

Hand in before **Friday 20 July**

Allowed formats: handwritten or printed or pdf [pdf by email]

Gjerrit Meinsma, g.meinsma@utwente.nl, Room 301, Lady Davis Bld.

(I will not be in my office during 12–19 July).

consult the July version of the lecture notes

1. Problem 2.7
2. Problem 3.18
3. Let $T : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be the time dependent temperature distribution in a wall of one meter thick. Assuming the material in the wall is uniform, the partial differential equation describing $T(x, t)$ is

$$\begin{cases} T_t(t, x) &= a T_{xx}(t, x), \\ \lambda T_x(t, 0) &= a_{\text{in}}(T(t, 0) - T_{\text{in}}) \\ \lambda T_x(t, 1) &= a_{\text{out}}(T_{\text{out}} - T(t, 1)) \end{cases}$$

where T_{in} and T_{out} are inside and outside temperatures (assumed constant) and $\lambda, a, a_{\text{in}}, a_{\text{out}}$ are positive constants and T_{xx} means second derivative with respect to place x and T_t first derivative with respect to time, etcetera.

Show that given $(T_{\text{in}}, T_{\text{out}}, \lambda, a, a_{\text{in}}, a_{\text{out}})$ the set of all possible temperatures T is an affine variety in some appropriate vector space. (You may impose smoothness assumptions on T .)

4. Consider the space of 2D polynomials $p : \mathbb{R}^2 \rightarrow \mathbb{R}$ of degree two

$$\begin{aligned} \mathbb{P}_2^2 &= \{p : \mathbb{R}^2 \rightarrow \mathbb{R} \mid \exists a, b, c, d, e, f \in \mathbb{R} : \\ &\quad p(x, y) = a + bx + cy + dxy + ex^2 + fy^2 \quad \forall x, y\} \end{aligned}$$

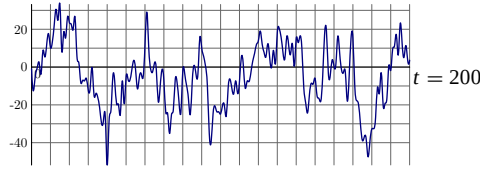
On this space we define the operator

$$q = \mathcal{F}(p) : \quad q(x, y) = p(y, x).$$

- a) Find a basis for $\mathbb{P}_2^2$
 - b) Determine the matrix $F_{\sharp}$ of $\mathcal{F}$ with respect to this basis
 - c) Determine the eigenvalues of $\mathcal{F}$ and specify for each eigenvalue all possible eigenfunctions $p \in \mathbb{P}_2^2$.
5. For which continuous $w : [0, 1] \rightarrow \mathbb{R}$ is $\langle f, g \rangle := \int_0^1 f(t)g(t)w(t) dt$ an inner product (on the vector space of continuous functions from $[0, 1]$ to $\mathbb{R}$)?
 6. Show that the stable causal delay-differential system

$$\dot{y}(t) + \frac{1}{2}y(t - \pi/2) = \frac{1}{2}u(t) - \dot{u}(t - \pi/2)$$

is $\mathbb{L}^2$ -norm preserving (and assume that u and y are both zero for negative time; this is just to ensure that y is function of u i.e. that there is no homogeneous part in y).



7. The above plot depicts a realization $y[0], \dots, y[200]$ of a stochastic process y . It was generated by *one* of the three schemes below with zero mean, unit variance white noise u :

$$y[t] = 1.8y[t-1] + u[t], \quad y[t] = 0.8y[t-1] + u[t], \quad y[t] = 0.8y[t-1] + 10u[t].$$

Explain which one the three was used and why the other two are unlikely.

8. Consider function space $\mathbb{L}_{\mathbb{R}}^2[0, 1]$. Use the Legendre polynomials to find the best approximation in $\text{span}\{1, t, t^2\}$ of t^3 . [So I insist you to do it using Legendre; you may use everything claimed in Example 5.12.3.]
9. Rewrite sections 10.2 and 10.3 (including proofs) for the case that *Problem-5* (page 101; also known as *Application-5*) is reformulated as:

Given matrices A, B and initial state $x(0) = x_0$ and final time $T > 0$, find an input $u : [0, T] \rightarrow \mathbb{R}^{n_u}$ that achieves $x(T) = 0$.

10. Consider Section 12.9 on matched filters in noise, but suppose now that the noise η is not white but is of the form

$$\eta = \frac{1}{1 + az^{-1}} w, \quad \text{with } w \text{ zero mean white noise and } |a| < 1.$$

What would the “optimal” filter $y = \mathcal{H}(v + \eta)$ now be (in terms of known x and number a)? [There may not be a unique “optimal” solution. Convince me why yours is good.]

11. Download the image 64×64 image `leonid.jpg` and the MATLAB script `coscos.m` from the course website. The script explains how to make, read and write images in MATLAB.

- a) Read the sample script. Take from now on $N = 64$ (as in the script). Are the basis functions

$$\phi_{kn}(t, s) = \cos(\pi k \frac{t+1/2}{N}) \cos(\pi n \frac{s+1/2}{N}), \quad t, s = \{0, 1, \dots, N-1\}$$

orthogonal in the standard inner product on $\mathbb{R}^{64 \times 64}$? [A matlab proof is fine!]

- b) normalize the basis elements to have unit norm.

- c) Determine the fourier coefficients $\hat{f}_{kn}$ —Eqn. (13.4) of lecture notes—of `leonid.jpg` for $k, n \in \{0, 1, \dots, 39\}$ (in MATLAB).

- d) Determine the chopped Fourier series

$$f_*(t, s) = \sum_{0 \leq k, n < 40} \hat{f}_{kn} \cos(\pi k \frac{t+1/2}{N}) \cos(\pi n \frac{s+1/2}{N}), \quad t, s = \{0, 1, \dots, N-1\}.$$

and show the result [send the final MATLAB script to me].

- e) Looking at the final result, what is “typical Fourier” about the above chopped Fourier series?

Problem:	1	2	3	4	5	6	7	8	9	10	11
Points:	8	8	10	10	8	8	8	10	10	10	10

Total number of points = 100.