be $\{-6.4505, -1.1444, 0, 5.9085\}$.

3. Assuming that the parameters are as in Table 1, design a state-feedback control law via formulating and solving an LQR problem. Specifically, consider the cost objectives

$$\gamma_\phi = \int_0^\infty \|\phi(t)\|^2 dt, \quad \gamma_\theta = \int_0^\infty \|\dot{\theta}(t)\|^2 dt, \quad \text{and} \quad \gamma_u = \int_0^\infty \|u(t)\|^2 dt$$

and construct Q , S , and R for the LQR formulation minimizing

$$\mathcal{J} = \lambda \gamma_\phi + (1 - \lambda) \gamma_\theta + r \gamma_u$$

for some $\lambda \in (0, 1)$ and $r > 0$. Tune λ and r so that the response of the closed-loop system to the initial condition $x(0) = [\pi/2 \ 0 \ 0 \ 0]'$ has the smallest $\max_t |\theta(t)|$ under the constraints

$$\text{a) } |u(t)| \leq 1 \text{ for all } t \geq 0,$$

$$\text{b) } t_s \leq 2 \text{ [sec], where } t_s \text{ is the smallest time instance such that } |\phi(t)| \leq 0.01\phi(0) \text{ for all } t \geq t_s$$

(make sure that $\max_t |\theta(t)| \leq 10^\circ$). Plot the response to the initial condition x_0 of ϕ (in deg), θ (in deg), and u using the time grid `t = linspace(0, 3, 501)`.

4. Present the Nyquist plot (nonnegative frequencies only, in the imaginary axis bounds $[-3, 4]$) of the loop $L(s) = -K(sI - A)^{-1}B$ designed in Item 3. Find its stability margins (gain and phase). Do these margins agree with those guaranteed by the return-difference inequality?
5. Find the transfer function $G_*(s)$ defined on Slide 27 of Lecture 5 for the design parameters in Item 3. Present the root locus of the closed-loop poles with respect to $1/r$, with the actual closed-loop poles for the choice of r in Item 3 marked on it as red dots (for the sake of visibility, keep the real axis in the interval $[-8, 8]$ and ignore it if some closed-loop poles go beyond this interval). Find the range of r for which all closed-loop poles have a damping ratio of at most $1/\sqrt{2}$ for the same λ .