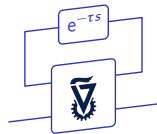


Time Delays in Control and Estimation (038806)



lecture no. 12

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Outline

Posicast control

Generating FIR responses

Input shaping

Optimal FIR tracking

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Problem

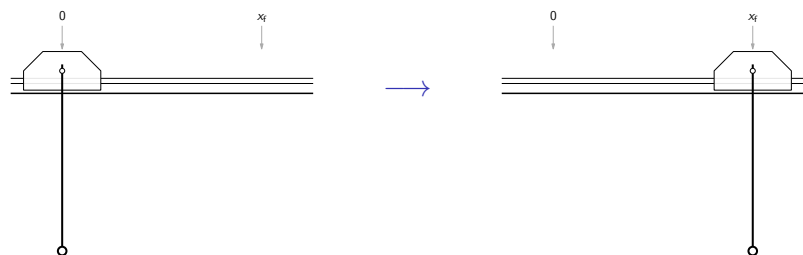
Undamped pendulum on a cart

- control input: cart position x
- output: pendulum angle θ

Transfer function of the linearized plant

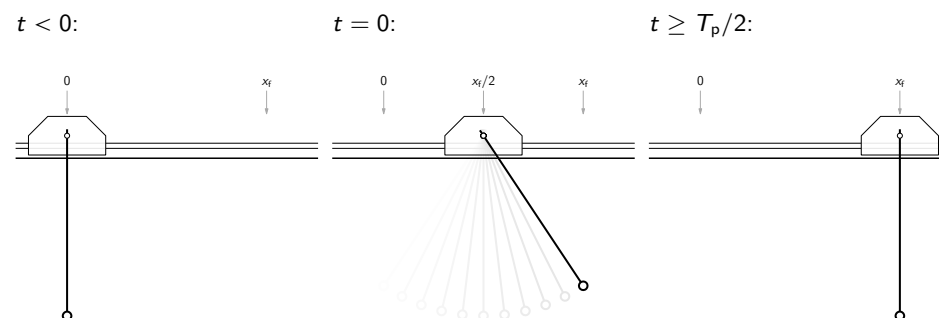
$$P(s) = \frac{s^2}{ls^2 + g},$$

where l is the pendulum length (period $T_p := 2\pi\sqrt{l/g}$). Our goal:



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Idea (by Otto J. M. Smith, again)



and we settle to steady state at the finite $t = T_p/2$.

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Analysis

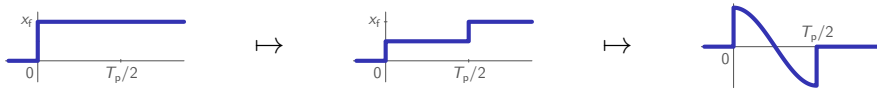
x is the response of

$$R(s) = \frac{1}{2} + \frac{1}{2} e^{-T_p s/2}$$

to (scaled step) $x_f \mathbb{1}$. Controlled dynamics

$$P(s)R(s) = \frac{s^2(1 + e^{-T_p s/2})}{2(ls^2 + g)}$$

have no poles, singularities at $s = \pm j2\pi/T_p$ are canceled by zeros of $R(s)$.



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Functions with no poles

Let

$$G(s) = \frac{N_0(s) + N_1(s)e^{-\tau_1 s} + \dots + N_n e^{-\tau_n s}}{D(s)} \quad (*)$$

for some polynomials $D(s)$ and $N_i(s)$ such that

- $\deg D(s) > \deg N_i(s)$, $\forall i = 0, \dots, n$,
- all roots s_i of $D(s)$ are simple¹,

and $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_n$. For every $j \in \mathbb{Z}_{0..n}$

$$\frac{N_j(s)}{D(s)} = \sum_i \frac{\alpha_{ij}}{s - s_i}$$

where

$$\alpha_{ij} = \text{Res}\left(\frac{N_j(s)}{D(s)}; s_i\right) := \lim_{s \rightarrow s_i} \frac{(s - s_i)N_j(s)}{D(s)}$$

is the residue of $N_j(s)/D(s)$ at its pole $s = s_i$.

¹This is assumed to shorten formulae, all arguments extend to the general case as well.

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Functions with no poles (contd)

Hence,

$$G(s) = \sum_{j=0}^n \frac{N_j(s)e^{-\tau_j s}}{D(s)} = \sum_{j=0}^n \sum_i \frac{\alpha_{ij} e^{-\tau_j s}}{s - s_i} = \sum_i \frac{\sum_{j=0}^n \alpha_{ij} e^{-\tau_j s}}{s - s_i}.$$

Let

$$\gamma_i(s) := \sum_{j=0}^n \alpha_{ij} e^{-\tau_j s} = \alpha_{i0} + \alpha_{i1} e^{-\tau_1 s} + \dots + \alpha_{in} e^{-\tau_n s}$$

The singularity $s = s_i$ of $G(s)$ is

- **removable** iff $\gamma_i(s_i) = \alpha_{i0} + \alpha_{i1} e^{-\tau_1 s_i} + \dots + \alpha_{in} e^{-\tau_n s_i} = 0$.

If all s_i are removable, then $G(s)$ is **entire**. It can also be shown that $G(s)$ is bounded in $\mathbb{C}_0$, hence

$$G \in H_\infty,$$

i.e. stable, whenever $\gamma_i(s_i) = 0$ for all i .

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Functions with no poles (contd)

Let $G(s)$ as in $(*)$ be entire. In this case $g = \mathcal{L}^{-1}\{G\}$ satisfies

$$\begin{aligned} g(t) &= \sum_i \sum_{j=0}^n \alpha_{ij} e^{s_i(t-\tau_j)} \mathbb{1}(t - \tau_j) = \sum_i e^{s_i t} \sum_{j=0}^n \alpha_{ij} e^{-\tau_j s_i} \mathbb{1}(t - \tau_j) \\ &= \sum_i e^{s_i t} \left(\sum_{j=0}^{n-1} \alpha_{ij} e^{-\tau_j s_i} (\mathbb{1}(t - \tau_j) - \mathbb{1}(t - \tau_n)) + \underbrace{\gamma_i(s_i)}_{=0} \mathbb{1}(t - \tau_n) \right) \\ &= \sum_i e^{s_i t} \sum_{j=0}^{n-1} \alpha_{ij} e^{-\tau_j s_i} (\mathbb{1}(t - \tau_j) - \mathbb{1}(t - \tau_n)). \end{aligned}$$

Thus, $g(t) = 0$ for all $t > \tau_n$. If $G(s)$ is transfer function of an LTI system G , its impulse response g has finite support, at most $[0, \tau_n]$ and G is **FIR**.

Remark: The truncation $\pi_\tau\{G\}$ and completion $\sigma_\tau\{G\bar{D}_\tau\}$ operators are special cases of such constructions for $n = 1$.

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Step responses of FIR systems

Let $G(s)$ as in (*) be the transfer function of an LTI G . Its step response

$$Y(s) = \frac{G(s)}{s} = \frac{G(0)}{s} + \frac{G(s) - G(0)}{s} =: \frac{G(0)}{s} + \tilde{G}(s),$$

where

$$\tilde{G}(s) = \frac{\overbrace{N_0(s) - G(0)D(s)}^{\tilde{N}_0(s)} + N_1(s)e^{-\tau_1 s} + \dots + N_\nu e^{-\tau_\nu s}}{sD(s)}$$

is also entire of form (*). Hence, the step response in the time domain,

$$y(t) = G(0)\mathbb{1}(t) + \tilde{g}(t),$$

converges to its steady-state value $G(0)$ in finite time (at most $t = \tau_\nu$).

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Damped pendulum

Extends to

$$P(s) = \frac{s^2}{ls^2 + 2cs + g}$$

with $c^2 < gl$ and poles at $s = -\sigma \pm j\omega$, where

$$\sigma := \frac{c}{l} \quad \text{and} \quad \omega := \sqrt{\frac{g}{l} - \sigma^2} = \frac{2\pi}{T_p},$$

Select

$$R(s) = \phi_0 + \phi_1 e^{-\tau s}$$

and try to meet

- $R(-\sigma \pm j\omega) = 0$, for dead-beat response
- $R(0) = 1$, for required steady state

in $\phi_0, \phi_1 \in \mathbb{R}$ and $\tau > 0$.

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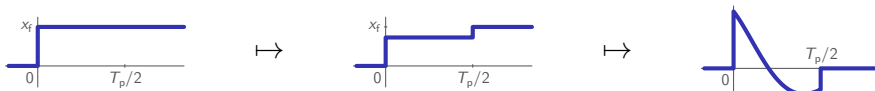
Design

Linear constraints

$$\begin{bmatrix} 1 & 1 \\ 1 & e^{\tau\sigma} e^{-j\tau\omega} \\ 1 & e^{\tau\sigma} e^{j\tau\omega} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & e^{\tau\sigma} \cos(\tau\omega) - je^{\tau\sigma} \sin(\tau\omega) \\ 1 & e^{\tau\sigma} \cos(\tau\omega) + je^{\tau\sigma} \sin(\tau\omega) \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

solvable iff $\sin(\tau\omega) = 0$. For example, if $\tau = \pi/\omega = T_p/2$, then

$$\begin{bmatrix} 1 & 1 \\ 1 & -e^{\sigma T_p/2} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} e^{\sigma T_p/2} \\ 1 \end{bmatrix} \frac{1}{1 + e^{\sigma T_p/2}}$$



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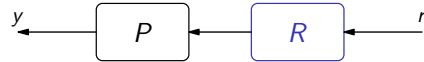
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General problem

Given a finite-dimensional P , find

$$R = \sum_{i=0}^v \phi_i \bar{D}_{\tau_i}, \quad 0 = \tau_0 < \tau_1 < \dots < \tau_v,$$

such that the series



is FIR.

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Solution

If $P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ is minimal, then PR is FIR iff

$$\underbrace{\begin{bmatrix} e^{A\tau_v} B & e^{A(\tau_v-\tau_1)} B & \dots & e^{A(\tau_v-\tau_{v-1})} B & B \end{bmatrix}}_{M_\tau} \underbrace{\begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{v-1} \\ \phi_v \end{bmatrix}}_{\Phi} = 0.$$

Proof: We know that (truncation)

$$P = \pi_{\tau_v-\tau_i} \{P\} + \hat{P}_i \bar{D}_{\tau_v-\tau_i}, \quad \text{where } \hat{P}(s) = \left[\begin{array}{c|c} A & e^{A(\tau_v-\tau_i)} B \\ \hline C & 0 \end{array} \right].$$

Hence,

$$P\phi_i \bar{D}_{\tau_i} = \pi_{\tau_v-\tau_i} \{P\} \phi_i \bar{D}_{\tau_i} + \hat{P}_i \phi_i \bar{D}_{\tau_v}.$$

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Solution (contd)

Proof (contd): Thus,

$$PK = \underbrace{\sum_{i=0}^v \pi_{\tau_v-\tau_i} \{P\} \phi_i \bar{D}_{\tau_i}}_{\text{FIR, in } [0, \tau_v]} + T \bar{D}_{\tau_v}$$

with the finite-dimensional

$$T(s) := \sum_{i=0}^v \hat{P}_i(s) \phi_i = C(sI - A)^{-1} \sum_{i=0}^v e^{A(\tau_v-\tau_i)} B \phi_i.$$

Therefore,

$$\sum_{i=0}^v e^{A(\tau_v-\tau_i)} B \phi_i = 0 \implies T = 0 \implies \text{FIR } PK.$$

Necessity follows by the observability of (C, A) . $\square$

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Solvability

If P is n -dimensional, then $M_\tau \in \mathbb{R}^{n \times (v+1)}$ and

- if $v \geq n$, then there are nontrivial solutions to $M_\tau \Phi = 0$ for all $\{\tau_i\}$,

$$\Phi \in \ker M_\tau$$

- if $v < n$, then solution might still exist for some $\{\tau_i\}$.

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Pendulum

Return to

$$P(s) = \frac{s^2}{ls^2 + 2cs + g} = \frac{1}{l} \left[\begin{array}{cc|c} -\sigma & \omega & 0 \\ -\omega & -\sigma & 1 \\ \hline \sigma^2/\omega - \omega & -2\sigma & 1 \end{array} \right],$$

where

$$\sigma := \frac{c}{l} \quad \text{and} \quad \omega := \sqrt{\frac{g}{l} - \sigma^2} = \frac{2\pi}{T_p}.$$

In this case

$$\exp\left(\left[\begin{array}{cc} -\sigma & \omega \\ -\omega & -\sigma \end{array}\right] t\right) = e^{-\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

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Pendulum: $\nu = 1$

In this case

$$\begin{bmatrix} e^{-\tau_1 \sigma} \sin(\tau_1 \omega) & 0 \\ e^{-\tau_1 \sigma} \cos(\tau_1 \omega) & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = 0$$

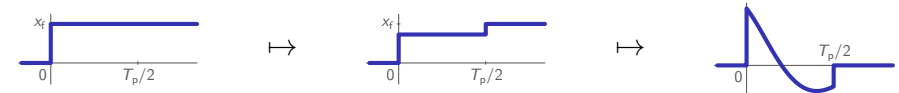
has a nontrivial solution iff

$$\sin(\tau_1 \omega) = 0 \iff \tau_1 = \frac{\pi}{\omega} k = \frac{T_p}{2} k, \quad k \in \mathbb{N}$$

which limits the choice of τ_1 . If $\tau_1 = \pi/\omega$, then two equations read

$$\begin{bmatrix} -e^{-\tau \sigma} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \iff \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} e^{\sigma T_p/2} \\ 1 \end{bmatrix} \frac{1}{1 + e^{\sigma T_p/2}}.$$

The response to $x_f \mathbb{1}(t)$ is



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Pendulum: $\nu = 2$

In this case

$$\begin{bmatrix} e^{-\tau_2 \sigma} \sin(\tau_2 \omega) & e^{-(\tau_2 - \tau_1) \sigma} \sin((\tau_2 - \tau_1) \omega) & 0 \\ e^{-\tau_2 \sigma} \cos(\tau_2 \omega) & e^{-(\tau_2 - \tau_1) \sigma} \cos((\tau_2 - \tau_1) \omega) & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix} = 0.$$

Unless $\sin(\tau_2 \omega) = \sin((\tau_2 - \tau_1) \omega) = 0$, this M_τ is a rank-two matrix with

$$\ker M_\tau = \text{Im} \begin{bmatrix} \sin((\tau_2 - \tau_1) \omega) \\ -e^{-\tau_1 \sigma} \sin(\tau_2 \omega) \\ e^{-\tau_2 \sigma} \sin(\tau_1 \omega) \end{bmatrix} \implies \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \sin((\tau_2 - \tau_1) \omega) \\ -e^{-\tau_1 \sigma} \sin(\tau_2 \omega) \\ e^{-\tau_2 \sigma} \sin(\tau_1 \omega) \end{bmatrix} \alpha$$

for any $\alpha \neq 0$. The static gain equation results in

$$\alpha = \frac{1}{\sin(\tau_2 \omega)(\cos(\tau_1 \omega) - e^{-\tau_1 \sigma}) + \sin(\tau_1 \omega)(e^{-\tau_2 \sigma} - \cos(\tau_2 \omega))},$$

which is well defined for almost all $\tau_2 > \tau_1 > 0$.

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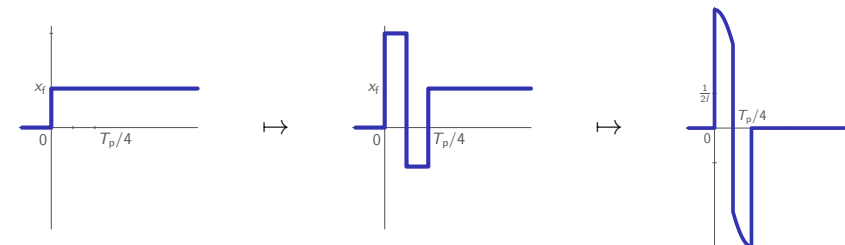
Pendulum: $\nu = 2$ (contd)

In

$$\begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \sin((\tau_2 - \tau_1) \omega) \\ -e^{-\tau_1 \sigma} \sin(\tau_2 \omega) \\ e^{-\tau_2 \sigma} \sin(\tau_1 \omega) \end{bmatrix} \alpha,$$

- if $\sin(\tau_1 \omega) = 0$, then $\phi_2 = 0$ and we effectively have $\nu = 1$;
- if $\sin(\tau_1 \omega) \neq 0$, then faster convergence is possible

let $\tau_1 = \pi/(4\omega) = T_p/8$ and $\tau_2 = \pi/(2\omega) = T_p/4$, then



If $\sin(\tau_2 \omega) = \sin((\tau_2 - \tau_1) \omega) = 0$, **additional degree of freedom** to exploit.

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Sensitivity of posicast

Let

$$P(s) = \frac{s^2}{ls^2 + 2\tilde{c}s + g} \quad \text{and} \quad R(s) = \frac{e^{\sigma T_p/2} + e^{-T_p s/2}}{1 + e^{\sigma T_p/2}}.$$

meaning assumed $\sigma = c/l$ is different from actual $\tilde{\sigma} = \tilde{c}/l$. In this case

- PR is no longer FIR $\implies$ residual oscillations

They are

$$\theta_{\text{res}}(t) = \frac{1 - e^{(\tilde{\sigma} - \sigma)T_p/2}}{1 + e^{-\sigma T_p/2}} \theta_{\text{step}}(t),$$

where $\theta_{\text{step}}(s)$ is the step response of the uncontrolled P . This is a

- linear function of $|\tilde{\sigma} - \sigma|$ for small $|\tilde{\sigma} - \sigma|$.

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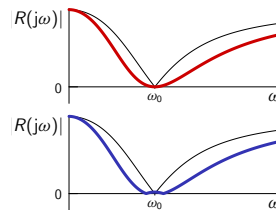
Improving robustness

Some ideas:

- multiple zeros of $R(s)$ at each plant pole²
- several zeros of $R(s)$ around each plant pole³
- adaptation mechanisms

Implications:

- ☹ increased complexity
- ☹ longer settling
- ☺ lower sensitivity to variations of parameters



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ZVD for pendulum

Now consider

$$R(s) = \phi_0 + \phi_1 e^{-\tau_1 s} + \phi_2 e^{-\tau_2 s}$$

and introduce additional constraint

$$R'(-\sigma + j\omega) = 0.$$

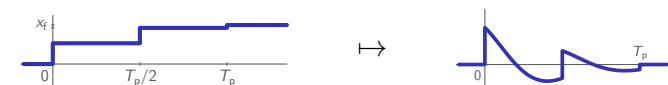
It can be imposed

- explicitly (like $-\tau_1 e^{\tau_1 \sigma} e^{\mp j\tau_1 \omega} \phi_1 - \tau_2 e^{\tau_2 \sigma} e^{\mp j\tau_2 \omega} \phi_2 = 0$)
- implicitly, design R for $P(s)/(ls^2 + 2cs + g)$

Both result in

$$R_{\text{zvd}}(s) = \left(\frac{e^{\sigma T_p/2} + e^{-T_p s/2}}{1 + e^{\sigma T_p/2}} \right)^2$$

and



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²Known as ZVD, for Zero Vibration and Derivative

³Known as EI for Extra-Insensitive

Sensitivity of ZVD

Let again

$$P(s) = \frac{s^2}{ls^2 + 2\tilde{c}s + g} \quad \text{and} \quad R(s) = \frac{e^{\sigma T_p/2} + e^{-T_p s/2}}{1 + e^{\sigma T_p/2}}.$$

meaning assumed $\sigma = c/l$ is different from actual $\tilde{\sigma} = \tilde{c}/l$. In this case

- PR is no longer FIR $\implies$ residual oscillations

They are

$$\theta_{\text{res}}(t) = \left(\frac{1 - e^{(\tilde{\sigma} - \sigma)T_p/2}}{1 + e^{-\sigma T_p/2}} \right)^2 \theta_{\text{step}}(t),$$

where $\theta_{\text{step}}(s)$ is the step response of the uncontrolled P . This is a

- quadratic function of $|\tilde{\sigma} - \sigma|$ for small $|\tilde{\sigma} - \sigma|$

and is more flat than the linear function for the posicast controller.

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Setup

Error system:

$$G_e = G_w + G_u R$$

for

$$\begin{bmatrix} G_w(s) & G_u(s) \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C & D_w & D_u \end{bmatrix}$$

assuming that (A, B_u) is stabilizable and $D_u' D_u = I$. The goal is to design R so that

- R is FIR, with the support in $[0, \tau]$
- G_e is FIR, with the support in $[0, \tau]$
- the H_2 -norm $\|G_e\|$ is minimized

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FIR condition

Lemma

If (C, A) is observable, then there is an FIR R with the impulse response r supported in $[0, \tau]$ such that G_e is also FIR with the same support iff

$$B_w + \int_0^\tau e^{-As} B_u r(s) ds = 0.$$

Proof: If R is FIR, then the impulse response of the error systems is

$$g_e(t) = D_w \delta(t) + C e^{At} B_w \mathbb{1}(t) + D_u r(t) + C \int_0^t e^{A(t-s)} B_u r(s) ds$$

If $t > \tau$, then

$$g_e(t) = C e^{At} \left(B_w + \int_0^\tau e^{-As} B_u r(s) ds \right),$$

whence the condition follows (necessity by the observability of (C, A)). $\square$

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Optimal FIR

Let

$$H = \begin{bmatrix} A & 0 \\ -C'C & -A' \end{bmatrix} + \begin{bmatrix} -B_u \\ C'D_u \end{bmatrix} \begin{bmatrix} D_u'C & B_u' \end{bmatrix}$$

Then the impulse response of the optimal controller is

$$r(t) = -(D_u'D_u)^{-1} \begin{bmatrix} D_u'C & B_u' \end{bmatrix} e^{Ht} \times \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} e^{H\tau} \right)^{-1} \begin{bmatrix} B_w \\ 0 \end{bmatrix} \mathbb{1}_{[0,\tau]}(t)$$

and the optimal cost

$$\|G_e\|_2^2 = \text{tr} \left(B_w' \begin{bmatrix} 0 & I \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} e^{H\tau} \right)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} B_w \right).$$

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Optimal FIR: example

With the pendulum P , form the regulated signal

$$\begin{bmatrix} \sqrt{1-\lambda}(1-x) \\ \sqrt{\lambda}\theta \\ \sqrt{\varrho}\dot{x} \end{bmatrix}$$

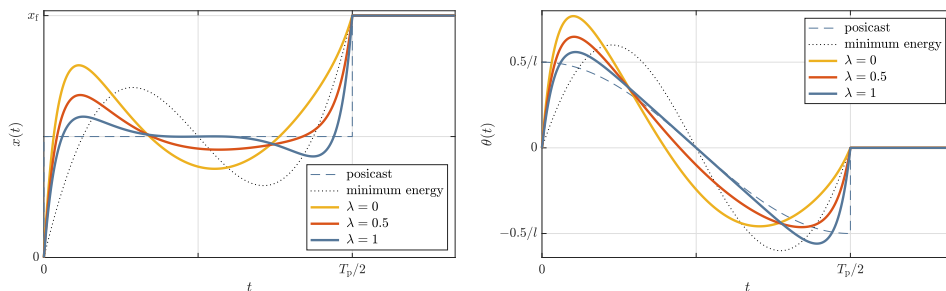
for $\lambda \in [0, 1]$ and $\varrho > 0$, so

$$\begin{bmatrix} G_w(s) & G_u(s) \end{bmatrix} = \begin{bmatrix} \sqrt{1-\lambda}/s & -\sqrt{1-\lambda}/s \\ 0 & \sqrt{\lambda}P(s)/s \\ 0 & \sqrt{\varrho} \end{bmatrix}$$

$$= \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & -1 \\ 0 & -\sigma & \omega & 0 & 0 \\ 0 & -\omega & -\sigma & 0 & 1 \\ \hline \sqrt{1-\lambda} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\lambda}\sigma/(l\omega) & \sqrt{\lambda}/l & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\varrho} \end{array} \right].$$

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Optimal FIR: example (contd)



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That's all ...

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