



STATE-SPACE CONTROL FUNDAMENTALS (00860721)

HOMEWORK 1

(submission deadline: 27/5/2026, 13:00; do make an effort to be concise, clear, and accurate)

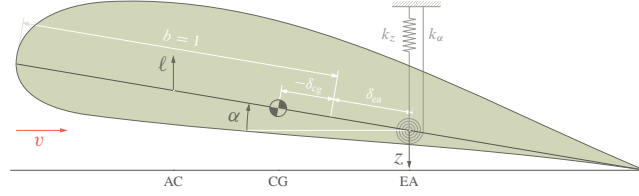


Fig. 1: Spring-suspended airfoil system.

Question 1. Consider the two-dimensional airfoil section in Fig. 1, where α is the pitch angle around the elastic axis (EA), z is the plunge displacement, ℓ is the circulatory constituent of the lift (force per unit length) acting at the aerodynamic center (AC), and v is the freestream velocity. It is assumed that the semi-chord of the airfoil is normalized, i.e. $b = 1$, and the distances of the EA and the center of gravity (CG) from the mid-chord are δ_{ea} and δ_{cg} , respectively ($-1/2 < \delta_{cg} < 1/2$).

Introducing the variable

$$q := \begin{bmatrix} z \\ \alpha \end{bmatrix},$$

the structural (plus effects of non-circulatory aerodynamic loads) model of the system is

$$(M + M_{nc})\ddot{q}(t) + C_{nc}\dot{q}(t) + Kq(t) = L\ell(t). \quad (1)$$

Here

$$M := \begin{bmatrix} m & -m(\delta_{ea} - \delta_{cg}) \\ -m(\delta_{ea} - \delta_{cg}) & J + m(\delta_{ea} - \delta_{cg})^2 \end{bmatrix}, \quad K := \begin{bmatrix} k_z & 0 \\ 0 & k_\alpha \end{bmatrix}, \quad \text{and} \quad L := \begin{bmatrix} -1 \\ 1/2 + \delta_{ea} \end{bmatrix}$$

are the mass, stiffness, and actuation matrices, respectively, that represent the structural model, where m is the mass per unit span (linear mass density), J is the mass moment of inertia per unit span about the CG point, and k_z and k_α are stiffness coefficients. The parameters

$$M_{nc} := \pi\rho \begin{bmatrix} 1 & -\delta_{ea} \\ -\delta_{ea} & 1/8 + \delta_{ea}^2 \end{bmatrix} \quad \text{and} \quad C_{nc} := \pi\rho v \begin{bmatrix} 0 & 1 \\ 0 & 1/2 - \delta_{ea} \end{bmatrix}$$

are the added-mass and aerodynamic damping, respectively, representing non-circulatory aerodynamic loads, where ρ is the air density.

Now, the input to (1), the circulatory constituent of the lift ℓ , is not an exogenous signal independent of the behavior of the airfoil. Rather, it depends on the signal

$$\alpha_e(t) := v^2\alpha(t) + v\dot{z}(t) + v(1/2 - \delta_{ea})\dot{\alpha}(t), \quad (2)$$

which is the effective aerodynamic angle of attack at 3/4-chord point scaled by a factor of v^2 . The dependence is conventionally assumed to be linear, but not static. Specifically,

$$\ell(t) = 2\pi\rho\left(\phi(t)\alpha_e(0) + \int_0^t \phi(t-s)\frac{d\alpha_e(s)}{ds}ds\right) + \ell_{\text{gust}}(t),$$

parameter	δ_{ea}	δ_{cg}	m	J	k_z	k_α	ρ	$\vdots$	a_1	a_2	k
value	0.2	-0.3	20	2	$3 \cdot 10^4$	$2 \cdot 10^4$	1.225	$\vdots$	0.0455	0.3	0.165

Table 1: Numerical values for simulations

where ϕ is a monotonically increasing function from $\phi(0) = 1/2$ to $\phi(\infty) = 1$, known as the Wagner function, and ℓ_{gust} is a signal representing unmodeled effects of wind gusts, which is treated as an exogenous input to the system (disturbance). The relation $\alpha_e \mapsto \ell$ can be written in a more conventional convolution form as

$$\ell(t) = 2\pi\rho\left(\phi(0)\alpha_e(t) - \int_0^t \frac{d\phi(t-s)}{ds}\alpha_e(s)ds\right) + \ell_{\text{gust}}(t) \quad (3)$$

(follows via integrating by parts). The Wagner function ϕ can be approximated (Jones' approximation) as

$$\phi(t) = 1 - ke^{-tv a_1} - (1/2 - k)e^{-tv a_2} \quad (4)$$

for some constants $a_1 > 0$, $a_2 > 0$, and k .

1. Prove that the matrix $M + M_{\text{nc}} > 0$ for all $m > 0$ and $J > 0$ (obviously, $\rho \geq 0$).
2. Find the state-space realization $\left(A_s, B_s, \begin{bmatrix} C_{s1} \\ C_{s2} \end{bmatrix}, \begin{bmatrix} D_{s1} \\ D_{s2} \end{bmatrix}\right)$ of the system $\ell \mapsto (q, \alpha_e)$ described by (1)–(2) for the choice of the state vector as $(q, \dot{q})$. Keep its parameters as block matrices, i.e. don't spell out M , M_{nc} , C_{nc} , K , and L .
3. Let ℓ_{step} be the response of (3) to the step $\alpha_e = \mathbb{1}$ under $\ell_{\text{gust}} = 0$. Find $y_{\text{step}}(0)$ and $\lim_{t \rightarrow \infty} y_{\text{step}}(t)$.
4. Find the transfer function $G_\ell(s)$ of the system $\alpha_e \mapsto \ell$ (assuming that $\ell_{\text{gust}} = 0$) described by (3)–(4).
5. Construct a state-space realization $(A_\ell, [B_{\ell1} \ B_{\ell2}], C_\ell, [D_{\ell1} \ D_{\ell2}])$ of the system $(\ell_{\text{gust}}, \alpha_e) \mapsto \ell$ described by (3)–(4).
6. Construct the realization of the overall system $G : \ell_{\text{gust}} \mapsto q$ in terms of “s” and “l” realizations in items 2 and 5, respectively.
7. Calculate the real part of the rightmost eigenvalues of the “A” matrix of the overall system in the previous item for $v = \text{linspace}(0, 65, 1001)$ when the other parameters are as given in Table 1. Present the result as the plot* of $\max_{\lambda \in \text{spec}(A)} \text{Re } \lambda$ vs. v . Submit *stand-alone* Matlab code (in *one file*) generating the result.

As a debugging hint, make sure that $\max_{\lambda \in \text{spec}(A)} \text{Re } \lambda = 0$ for both $v = 0$ and $v \approx 60.9267$.

*Produce PDF plots with `exportgraphics(gca, 'filename.pdf', 'ContentType', 'vector')` from your code, rather than saving the plot window.