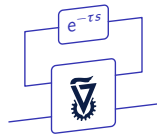


Time Delays in Control and Estimation (038806)



lecture no. 11

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

1/36

Outline

Delays that help (or so we thought)

Repetitive control

Repetitive control: stabilization

Case studies

2/36

... because we can

Essentially, two directions:

1. approximating derivatives, like

$$s \approx \Delta_\tau(s) := \frac{1 - e^{-\tau s}}{\tau}$$

like

- proportional minus delay controller instead of PID
- pseudo-paradoxical stabilization by P+delay where P does not succeed
- ...

2. control of lightly-damped systems (phase lag might be advantageous)
proposed by Wie and Byun (1989)

3/36

Approximating derivatives

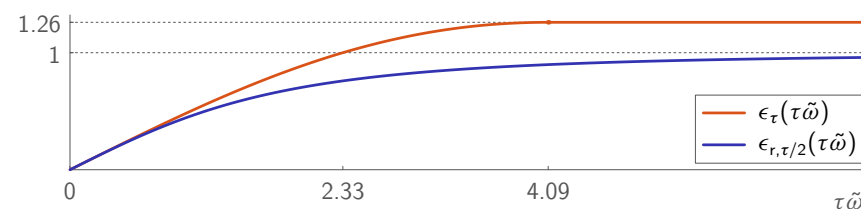
Consider two options:

$$s \approx \Delta_\tau(s) := \frac{1 - e^{-\tau s}}{\tau} \quad \text{vs.} \quad s \approx \Delta_{r,T}(s) := \frac{s}{Ts + 1}.$$

They have the same high-frequency gain iff $T = \tau/2$.

Accuracy over finite frequency range $\omega \in [0, \tilde{\omega}]$:

$$\epsilon_\tau(\tilde{\omega}) := \max_{\omega \in [0, \tilde{\omega}]} \left| 1 - \frac{\Delta_\tau(j\omega)}{j\omega} \right| \quad \text{and} \quad \epsilon_{r,\tau/2}(\tilde{\omega}) := \max_{\omega \in [0, \tilde{\omega}]} \left| 1 - \frac{\Delta_{r,\tau/2}(j\omega)}{j\omega} \right|$$



4/36

Stabilizing delays

Let

$$P(s) = \frac{1}{ms^2 + k} \quad \text{and} \quad R(s) = -k_p e^{-\tau s}.$$

Its

$$\chi_\tau(s) = ms^2 + k + k_p e^{-\tau s}$$

is unstable for $\tau = 0$. Gain and phase relations:

$$|k - m\omega^2| = |k_p| \quad \text{and} \quad \tau\omega = \arg k_p - \arg(k - m\omega^2) + (2i - 1)\pi.$$

Crossing frequencies

$$\omega^2 = \frac{k \pm |k_p|}{m}.$$

Hence,

- if $|k_p| \geq k \Rightarrow$ one nonzero crossover $\Rightarrow$ switch $\Rightarrow$ unstable $\forall \tau$
- if $|k_p| < k \Rightarrow$ two crossover frequencies, $\omega_{1,2} = \sqrt{(k \pm |k_p|)/m}$

5/36

Stabilizing delays (contd)

Crossing delays (switches and reversals):

$$\tau_{1i} = \frac{2\pi}{\omega_1} i + \frac{1}{\omega_1} \begin{cases} \pi & \text{if } k_p < 0 \\ 0 & \text{if } k_p > 0 \end{cases} \quad \text{and} \quad \tau_{2i} = \frac{2\pi}{\omega_2} i + \frac{1}{\omega_2} \begin{cases} 0 & \text{if } k_p < 0 \\ \pi & \text{if } k_p > 0 \end{cases}$$

As $\tau_{1,i+1} - \tau_{1i} < \tau_{2,i+1} - \tau_{2i}$, $\chi_\tau(s)$, stable only if $\tau_{20} < \tau_{10}$. Hence,

- never stable if $k_p > 0$
- always stable for sufficiently small τ if $k_p < 0$

Thus, if $-k < k_p < 0$, then

- system stable at least in $\tau \in (0, \pi \sqrt{m/(k - k_p)})$, but not for $\tau = 0$.

Although made a splash,

- easily stabilizable by 1-order feedback, with no apparent advantages of delayed feedback.

6/36

Outline

Delays that help (or so we thought)

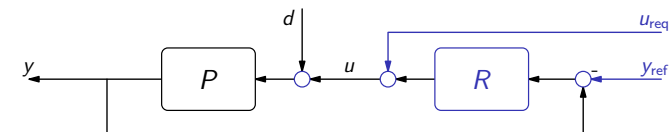
Repetitive control

Repetitive control: stabilization

Case studies

7/36

2DOF control architecture



has

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} y_{\text{ref}} \\ u_{\text{req}} \end{bmatrix} + \begin{bmatrix} I \\ R \end{bmatrix} S(Pd - y_{\text{ref}} + Pu_{\text{req}}), \quad S = (I - PR)^{-1}$$

Nominal case (P known, $d = 0$): choose consistent $y_{\text{ref}} = Pu_{\text{req}}$, for which

$$y = y_{\text{ref}} \quad \text{and} \quad u = u_{\text{req}}$$

regardless the choice of R (provided it's stabilizing).

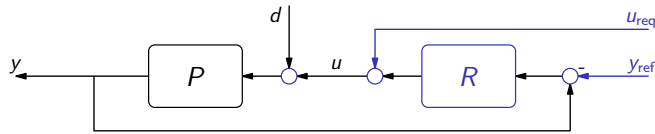
Realistic case (P uncertain, $d \neq 0$): if the closed-loop system stable, then

$$e_y := y - y_{\text{ref}} = S(Pd - y_{\text{ref}} + Pu_{\text{req}}) = T_d d - Sy_{\text{ref}} + T_d u_{\text{req}}$$

can be affected by R .

8/36

The marvel of high-gain feedback



If $R \rightarrow \infty$, in loose terms, then

$$S = (I - PR)^{-1} \rightarrow 0 \quad \text{and} \quad T_d = (I - PR)^{-1}P \rightarrow 0$$

and therefore

$$e_y \rightarrow 0$$

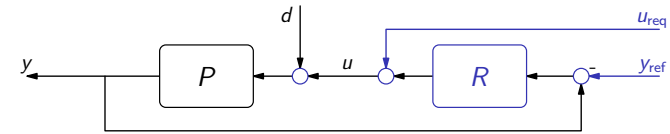
as if we knew P and measured d .

But,

- stability
- measurement noise sensitivity

9/36

Relaxing requirements: (asymptotic) regulator problem



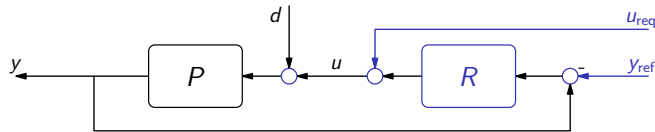
Design R , which

- (internally) stabilizes the system
- guarantees $e_{ss} := \limsup_{t \rightarrow \infty} |e_y(t)| = 0$
- for **persistent** r and / or d from a **known class**

i.e. asymptotically perfect tracking and asymptotic disturbance rejection for signals from a limited class.

10/36

Integral action



If $\lim_{t \rightarrow \infty} y_{ref}(t) = y_{ss}$ and $\lim_{t \rightarrow \infty} d(t) = d_{ss}$ for some y_{ss} and (unknown) d_{ss} , then $\lim_{t \rightarrow \infty} u_{req}(t) = u_{ss} = y_{ss}/P_{nom}(0)$ and

$$e_{ss} = |T_d(0)d_{ss} - S(0)y_{ss} + T_d(0)u_{ss}| = |T_d(0)| \left| d_{ss} - \frac{y_{ss}}{P(0)} + u_{ss} \right|$$

(we naturally assume that $P(0) \neq 0$). To have $e_{ss} = 0$ for all y_{ss} and d_{ss} we need

$$T_d(0) = \lim_{s \rightarrow 0} \frac{P(s)}{1 - P(s)R(s)} = 0$$

for all possible plants P .

11/36

Integral action (contd)

If

$$R(s) = \frac{1}{s} R_s(s)$$

for some **stabilizer** R_s with $R(0) \neq 0$, then (again, assuming $P(0) \neq 0$)

$$\lim_{s \rightarrow 0} \frac{P(s)}{1 - P(s)R(s)} = \lim_{s \rightarrow 0} \frac{s}{s/P(s) - R_s(s)} = 0$$

indeed. This supports well-known fact that

- integral action in the controller
- asymptotically rejects DC components of disturbances
- yields asymptotically perfect set-point tracking despite uncertainty in P

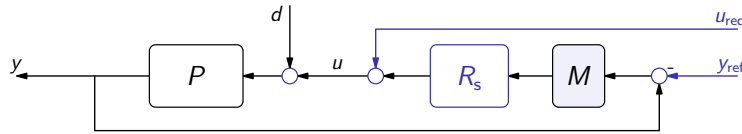
Rationale:

- infinite controller gain at the frequency of persistent parts of y_{ref} and d (which is $\omega = 0$ in this case).

12/36

Internal model principle (Francis & Wonham, 75)

All unstable poles of $Y_{\text{ref}}(s)$ and $D(s)$ must be in the controller:



and the regulator problem is solvable iff

- no unstable cancellations in MP
- R_s stabilizes MP

Again,

- infinite controller gain at dominant modes of y_{ref} and d modes of M

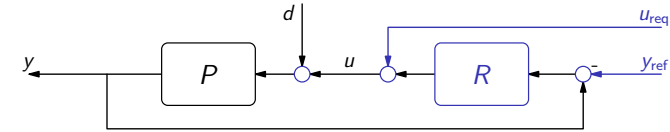
Example (harmonic load disturbance): Let

$$d(t) = d_a \sin(\omega_0 t + \phi) \implies D(s) = d_a \frac{s \sin \phi + \omega_0 \cos \phi}{s^2 + \omega_0^2}$$

Hence, $M(s)$ must have poles at $s = \pm j\omega_0$.

13/36

Repetitive control



(Ambitious) goal:

- solve the regulator problem for arbitrary τ -periodic y_{ref} and / or d for a given $\tau > 0$.

Rationale: All “reasonable” τ -periodic signals can be expanded as

$$x(t) = \sum_{i \in \mathbb{Z}} x_i e^{j\omega_i t}, \quad \text{where } \omega_i := \frac{2\pi}{\tau} i$$

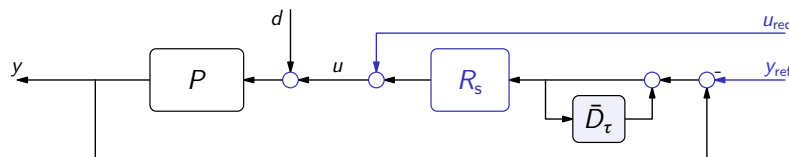
so by the internal model principle

- the regulator should have poles at $s = j\omega_i$, for all $i \in \mathbb{Z}$.

14/36

Repetitive control: configuration

Elegant solution (Inoue, Nakano, & Iwai, 1981):



where the transfer function of the repetitive element,

$$M(s) = \frac{1}{1 - e^{-\tau s}},$$

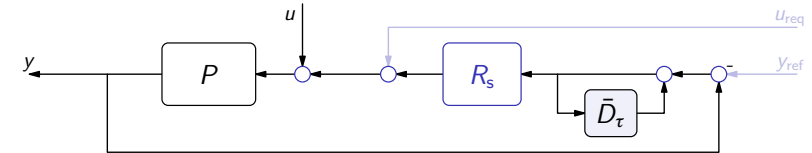
has its poles at $s = j\omega_i$ for all $i \in \mathbb{Z}$, indeed. Consequently,

$$T_d(s) = \frac{P(s)(1 - e^{-\tau s})}{1 - e^{-\tau s} - P(s)R_s(s)}$$

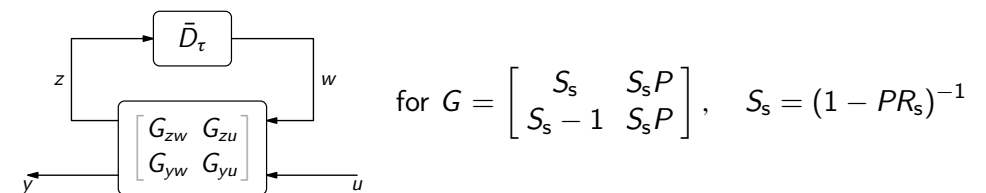
has zeros at all $s = j\omega_i$, asymptotically blocking all τ -periodic y_{ref} and d).

15/36

Repetitive control: catch



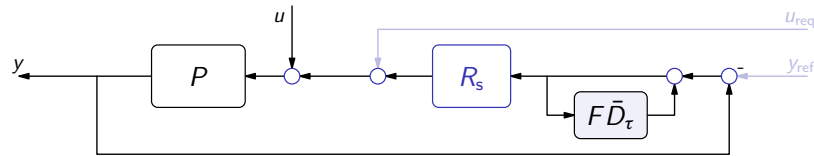
This feedback is equivalent to



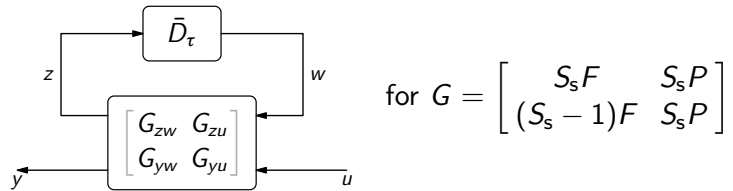
If $P(s)R_s(s)$ is strictly proper (reasonable), then $|G_{zw}(\infty)| = 1 \neq 1$ and
 ☹ no (practically) stabilizing R_s exists.

16/36

Repetitive control: workaround



This feedback is equivalent to

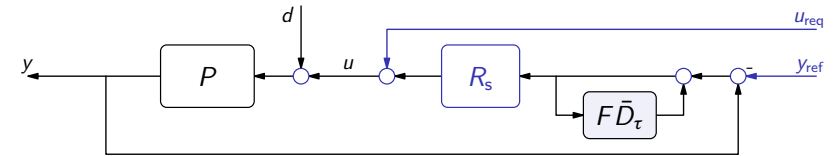


If F stable and such that

$$|F(\infty)| < 1,$$

then $|G_{zw}(\infty)| = |S_s(\infty)F(\infty)| = |F(\infty)| < 1$ and a stabilizing R_s exists.

Repetitive control: price to pay



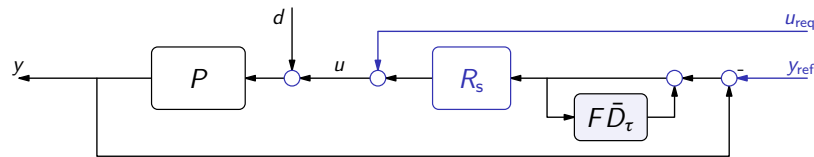
But now

$$M(s) = \frac{1}{1 - F(s)e^{-\tau s}}$$

no longer has poles at all $j\omega_i$, so

- no perfect regulation.

Repetitive control: choice of F



If

$$F(j\omega_i) \approx 1 \quad \text{for } |i| \leq n$$

then still

- “good” regulation for all “slow” τ -periodic d and / or $r \leftarrow$ reasonable

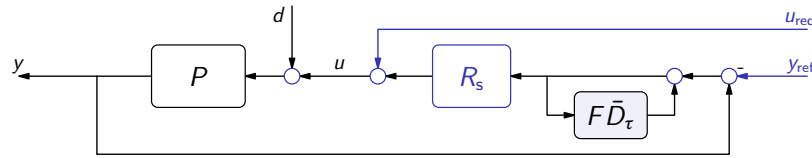
The name of the game:

- high-bandwidth low-pass F for which stabilizing R_s can be designed.

Outline

Repetitive control: stabilization

Robust control approach



Robustness arguments, like the small-gain condition

$$\|G_{zw}\| = \left\| \frac{F}{1 - PR_s} \right\|_\infty < 1$$

for all $\omega \in \mathbb{R}$.

- 😊 delay-free H_∞ problem (weighted sensitivity) $\implies$ **standard** tools
e.g. if $P(z_i) = 0$ for $z_i \in \mathbb{C}_0$ and $P(s)$ has n_+ poles p_j in $\mathbb{C}_0$, then holds only if

$$|F(z_i)| \leq \prod_{j=1}^{n_+} |z_i - p_j| / |z_i + p_j|$$

- 😞 **overkill**, stabilizes for all constant $\tau \geq 0$

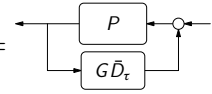
21/36

Loop shifting: preliminary

Lemma

If $FP^{-1} \in H_\infty$, then there is $G \in H_\infty$ such that

$$(1 - F\bar{D}_\tau)^{-1}P = (1 - PG\bar{D}_\tau)^{-1}P =$$



Proof: The equality holds if $\exists G$ such that

$$1 - F\bar{D}_\tau = 1 - PG\bar{D}_\tau \implies G = FP^{-1} \in H_\infty$$

which is what we need. □

Remark: Applies seamlessly if

- $P(s)$ is minimum phase and pole excess $P(s)$ does not exceed pole excess of $F(s)$.

If $P(s)$ is nonminimum-phase, then

- all nonminimum-phase zeros of $P(s)$ must be those of $F(s)$,

which might hamper aspirations to have $F(j\omega_i) \approx 1$.

22/36

Loop shifting: design

If $G = FP^{-1} \in H_\infty$, then

- R_s stabilizes $(1 - F\bar{D}_\tau)^{-1}P \iff \tilde{R} = R_s + G\bar{D}_\tau$ stabilizes P .

Hence, the design has two steps:

1. design $\tilde{R}$ stabilizing P w/o the repetitive block
2. implement the controller

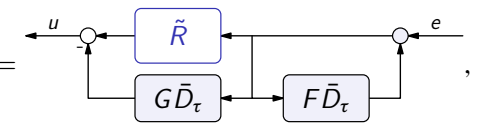
$$R = (\tilde{R} - G\bar{D}_\tau)(1 - F\bar{D}_\tau)^{-1}$$

23/36

Loop shifting: implementation

Delays in the overall controller,

$$R = (\tilde{R} - G\bar{D}_\tau)(1 - F\bar{D}_\tau)^{-1} =$$



have to be kept equal. R can be implemented with only **one delay element**:

$$R = \mathcal{F}_l \left(\begin{bmatrix} \tilde{R} & \tilde{R}F - G \\ 1 & F \end{bmatrix}, \bar{D}_\tau \right)$$

which is more robust.

24/36

Loop shifting: closed-loop system

Disturbance sensitivity:

$$\begin{aligned}
 T_d &= \mathcal{F}_l \left(\begin{bmatrix} P & P \\ P & P \end{bmatrix}, R \right) \\
 &= \mathcal{F}_l \left(\begin{bmatrix} P & P \\ P & P \end{bmatrix}, \mathcal{F}_l \left(\begin{bmatrix} \tilde{R} & \tilde{R}F - G \\ 1 & F \end{bmatrix}, \bar{D}_\tau \right) \right) \\
 &= \mathcal{F}_l \left(\begin{bmatrix} P & P \\ P & P \end{bmatrix} \star \begin{bmatrix} \tilde{R} & \tilde{R}F - G \\ 1 & F \end{bmatrix}, \bar{D}_\tau \right) \\
 &= \mathcal{F}_l \left(\begin{bmatrix} (1 - P\tilde{R})^{-1}P & -F \\ (1 - P\tilde{R})^{-1}P & 0 \end{bmatrix}, \bar{D}_\tau \right) \\
 &= (1 - P\tilde{R})^{-1}P(1 - F\bar{D}_\tau),
 \end{aligned}$$

which is an affine function of the controller delay element.

25/36

Loop shifting: nonminimum-phase plants

Like in the MSP case, replace $(1 - F\bar{D}_\tau)^{-1}P = (1 - PG\bar{D}_\tau)^{-1}P$ with

$$(1 - F\bar{D}_\tau)^{-1}P = (1 - \tilde{P}\Pi)^{-1}\tilde{P}$$

for

- $\Pi \in H_\infty$
- delay-free $\tilde{P}$

In square case,

$$P^{-1}(1 - F\bar{D}_\tau) = \tilde{P}^{-1}(1 - \tilde{P}\Pi) \iff \Pi = \tilde{P}^{-1} - P^{-1} + P^{-1}F\bar{D}_\tau$$

If $G := P^{-1}(s)F(s)$ proper, then

- $\Pi = -\sigma_\tau \{G\bar{D}_\tau\}$
- $\tilde{P} = (I - P\tilde{G})^{-1}P$ $\tilde{G} = G\bar{D}_\tau + \sigma_\tau \{G\bar{D}_\tau\}$

The resulting $\tilde{P}(s)$ has the same zeros / pole excess as $P(s)$. However

- ☹ FIR Π harder to implement doesn't blend with repetitive block well

26/36

Outline

Delays that help (or so we thought)

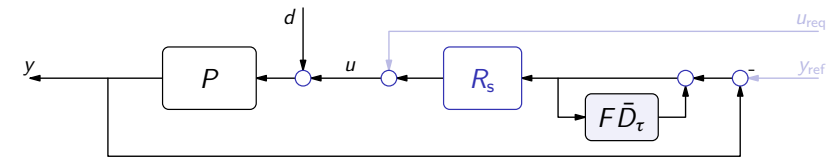
Repetitive control

Repetitive control: stabilization

Case studies

27/36

Setup



Let

$$F(s) = \frac{\omega_1 \alpha}{s + \omega_1 \alpha},$$

where $\omega_1 = 2\pi/\tau$ is the fundamental frequency of d . In this case

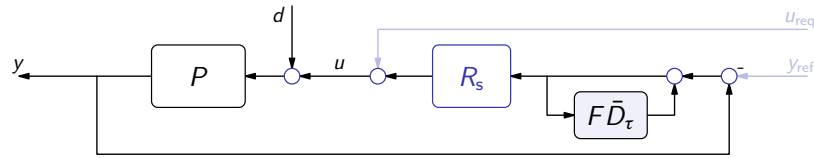
$$|1 - F(j\omega_i)| = \frac{1}{\sqrt{1 + \alpha^2/i^2}} = \frac{1}{\sqrt{1 + \alpha^2/\omega_i^2}}$$

where $\omega_i := i2\pi/\tau$, and

- range of attenuated frequencies is proportional to α .

28/36

Example 1



Let

$$P(s) = \frac{1}{s} \implies G(s) = \frac{F(s)}{P(s)} = \frac{\omega_1 \alpha s}{s + \omega_1 \alpha}$$

is stable and we stabilize P by $\tilde{R} = k_p > 0$, so that

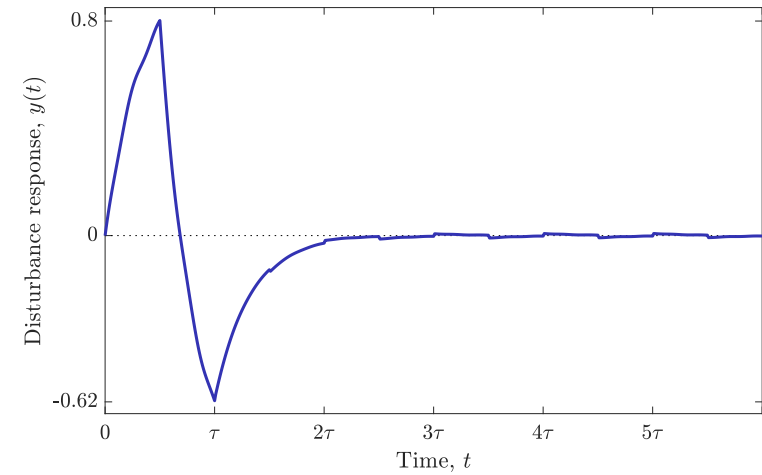
$$R_s(s) = k_p + \frac{\omega_1 \alpha s}{s + \omega_1 \alpha} e^{-\tau s}$$

stabilizes the system.

29/36

Example 1: square wave d

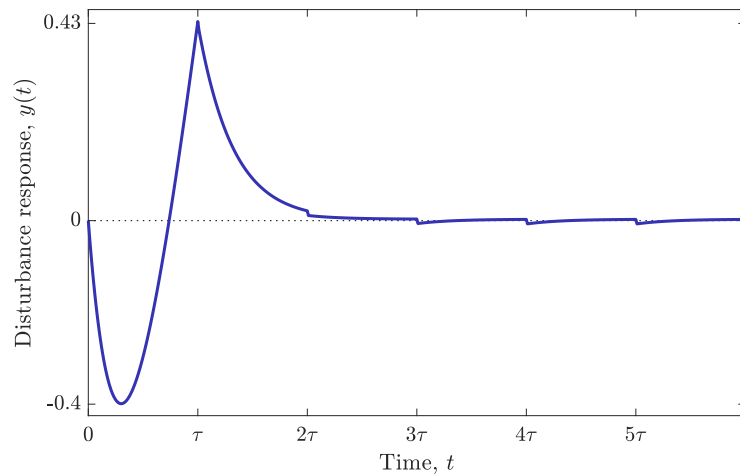
If $\alpha = 100$, then



30/36

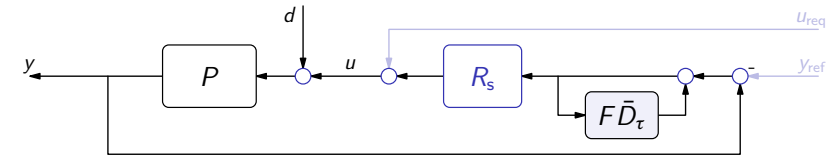
Example 1: sawtooth d

If $\alpha = 100$, then



31/36

Example 2



If

$$P(s) = \frac{s-1}{(s+1)(s-p)}, \quad p \in (0,1)$$

then

$$\left\| \frac{F}{1 - PR_s} \right\|_\infty \geq |F(1)| \left| \frac{s+p}{s-p} \right|_{s=1} = \frac{\omega_1 \alpha}{\omega_1 \alpha + 1} \frac{1+p}{1-p}$$

and

$$\left\| \frac{F}{1 - PR_s} \right\|_\infty < 1 \quad \text{only if} \quad \alpha < \frac{1}{\omega_1} \frac{1-p}{2p},$$

which is not practical for $p \uparrow 1$.

32/36

Example 2: construction of Π

In this case

$$G(s) = \frac{F(s)}{P(s)} = \frac{\omega_1 \alpha (s+1)(s-p)}{(s-1)(s+\omega_1 \alpha)} = \frac{(1-p)\omega_1 \alpha}{\omega_1 \alpha + 1} \frac{2}{s-1} + G_{\text{st}}(s),$$

where

$$G_{\text{st}}(s) = \omega_1 \alpha \left(1 - \frac{\omega_1 \alpha - 1}{\omega_1 \alpha + 1} \frac{p + \omega_1 \alpha}{s + \omega_1 \alpha} \right),$$

has an unstable pole. The choice

$$\tilde{P}(s) = \frac{s-1}{(s+1)(s-\tilde{p})},$$

where $\tilde{p} := p + e^{-\tau}(1-p)\omega_1 \alpha / (1 + \omega_1 \alpha) \in (p, 1)$, yields stable

$$\begin{aligned} \Pi(s) &= \left(\frac{1}{\tilde{P}(s)} - \frac{1}{P(s)} \right) + \frac{F(s)}{P(s)} e^{-\tau s} \\ &= \frac{(1-p)\omega_1 \alpha}{\omega_1 \alpha + 1} \frac{e^{-\tau}(s+1) - 2e^{-\tau s}}{s-1} + G_{\text{st}}(s)e^{-\tau s}. \end{aligned}$$

33/36

Example 2: controller design

Design for

$$\tilde{P}(s) = \frac{s-1}{(s+1)(s-\tilde{p})} \quad \text{with } \tilde{p} \in (p, 1)$$

is qualitatively the same as that for

$$P(s) = \frac{s-1}{(s+1)(s-p)}.$$

If $p = 0.8$, then

$$\alpha \in [10, 1000] \implies \tilde{p} \in [0.8082, 0.8086]$$

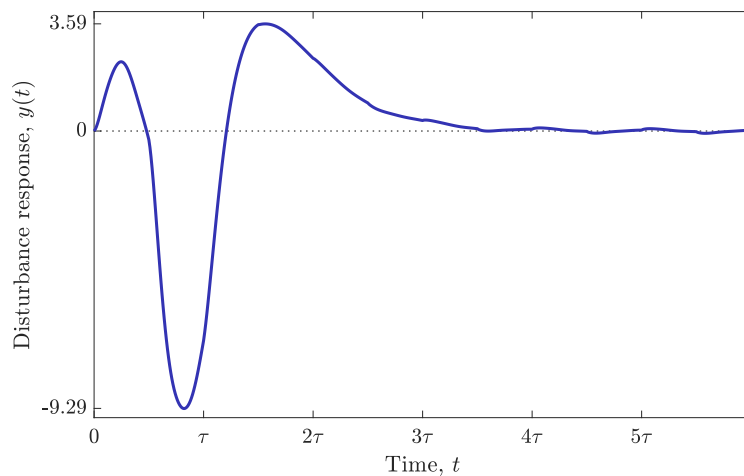
and P and $\tilde{P}$ are practically identical, stabilizable by

$$\tilde{R}(s) = -0.88 \frac{s+1}{0.02s+1}.$$

34/36

Example 2: square wave d

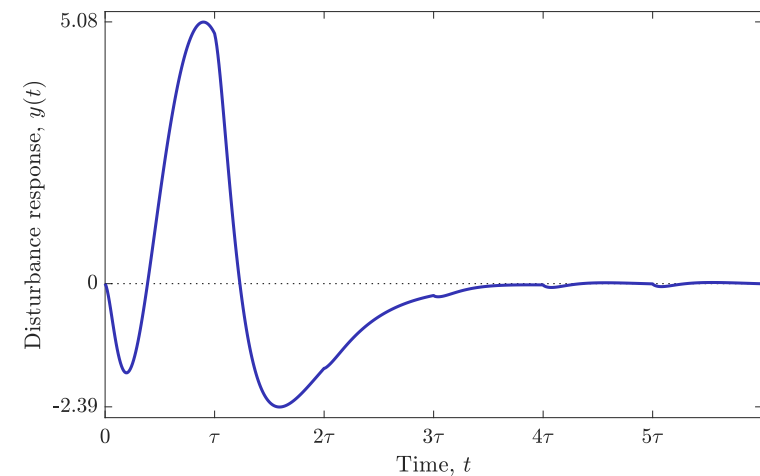
If $\alpha = 100$, then



35/36

Example 2: sawtooth d

If $\alpha = 100$, then



36/36