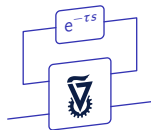


Time Delays in Control and Estimation (038806)



lecture no. 12

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Outline

Posicast control

Generating FIR responses

Input shaping

Optimal FIR tracking

Outline

Posicast control

Generating FIR responses

Input shaping

Optimal FIR tracking

Problem

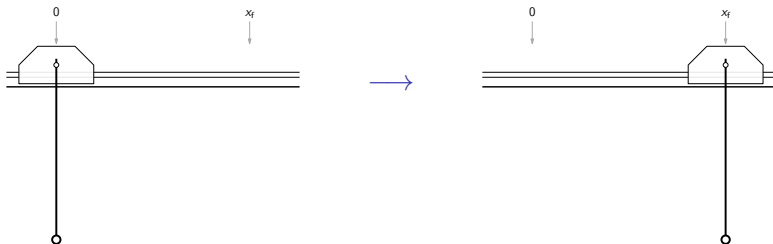
Undamped pendulum on a cart

- control input: cart position x
- output: pendulum angle θ

Transfer function of the linearized plant

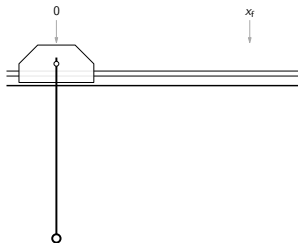
$$P(s) = \frac{s^2}{ls^2 + g},$$

where l is the pendulum length (period $T_p := 2\pi\sqrt{l/g}$). Our goal:



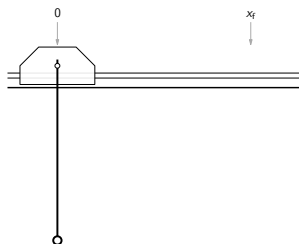
Idea (by Otto J. M. Smith, again)

$t < 0$:

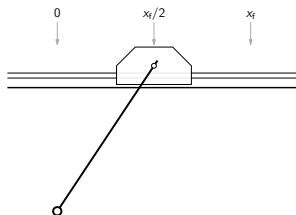


Idea (by Otto J. M. Smith, again)

$t < 0$:

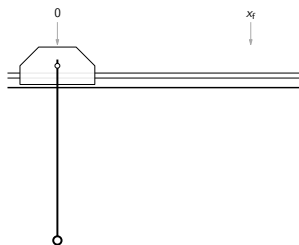


$t = 0$:

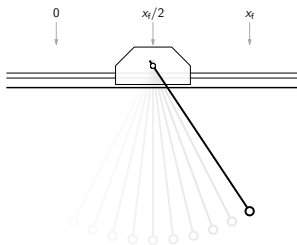


Idea (by Otto J. M. Smith, again)

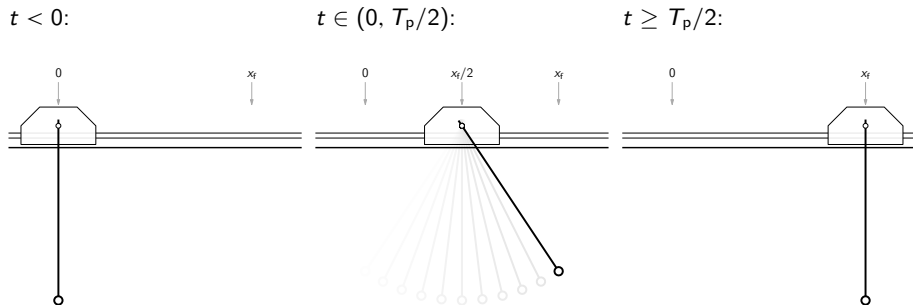
$t < 0$:



$t \in (0, T_p/2)$:



Idea (by Otto J. M. Smith, again)



and we settle to steady state at the finite $t = T_p/2$.

Analysis

x is the response of

$$R(s) = \frac{1}{2} + \frac{1}{2} e^{-T_p s/2}$$

to (scaled step) $x_f \mathbb{1}$. Controlled dynamics

$$P(s)R(s) = \frac{s^2(1 + e^{-T_p s/2})}{2(ls^2 + g)}$$

have no poles, singularities at $s = \pm j2\pi/T_p$ are canceled by zeros of $R(s)$.

Analysis

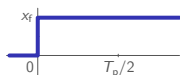
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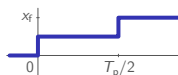
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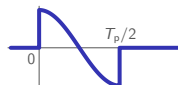
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$\mapsto$



$\mapsto$



Functions with no poles

Let

$$G(s) = \frac{N_0(s) + N_1(s)e^{-\tau_1 s} + \dots + N_n e^{-\tau_v s}}{D(s)} \quad (*)$$

for some polynomials $D(s)$ and $N_i(s)$ such that

- $\deg D(s) > \deg N_i(s)$, $\forall i = 0, \dots, v$,
- all roots s_i of $D(s)$ are simple¹,

and $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_v$. For every $j \in \mathbb{Z}_{0..v}$

$$\frac{N_j(s)}{D(s)} = \sum_i \frac{\alpha_{ij}}{s - s_i}$$

where

$$\alpha_{ij} = \text{Res}\left(\frac{N_j(s)}{D(s)}; s_i\right) := \lim_{s \rightarrow s_i} \frac{(s - s_i)N_j(s)}{D(s)}$$

is the residue of $N_j(s)/D(s)$ at its pole $s = s_i$.

¹This is assumed to shorten formulae, all arguments extend to the general case as well.

Functions with no poles (contd)

Hence,

$$G(s) = \sum_{j=0}^{\nu} \frac{N_j(s) e^{-\tau_j s}}{D(s)} = \sum_{j=0}^{\nu} \sum_i \frac{\alpha_{ij} e^{-\tau_j s}}{s - s_i} = \sum_i \frac{\sum_{j=0}^{\nu} \alpha_{ij} e^{-\tau_j s}}{s - s_i}.$$

Let

$$\gamma_i(s) := \sum_{j=0}^{\nu} \alpha_{ij} e^{-\tau_j s} = \alpha_{i0} + \alpha_{i1} e^{-\tau_1 s} + \cdots + \alpha_{i\nu} e^{-\tau_\nu s}$$

The singularity $s = s_i$ of $G(s)$ is

- **removable** iff $\gamma_i(s_i) = \alpha_{i0} + \alpha_{i1} e^{-\tau_1 s_i} + \cdots + \alpha_{i\nu} e^{-\tau_\nu s_i} = 0$.

If all s_i are removable, then $G(s)$ is **entire**. It can also be shown that $G(s)$ is bounded in $\mathbb{C}_0$, hence

$$G \in H_{\infty},$$

i.e. stable, whenever $\gamma_i(s_i) = 0$ for all i .

Functions with no poles (contd)

Hence,

$$G(s) = \sum_{j=0}^v \frac{N_j(s)e^{-\tau_j s}}{D(s)} = \sum_{j=0}^v \sum_i \frac{\alpha_{ij} e^{-\tau_j s}}{s - s_i} = \sum_i \frac{\sum_{j=0}^v \alpha_{ij} e^{-\tau_j s}}{s - s_i}.$$

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Functions with no poles (contd)

Let $G(s)$ as in (*) be entire. In this case $g = \mathcal{L}^{-1}\{G\}$ satisfies

$$\begin{aligned}g(t) &= \sum_i \sum_{j=0}^v \alpha_{ij} e^{s_i(t-\tau_j)} \mathbb{1}(t - \tau_j) = \sum_i e^{s_i t} \sum_{j=0}^v \alpha_{ij} e^{-\tau_j s_i} \mathbb{1}(t - \tau_j) \\&= \sum_i e^{s_i t} \left(\sum_{j=0}^{v-1} \alpha_{ij} e^{-\tau_j s_i} (\mathbb{1}(t - \tau_j) - \mathbb{1}(t - \tau_v)) + \underbrace{\gamma_i(s_i)}_{=0} \mathbb{1}(t - \tau_v) \right) \\&= \sum_i e^{s_i t} \sum_{j=0}^{v-1} \alpha_{ij} e^{-\tau_j s_i} (\mathbb{1}(t - \tau_j) - \mathbb{1}(t - \tau_v)).\end{aligned}$$

Thus, $g(t) = 0$ for all $t > \tau_v$. If $G(s)$ is transfer function of an LTI system G , its impulse response g has finite support, at most $[0, \tau_v]$ and G is FIR.

Functions with no poles (contd)

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 &= \sum_i e^{s_i t} \sum_{j=0}^{v-1} \alpha_{ij} e^{-\tau_j s_i} (\mathbb{1}(t-\tau_j) - \mathbb{1}(t-\tau_v)).
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Thus, $g(t) = 0$ for all $t > \tau_v$. If $G(s)$ is transfer function of an LTI system G , its impulse response g has finite support, at most $[0, \tau_v]$ and G is **FIR**.

Remark: The truncation $\pi_\tau \{G\}$ and completion $\sigma_\tau \{G \bar{D}_\tau\}$ operators are special cases of such constructions for $v = 1$.

Step responses of FIR systems

Let $G(s)$ as in $(*)$ be the transfer function of an LTI G . Its step response

$$Y(s) = \frac{G(s)}{s} = \frac{G(0)}{s} + \frac{G(s) - G(0)}{s} =: \frac{G(0)}{s} + \tilde{G}(s),$$

where

$$\tilde{G}(s) = \frac{\overbrace{N_0(s) - G(0)D(s)}^{\tilde{N}_0(s)} + N_1(s)e^{-\tau_1 s} + \dots + N_\nu e^{-\tau_\nu s}}{sD(s)}$$

is also entire of form $(*)$. Hence, the step response in the time domain,

$$y(t) = G(0)\mathbb{1}(t) + \tilde{g}(t),$$

converges to its steady-state value $G(0)$ in finite time (at most $t = \tau_\nu$).

Damped pendulum

Extends to

$$P(s) = \frac{s^2}{ls^2 + 2cs + g}$$

with $c^2 < gl$ and poles at $s = -\sigma \pm j\omega$, where

$$\sigma := \frac{c}{l} \quad \text{and} \quad \omega := \sqrt{\frac{g}{l} - \sigma^2} = \frac{2\pi}{T_p},$$

Select

$$R(s) = \phi_0 + \phi_1 e^{-\tau s}$$

and try to meet

- $R(-\sigma \pm j\omega) = 0$, for dead-beat response
- $R(0) = 1$, for required steady state

in $\phi_0, \phi_1 \in \mathbb{R}$ and $\tau > 0$.

Design

Linear constraints

$$\begin{bmatrix} 1 & 1 \\ 1 & e^{\tau\sigma} e^{-j\tau\omega} \\ 1 & e^{\tau\sigma} e^{j\tau\omega} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & e^{\tau\sigma} \cos(\tau\omega) - j e^{\tau\sigma} \sin(\tau\omega) \\ 1 & e^{\tau\sigma} \cos(\tau\omega) + j e^{\tau\sigma} \sin(\tau\omega) \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

solvable iff $\sin(\tau\omega) = 0$. For example, if $\tau = \pi/\omega = T_p/2$, then

$$\begin{bmatrix} 1 & 1 \\ 1 & -e^{\sigma T_p/2} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} e^{\sigma T_p/2} \\ 1 \end{bmatrix} \frac{1}{1 + e^{\sigma T_p/2}}$$

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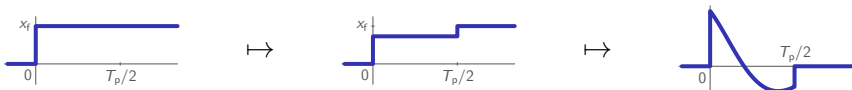
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Posicast control

Generating FIR responses

Input shaping

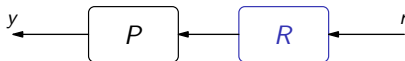
Optimal FIR tracking

General problem

Given a finite-dimensional P , find

$$R = \sum_{i=0}^v \phi_i \bar{D}_{\tau_i}, \quad 0 = \tau_0 < \tau_1 < \cdots < \tau_v,$$

such that the series



is FIR.

Solution

If $P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ is minimal, then PR is FIR iff

$$\underbrace{\left[e^{A\tau_v} B \quad e^{A(\tau_v - \tau_1)} B \quad \dots \quad e^{A(\tau_v - \tau_{v-1})} B \quad B \right]}_{M_\tau} \underbrace{\begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{v-1} \\ \phi_v \end{bmatrix}}_{\Phi} = 0.$$

Proof: We know that (truncation)

$$P = \pi_{\tau_v - \tau_1} \{P\} + \hat{P}_1 \bar{D}_{\tau_v - \tau_1}, \quad \text{where } \hat{P}(s) = \left[\begin{array}{c|c} A & e^{A(\tau_v - \tau_1)} B \\ \hline C & 0 \end{array} \right].$$

Hence,

$$P\phi_1 \bar{D}_{\tau_1} = \pi_{\tau_v - \tau_1} \{P\} \phi_1 \bar{D}_{\tau_1} + \hat{P}_1 \phi_1 \bar{D}_{\tau_v - \tau_1}.$$

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Solution (contd)

Proof (contd): Thus,

$$PK = \underbrace{\sum_{i=0}^v \pi_{\tau_v - \tau_i} \{P\} \phi_i \bar{D}_{\tau_i}}_{\text{FIR, in } [0, \tau_v]} + T \bar{D}_{\tau_v}$$

with the finite-dimensional

$$T(s) := \sum_{i=0}^v \hat{P}_i(s) \phi_i = C(sI - A)^{-1} \sum_{i=0}^v e^{A(\tau_v - \tau_i)} B \phi_i.$$

Therefore,

$$\sum_{i=0}^v e^{A(\tau_v - \tau_i)} B \phi_i = 0 \implies T = 0 \implies \text{FIR } PK.$$

Necessity follows by the observability of (C, A) .



Solvability

If P is n -dimensional, then $M_\tau \in \mathbb{R}^{n \times (\nu+1)}$ and

- if $\nu \geq n$, then there are nontrivial solutions to $M_\tau \Phi = 0$ for all $\{\tau_i\}$,

$$\Phi \in \ker M_\tau$$

- if $\nu < n$, then solution might still exist for some $\{\tau_i\}$.

Pendulum

Return to

$$P(s) = \frac{s^2}{ls^2 + 2cs + g} = \frac{1}{l} \left[\begin{array}{cc|c} -\sigma & \omega & 0 \\ -\omega & -\sigma & 1 \\ \hline \sigma^2/\omega - \omega & -2\sigma & 1 \end{array} \right],$$

where

$$\sigma := \frac{c}{l} \quad \text{and} \quad \omega := \sqrt{\frac{g}{l} - \sigma^2} = \frac{2\pi}{T_p}.$$

In this case

$$\exp\left(\left[\begin{array}{cc} -\sigma & \omega \\ -\omega & -\sigma \end{array}\right] t\right) = e^{-\sigma t} \left[\begin{array}{cc} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{array}\right].$$

Pendulum: $\nu = 1$

In this case

$$\begin{bmatrix} e^{-\tau_1\sigma} \sin(\tau_1\omega) & 0 \\ e^{-\tau_1\sigma} \cos(\tau_1\omega) & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = 0$$

has a nontrivial solution iff

$$\sin(\tau_1\omega) = 0 \iff \tau_1 = \frac{\pi}{\omega}k = \frac{T_p}{2}k, \quad k \in \mathbb{N}$$

which limits the choice of τ_1 . If $\tau_1 = \pi/\omega$, then two equations read

$$\begin{bmatrix} -e^{-\tau\sigma} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \iff \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} e^{\sigma T_p/2} \\ 1 \end{bmatrix} \frac{1}{1 + e^{\sigma T_p/2}}$$

The response to $x_1(t)$ is

Pendulum: $\nu = 1$

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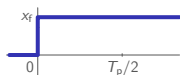
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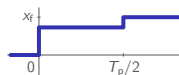
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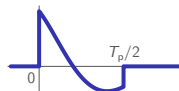
The response to $x_f \mathbb{1}(t)$ is



$\mapsto$



$\mapsto$



Pendulum: $\nu = 2$

In this case

$$\begin{bmatrix} e^{-\tau_2\sigma} \sin(\tau_2\omega) & e^{-(\tau_2-\tau_1)\sigma} \sin((\tau_2-\tau_1)\omega) & 0 \\ e^{-\tau_2\sigma} \cos(\tau_2\omega) & e^{-(\tau_2-\tau_1)\sigma} \cos((\tau_2-\tau_1)\omega) & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix} = 0.$$

Unless $\sin(\tau_2\omega) = \sin((\tau_2-\tau_1)\omega) = 0$, this M_τ is a rank-two matrix with

$$\ker M_\tau = \text{Im} \begin{bmatrix} \sin((\tau_2-\tau_1)\omega) \\ -e^{-\tau_1\sigma} \sin(\tau_2\omega) \\ e^{-\tau_2\sigma} \sin(\tau_1\omega) \end{bmatrix} \implies \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \sin((\tau_2-\tau_1)\omega) \\ -e^{-\tau_1\sigma} \sin(\tau_2\omega) \\ e^{-\tau_2\sigma} \sin(\tau_1\omega) \end{bmatrix} \alpha$$

for any $\alpha \neq 0$. The static gain equation results in

$$\alpha = \frac{1}{\sin(\tau_2\omega)(\cos(\tau_1\omega) - e^{-\tau_1\sigma}) + \sin(\tau_1\omega)(e^{-\tau_2\sigma} - \cos(\tau_2\omega))},$$

which is well defined for almost all $\tau_2 > \tau_1 > 0$.

Pendulum: $\nu = 2$ (contd)

In

$$\begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \sin((\tau_2 - \tau_1)\omega) \\ -e^{-\tau_1\sigma} \sin(\tau_2\omega) \\ e^{-\tau_2\sigma} \sin(\tau_1\omega) \end{bmatrix} \alpha,$$

- if $\sin(\tau_1\omega) = 0$, then $\phi_2 = 0$ and we effectively have $\nu = 1$;
- if $\sin(\tau_1\omega) \neq 0$, then faster convergence is possible

let $\tau_1 = \pi/(4\omega) = T_p/8$ and $\tau_2 = \pi/(2\omega) = T_p/4$, then

If $\sin(\tau_2\omega) = \sin((\tau_2 - \tau_1)\omega) = 0$, additional degree of freedom to exploit.

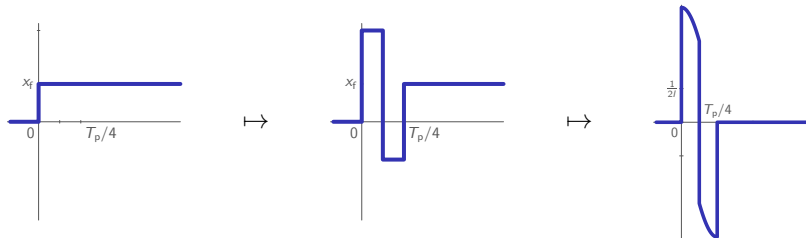
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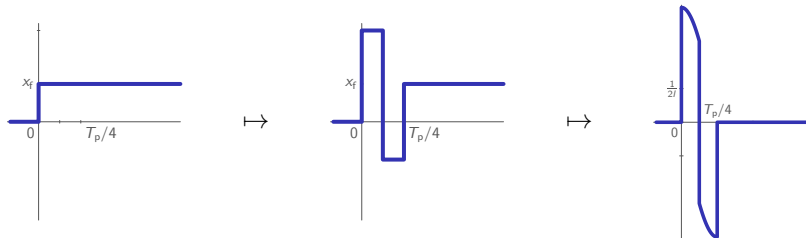
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If $\sin(\tau_2\omega) = \sin((\tau_2 - \tau_1)\omega) = 0$, **additional degree of freedom** to exploit.

Outline

Posicast control

Generating FIR responses

Input shaping

Optimal FIR tracking

Sensitivity of posicast

Let

$$P(s) = \frac{s^2}{ls^2 + 2\tilde{c}s + g} \quad \text{and} \quad R(s) = \frac{e^{\sigma T_p/2} + e^{-T_p s/2}}{1 + e^{\sigma T_p/2}}.$$

meaning assumed $\sigma = c/l$ is different from actual $\tilde{\sigma} = \tilde{c}/l$. In this case

- PR is no longer FIR $\implies$ residual oscillations

They are

$$\theta_{\text{res}}(t) = \frac{1 - e^{(\tilde{\sigma} - \sigma)T_p/2}}{1 + e^{-\sigma T_p/2}} \theta_{\text{step}}(t),$$

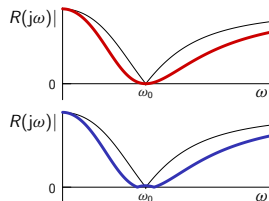
where $\theta_{\text{step}}(s)$ is the step response of the uncontrolled P . This is a

- linear function of $|\tilde{\sigma} - \sigma|$ for small $|\tilde{\sigma} - \sigma|$.

Improving robustness

Some ideas:

- multiple zeros of $R(s)$ at each plant pole²
- several zeros of $R(s)$ around each plant pole³
- adaptation mechanisms



Implications:

- increased complexity
- longer settling
- lower sensitivity to variations of parameters

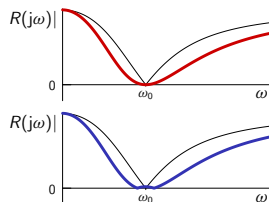
²Known as ZVD, for Zero Vibration and Derivative

³Known as EI for Extra-Insensitive

Improving robustness

Some ideas:

- multiple zeros of $R(s)$ at each plant pole²
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Implications:

- ☹ increased complexity
- ☹ longer settling
- ☺ lower sensitivity to variations of parameters

²Known as ZVD, for Zero Vibration and Derivative

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ZVD for pendulum

Now consider

$$R(s) = \phi_0 + \phi_1 e^{-\tau_1 s} + \phi_2 e^{-\tau_2 s}$$

and introduce additional constraint

$$R'(-\sigma + j\omega) = 0.$$

It can be imposed

- explicitly (like $-\tau_1 e^{\tau_1 \sigma} e^{\mp j \tau_1 \omega} \phi_1 - \tau_2 e^{\tau_2 \sigma} e^{\mp j \tau_2 \omega} \phi_2 = 0$)
- implicitly, design R for $P(s)/(ls^2 + 2cs + g)$

Both result in

$$R_{\text{zvd}}(s) = \left(\frac{e^{\sigma T_0/2} + e^{-T_0 s/2}}{1 + e^{\sigma T_0/2}} \right)^2$$

and

ZVD for pendulum

Now consider

$$R(s) = \phi_0 + \phi_1 e^{-\tau_1 s} + \phi_2 e^{-\tau_2 s}$$

and introduce additional constraint

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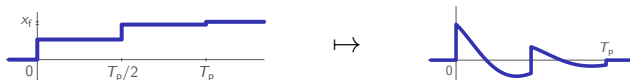
It can be imposed

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$$R_{\text{zvd}}(s) = \left(\frac{e^{\sigma T_p/2} + e^{-T_p s/2}}{1 + e^{\sigma T_p/2}} \right)^2$$

and



Sensitivity of ZVD

Let again

$$P(s) = \frac{s^2}{ls^2 + 2\tilde{\sigma}s + g} \quad \text{and} \quad R(s) = \frac{e^{\sigma T_p/2} + e^{-T_p s/2}}{1 + e^{\sigma T_p/2}}.$$

meaning assumed $\sigma = c/l$ is different from actual $\tilde{\sigma} = \tilde{c}/l$. In this case

- PR is no longer FIR $\implies$ residual oscillations

They are

$$\theta_{\text{res}}(t) = \left(\frac{1 - e^{(\tilde{\sigma} - \sigma)T_p/2}}{1 + e^{-\sigma T_p/2}} \right)^2 \theta_{\text{step}}(t),$$

where $\theta_{\text{step}}(s)$ is the step response of the uncontrolled P . This is a

- quadratic function of $|\tilde{\sigma} - \sigma|$ for small $|\tilde{\sigma} - \sigma|$

and is more flat than the linear function for the posicast controller.

Outline

Posicast control

Generating FIR responses

Input shaping

Optimal FIR tracking

Setup

Error system:

$$G_e = G_w + G_u R$$

for

$$\begin{bmatrix} G_w(s) & G_u(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C & D_w & D_u \end{array} \right]$$

assuming that (A, B_u) is stabilizable and $D_u' D_u = I$. The goal is to design R so that

- R is FIR, with the support in $[0, \tau]$
- G_e is FIR, with the support in $[0, \tau]$
- the H_2 -norm $\|G_e\|$ is minimized

FIR condition

Lemma

If (C, A) is observable, then there is an FIR R with the impulse response r supported in $[0, \tau]$ such that G_e is also FIR with the same support iff

$$B_w + \int_0^\tau e^{-As} B_u r(s) ds = 0.$$

Proof: If R is FIR, then the impulse response of the error systems is

$$g_e(t) = D_w \delta(t) + C e^{At} B_w \mathbb{1}(t) + D_u r(t) + C \int_0^t e^{A(t-s)} B_u r(s) ds$$

If $t > \tau$, then

$$g_e(t) = C e^{At} \left(B_w + \int_0^\tau e^{-As} B_u r(s) ds \right),$$

whence the condition follows (necessity by the observability of (C, A)). $\square$

Optimal FIR

Let

$$H = \begin{bmatrix} A & 0 \\ -C'C & -A' \end{bmatrix} + \begin{bmatrix} -B_u \\ C'D_u \end{bmatrix} \begin{bmatrix} D_u'C & B_u' \end{bmatrix}$$

Then the impulse response of the optimal controller is

$$\begin{aligned} r(t) = & -(D_u'D_u)^{-1} \begin{bmatrix} D_u'C & B_u' \end{bmatrix} e^{Ht} \\ & \times \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} e^{H\tau} \right)^{-1} \begin{bmatrix} B_w \\ 0 \end{bmatrix} \mathbb{1}_{[0,\tau]}(t) \end{aligned}$$

and the optimal cost

$$\|G_e\|_2^2 = \text{tr} \left(B_w' \begin{bmatrix} 0 & I \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} e^{H\tau} \right)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} B_w \right).$$

Optimal FIR: example

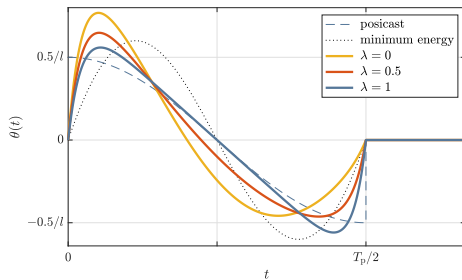
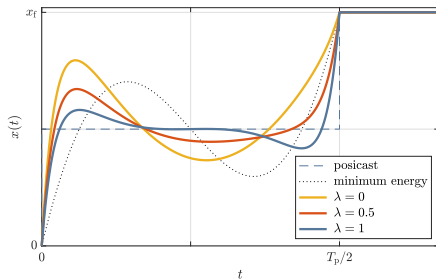
With the pendulum P , form the regulated signal

$$\begin{bmatrix} \sqrt{1-\lambda}(1-x) \\ \sqrt{\lambda}\theta \\ \sqrt{\varrho}\dot{x} \end{bmatrix}$$

for $\lambda \in [0, 1]$ and $\varrho > 0$, so

$$\begin{aligned} \begin{bmatrix} G_w(s) & G_u(s) \end{bmatrix} &= \begin{bmatrix} \sqrt{1-\lambda}/s & -\sqrt{1-\lambda}/s \\ 0 & \sqrt{\lambda}P(s)/s \\ 0 & \sqrt{\varrho} \end{bmatrix} \\ &= \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & -1 \\ 0 & -\sigma & \omega & 0 & 0 \\ 0 & -\omega & -\sigma & 0 & 1 \\ \hline \sqrt{1-\lambda} & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\lambda}\sigma/(l\omega) & \sqrt{\lambda}/l & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\varrho} \end{array} \right]. \end{aligned}$$

Optimal FIR: example (contd)



That's all ...