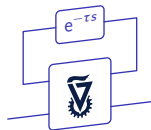


Time Delays in Control and Estimation (038806)



lecture no. 10

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Outline

Optimization-based design: introduction

H_2 prediction: gaining insight via a simple problem

H_2 standard problem: single-delay case

H_2 standard problem: multiple-delay extensions

H_∞ standard problem

Outline

Optimization-based design: introduction

H_2 prediction: gaining insight via a simple problem

H_2 standard problem: single-delay case

H_2 standard problem: multiple-delay extensions

H_∞ standard problem

Optimal control

Problem:

- minimize **cost function** (criterion) subject to **constraints** imposed by process dynamics

Hope:

- solution results in “good” (in whatever sense) control system

Advantages:

- analytic design method, with strong theoretic justification
- important byproducts (like stability, robustness, etc)

Things to remember:

- no criterion can ever reflect all our requirements
- more comprehensive cost functions result in less transparent solutions
- “optimal” might have nothing to do with “good”
- optimization should be used as a tool rather than as the goal

Optimal control

Problem:

- minimize **cost function** (criterion) subject to **constraints** imposed by process dynamics

Hope:

- solution results in “good” (in whatever sense) control system

Advantages:

- analytic design method, with strong theoretic justification
- important byproducts (like stability, robustness, etc)

Things to remember

- no criterion can ever reflect all our requirements
- more comprehensive cost functions result in less transparent solutions
- “optimal” might have nothing to do with “good”
- optimization should be used as a tool rather than as the goal

Optimal control

Problem:

- minimize **cost function** (criterion) subject to **constraints** imposed by process dynamics

Hope:

- solution results in “good” (in whatever sense) control system

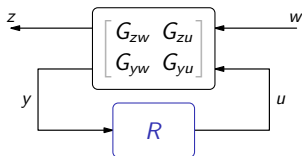
Advantages:

- analytic design method, with strong theoretic justification
- important byproducts (like stability, robustness, etc)

Things to remember:

- no criterion can ever reflect all our requirements
- more comprehensive cost functions result in less transparent solutions
- “optimal” might have nothing to do with “good”
- optimization should be used as a **tool** rather than as the goal

Generalized plant paradigm



Systems:

- G is generalized plant
- R is controller

given components
components we design

Signals:

- w is exogenous input
- u is control input
- z is regulated output
- y is measured output

reference, disturbances, noise, etc
output of controller
collection of signals we want to keep “small”
input of controller

Performance measure:

- size (**norm**) of the closed-loop **system** $T_{zw} : w \mapsto z$

H_2 system norm

Let

$$H_2 := \{G(s) : G(s) \text{ holomorphic in } \mathbb{C}_0 \text{ and } \|G\|_2 < \infty\}$$

where

$$\|G\|_2 := \sup_{\sigma > 0} \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(\sigma + j\omega)\|_F^2 d\omega}$$

is the H_2 norm and $\|\cdot\|_F$ is Frobenius matrix norm. If $G \in H_2$, then

$$\|G\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 d\omega} = \sqrt{\langle G, G \rangle_2},$$

where

$$\langle G_1, G_2 \rangle_2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}(G_1(j\omega)[G_2(j\omega)]') d\omega$$

is the inner product (H_2 is a Hilbert space).

H_2 system norm (contd)

By Parseval, if $G_1, G_2 \in H_2$, then

$$\langle G_1, G_2 \rangle_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}(G_1(j\omega)[G_2(j\omega)]') d\omega = \int_0^{\infty} \text{tr}(g_1(t)g_2'(t)) dt,$$

where $g_i = \mathfrak{F}^{-1}\{G_i\}$, and

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 d\omega = \int_0^{\infty} \|g(t)\|_F^2 dt.$$

Signal interpretations: In SISO case $\|T_{zw}\|_2^2$ is

- energy of z when $w = \delta$ (energy of impulse response)
- variance of z when w zero-mean unit intensity white noise

We say that $G_1 \perp G_2$ if $\langle G_1, G_2 \rangle_2 = 0$. Pythagoras:

$$G_1 \perp G_2 \implies \|G_1 + G_2\|_2^2 = \|G_1\|_2^2 + \|G_2\|_2^2$$

H_2 system norm (contd)

By Parseval, if $G_1, G_2 \in H_2$, then

$$\langle G_1, G_2 \rangle_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}(G_1(j\omega)[G_2(j\omega)]') d\omega = \int_0^{\infty} \text{tr}(g_1(t)g_2'(t)) dt,$$

where $g_i = \mathfrak{F}^{-1}\{G_i\}$, and

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(j\omega)\|_F^2 d\omega = \int_0^{\infty} \|g(t)\|_F^2 dt.$$

Signal interpretations: In SISO case $\|T_{zw}\|_2^2$ is

- energy of z when $w = \delta$ (energy of impulse response)
- variance of z when w zero-mean unit intensity white noise

We say that $G_1 \perp G_2$ if $\langle G_1, G_2 \rangle_2 = 0$. Pythagoras:

$$G_1 \perp G_2 \implies \|G_1 + G_2\|_2^2 = \|G_1\|_2^2 + \|G_2\|_2^2$$

H_∞ system norm

We have already defines space

$$H_\infty := \left\{ G(s) : G(s) \text{ holomorphic in } \mathbb{C}_0 \text{ and } \|G\|_\infty < \infty \right\}$$

where

$$\|G\|_\infty := \sup_{s \in \mathbb{C}_0} \|G(s)\|$$

is the H_∞ norm and $\|\cdot\|$ is spectral matrix norm. If $G \in H_\infty$, then

$$\|G\|_\infty = \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \|G(j\omega)\|$$

Signal interpretation: $\|T_{zw}\|_\infty^2$ is

- the upper bound on energy of z over all $\|w\|_2 = 1$

or, equivalently,

- energy of z for worst-case normalized w

Inner transfer functions

$G \in H_\infty$ is said to be **inner** if

$$G^\sim(s)G(s) = I \quad \text{or} \quad [G(j\omega)]^* G(j\omega) = I,$$

where **conjugate** $G^\sim(s) := [G(-s)]'$. In the scalar case **inner** means **stable with unit magnitude** for all frequencies (as $G^\sim(j\omega) = \overline{G(j\omega)}$). Clearly

- $e^{-\tau s}$ is inner for all $\tau \geq 0$
- $G^\sim(s)/G(s)$ is always inner

Important property of LTI systems with inner functions is that they are

- energy preserving,

i.e. if $y = Gu$ for an inner G , then $\|y\|_2 = \|u\|_2$ for all $u \in L_2(\mathbb{R})$. Hence,

$$G(s) \text{ inner} \implies \begin{cases} \|GT_{zw}\|_2 = \|T_{zw}\|_2 \\ \|GT_{zw}\|_\infty = \|T_{zw}\|_\infty \end{cases}$$

i.e. multiplication by an inner transfer function preserves H_2 and H_∞ norms.

Inner transfer functions

$G \in H_\infty$ is said to be **inner** if

$$G^\sim(s)G(s) = I \quad \text{or} \quad [G(j\omega)]^* G(j\omega) = I,$$

where **conjugate** $G^\sim(s) := [G(-s)]'$. In the scalar case **inner** means **stable with unit magnitude** for all frequencies (as $G^\sim(j\omega) = \overline{G(j\omega)}$). Clearly

- $e^{-\tau s}$ is inner for all $\tau \geq 0$
- $G^\sim(s)/G(s)$ is always inner

Important property of LTI systems with inner functions is that they are

- energy preserving,

i.e. if $y = Gu$ for an inner G , then $\|y\|_2 = \|u\|_2$ for all $u \in L_2(\mathbb{R})$. Hence,

$$G(s) \text{ inner} \implies \begin{cases} \|GT_{zw}\|_2 = \|T_{zw}\|_2 \\ \|GT_{zw}\|_\infty = \|T_{zw}\|_\infty \end{cases}$$

i.e. multiplication by an inner transfer function preserves H_2 and H_∞ norms.

H_2 and H_∞ system norms for simple systems

If

$$G(s) = \frac{k}{\tau s + 1}, \quad \tau > 0,$$

then

$$\|G\|_2 = \frac{|k|}{\sqrt{2\tau}} \quad \text{and} \quad \|G\|_\infty = |k|$$

If

$$G(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta > 0, \omega_n > 0,$$

then

$$\|G\|_2 = \frac{|k|\sqrt{\omega_n}}{2\sqrt{\zeta}} \quad \text{and} \quad \|G\|_\infty = \begin{cases} \frac{|k|}{2\zeta\sqrt{1-\zeta^2}} & \text{if } 0 < \zeta < \frac{1}{\sqrt{2}} \\ |k| & \text{if } \zeta \geq \frac{1}{\sqrt{2}} \end{cases}$$

Hmm

— smaller τ / larger ω_n yield larger H_2 -norm

counterintuitive

H_2 and H_∞ system norms for simple systems (contd)

If

$$G(s) = \left(1 - \frac{1}{\tau s + 1}\right) \frac{k}{s} = \frac{k\tau}{\tau s + 1}, \quad \tau > 0,$$

then

$$\|G\|_2 = \frac{|k|\sqrt{\tau}}{\sqrt{2}} \quad \text{and} \quad \|G\|_\infty = |k|\tau$$

If

$$G(s) = \left(1 - \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \frac{k}{s} = \frac{k(s + 2\zeta\omega_n)}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta > 0, \omega_n > 0,$$

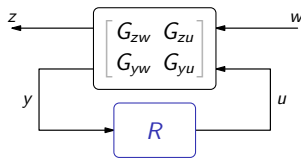
then

$$\|G\|_2 = \frac{|k|\sqrt{1 + 4\zeta^2}}{2\sqrt{\zeta}\omega_n} \quad \text{and} \quad \|G\|_\infty = \begin{cases} \frac{|k|}{\alpha(\zeta)\omega_n} & \text{if } 0 < \zeta < \frac{\sqrt{1 + \sqrt{2}}}{2} \\ \frac{2|k|\zeta}{\omega_n} & \text{if } \zeta \geq \frac{\sqrt{1 + \sqrt{2}}}{2} \end{cases}$$

for $\alpha(\zeta) := \sqrt{2(\sqrt{1 + 8\zeta^2} - 1 - 2\zeta^2)}$.

- minimum $\|G\|_2$ at $\zeta = 1/2$, minimum $\|G\|_\infty$ at $\zeta = \sqrt{6}/4 \approx 0.61237$.

The standard problem



H_2 version: Given G , design causal stabilizing R that minimizes

$$\|T_{zw}\|_2, \quad \text{where } T_{zw} = \mathcal{F}_l(G, R).$$

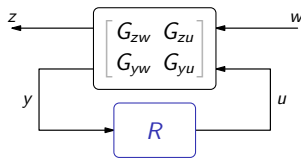
Examples: LQR, Kalman filter, LQG...

H_∞ version: Given G and $\gamma > 0$, design causal stabilizing R that renders

$$\|T_{zw}\|_\infty < \gamma, \quad \text{where } T_{zw} = \mathcal{F}_l(G, R).$$

Examples: Weighted sensitivity, mixed sensitivity, robust stability, H_∞ loop shaping, ...

The standard problem: standard assumptions



Let

$$G(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array} \right].$$

$\mathcal{A}_1$: (A, B_u, C_y, D_{yu}) has no unstable hidden modes

$\mathcal{A}_2$: (A, B_u, C_z, D_{zu}) has no invariant zeros in $j\mathbb{R}$ and $D'_{zu}D_{zu} = I$

$\mathcal{A}_3$: (A, B_w, C_y, D_{yw}) has no invariant zeros in $j\mathbb{R}$ and $D_{yw}D'_{yw} = I$

Remark: What we actually need in $\mathcal{A}_2$ and $\mathcal{A}_3$ is $D'_{zu}D_{zu} > 0$ and $D_{yw}D'_{yw} > 0$, respectively. But normalizing D_{zu} and D_{yw} is routine. It is sometimes also assumed, w.l.o.g., that $D'_{zu}C_z = 0$ and $B_wD'_{yw} = 0$. Those assumptions simplify the formulae.

H_2 standard problem: Riccati equations

If $\mathcal{A}_{1-3}$ hold, then AREs

$$XA + A'X + C_z' C_z - (XB_u + C_z' D_{zu})(B_u' X + D_{zu}' C_z) = 0$$

and

$$AY + YA' + B_w B_w' - (YC_y' + B_w D_{yw}') (C_y Y + D_{yw} B_w') = 0$$

have stabilizing solutions $X \geq 0$ and $Y \geq 0$, for which

$$A_K = A + B_u K \quad \text{and} \quad A_L = A + LC_y$$

are Hurwitz stable, where

$$K = -(B_u' X + D_{zu}' C_z) \quad \text{and} \quad L = -(YC_y' + B_w D_{yw}').$$

H_2 standard problem: solution

Every stabilizing controller of the form $R = \mathcal{F}_I(J_2, Q)$ for

$$J_2(s) = \left[\begin{array}{c|cc} \frac{A + B_u K + L C_y}{K} & -L & B_u \\ \hline & 0 & I \\ -C_y & I & 0 \end{array} \right]$$

renders

$$\|T_{zw}\|_2^2 = \gamma_{\text{opt}}^2 + \|Q\|_2^2,$$

where

$$\gamma_{\text{opt}}^2 = \text{tr}(B'_w X B_w) + \text{tr}(K Y K').$$

The unique optimal controller corresponds to $Q = 0$ and is

$$R_{\text{opt}}(s) = \left[\begin{array}{c|c} \frac{A + B_u K + L C_y}{K} & -L \\ \hline & 0 \end{array} \right]$$

H_∞ standard problem: Riccati equations

If $\mathcal{A}_{1-3}$ hold, then the problem is solvable for a given $\gamma > 0$ iff AREs

$$XA + A'X + C_z' C_z - (XB_u + C_z' D_{zu})(B_u' X + D_{zu}' C_z) + \gamma^{-2} X B_w B_w' X = 0$$

and

$$AY + YA' + B_w B_w' - (YC_y' + B_w D_{yw}') (C_y Y + D_{yw} B_w') + \gamma^{-2} Y C_z' C_z Y = 0$$

have stabilizing solutions X and Y such that

$$X \geq 0, \quad Y \geq 0, \quad \text{and} \quad \rho(YX) < \gamma^2.$$

Here

$$A_K = A + \gamma^{-2} B_w B_w' X + B_u K \quad \text{and} \quad A_L = A + \gamma^{-2} Y C_z' C_z + L C_y$$

are Hurwitz stable, where

$$K = -(B_u' X + D_{zu}' C_z) \quad \text{and} \quad L = -(Y C_y' + B_w D_{yw}')$$

($A + B_u K$ and $A + L C_y$ are also Hurwitz then).

H_∞ standard problem: solution

All solutions are of the form $\mathcal{F}_l(J_\infty, Q)$ for

$$J_\infty(s) = \left[\begin{array}{c|cc} A + \gamma^{-2}B_w B_w' X + B_u K + Z_\gamma L \tilde{C}_y & -Z_\gamma L & \tilde{B}_u \\ \hline K & 0 & I \\ -\tilde{C}_y & I & 0 \end{array} \right]$$

and any $\|Q\|_\infty < \gamma$, where

$$Z_\gamma = (I - \gamma^{-2} Y X)^{-1}, \quad \tilde{B}_u = B_u + \gamma^{-2} Y C_z' D_{zu}, \quad \tilde{C}_y = C_y + \gamma^{-2} D_{yw} B_w' X$$

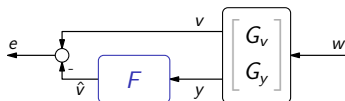
The controller corresponding to $Q = 0$,

$$R(s) = \left[\begin{array}{c|c} \frac{A + \gamma^{-2}B_w B_w' X + B_u K + Z_\gamma L (C_y + \gamma^{-2} D_{yw} B_w' X)}{K} & \frac{-Z_\gamma L}{0} \end{array} \right],$$

is known as the central controller.

Remark: We may guarantee that $R = \mathcal{F}_l(J_\infty, Q)$ is stabilizing only if $\|Q\|_\infty < \gamma$. As $\gamma \rightarrow \infty$ we recover the H_2 solution.

H_2 estimation problem



for a signal to be estimated v , measurement y , and

$$G(s) = \left[\begin{array}{c|c} A & B_w \\ \hline C_v & 0 \\ C_y & D_{yw} \end{array} \right]$$

such that $\mathcal{A}_3$ holds and

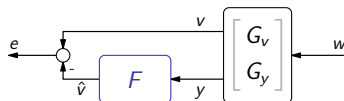
$\mathcal{A}_5$: (C_y, A) detectable

Problem: design $F \in H_\infty$ that minimizes $\|T_e\|_2$, where $T_e = G_v - FG_y$

Remark: $\|T_e\|_2$ is the H_2 norm of the error transfer function T_e . G_v and G_y are assumed to be stable.

Reference: [1, Ch. 10]

H_2 estimation problem



for a signal to be estimated v , measurement y , and

$$G(s) = \left[\begin{array}{c|c} A & B_w \\ \hline C_v & 0 \\ C_y & D_{yw} \end{array} \right]$$

such that $\mathcal{A}_3$ holds and

$\mathcal{A}_5$: (C_y, A) detectable

Problem: design $F \in H_\infty$ that minimizes $\|T_e\|_2$, where $T_e = G_v - FG_y$

Remark: There is no internal stability requirement, instabilities in G_v and G_y are normally fictitious, introduced to impose steady-state requirements.

H_2 estimation problem: solution

If $\mathcal{A}_{5,3}$ hold, then CARE

$$AY + YA' + B_w B_w' - (YC_y' + B_w D_{yw}') (C_y Y + D_{yw} B_w') = 0$$

has a stabilizing solution $Y \geq 0$ for which $A_L = A + LC_y$ is Hurwitz, where $L = -(YC_y' + B_w D_{yw}')$. The unique optimal estimator

$$F_{\text{opt}}(s) = \left[\frac{A + LC_y}{C_v} \middle| \frac{-L}{0} \right]$$

attains the performance level

$$\|T_e\|_2^2 = \gamma_{\text{opt}}^2 = \text{tr}(C_v Y C_v').$$

Outline

Optimization-based design: introduction

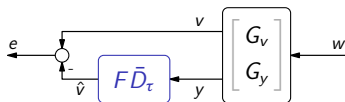
H_2 prediction: gaining insight via a simple problem

H_2 standard problem: single-delay case

H_2 standard problem: multiple-delay extensions

H_∞ standard problem

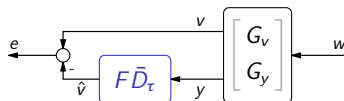
The problem



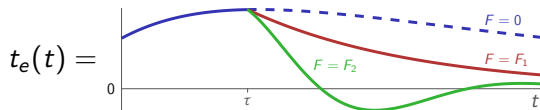
i.e.

- delayed output $\bar{D}_\tau y$ is observed

Key observation



The impulse response of the error system $T_e = G_v - F\bar{D}_\tau G_y = G_v - \bar{D}_\tau F G_y$



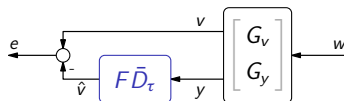
Hence,

$$\|T_e\|_2^2 \geq \gamma_{[0,\tau]}^2 := \int_0^\tau \|g_v(t)\|_F^2 dt$$

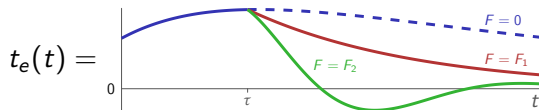
So the solution

- aims at extracting the part contributing to F -independent $\gamma_{[0,\tau]}$

Key observation



The impulse response of the error system $T_e = G_v - F\bar{D}_\tau G_y = G_v - \bar{D}_\tau F G_y$



Hence,

$$\|T_e\|_2^2 \geq \gamma_{[0,\tau]}^2 := \int_0^\tau \|g_v(t)\|_F^2 dt$$

So the solution

- aims at extracting the part contributing to F -independent $\gamma_{[0,\tau]}$

Technicalities

We know that

$$G_v = \pi_\tau \{ G_v \} + \bar{D}_\tau \hat{G}_v$$

so

$$T_e = G_v - \bar{D}_\tau F G_y = \pi_\tau \{ G_v \} + \bar{D}_\tau (\hat{G}_v - F G_y)$$

Lemma

If $T_e \in H_2$, then $\pi_\tau \{ G_v \} \perp \bar{D}_\tau (\hat{G}_v - F G_y)$.

Proof: Let $G_1 := \pi_\tau \{ G_v \}$ and $G_2 := \bar{D}_\tau (\hat{G}_v - F G_y)$.

$$\begin{aligned} \langle G_1, G_2 \rangle_2 &= \int_0^\infty \text{tr}(g_1(t)g_2'(t))dt \\ &= \int_0^\tau \text{tr}(g_1(t)g_2'(t))dt + \int_\tau^\infty \text{tr}(g_1(t)g_2'(t))dt = 0 \end{aligned}$$

because g_1 has support in $[0, \tau]$ and g_2 has support in $[\tau, \infty)$.



Reduction

By Pythagoras and the inner property of $e^{-\tau s}$,

$$\|T_e\|_2^2 = \|\pi_\tau\{G_v\}\|_2^2 + \|\bar{D}_\tau(\hat{G}_v - FG_y)\|_2^2 = \|\pi_\tau\{G_v\}\|_2^2 + \|\hat{G}_v - FG_y\|_2^2$$

Theorem

F solves the prediction problem iff it solves the filtering problem for

$$G_\tau(s) = \begin{bmatrix} \hat{G}_v(s) \\ G_y(s) \end{bmatrix} = \left[\begin{array}{c|c} A & B_w \\ \hline C_v e^{A\tau} & 0 \\ G_y & D_{yw} \end{array} \right]$$

The optimal cost is then

$$\gamma_{opt,\tau}^2 = \underbrace{\int_0^\tau \|C_v e^{At} B_w\|_F^2 dt}_{\|\pi_\tau\{G_v\}\|_2^2} + \text{tr}(C_v e^{A\tau} Y e^{A'\tau} C_v').$$

Optimal predictor

Riccati, based on **unchanged** G_y ,

$$AY + YA' + B_w B_w' - (YC_y' + B_w D_{yw}') (C_y Y + D_{yw} B_w') = 0$$

Optimal predictor (mechanical substitution):

$$F_{\text{opt},\tau}(s) = \left[\frac{A + LC_y}{C_v e^{A\tau}} \middle| \frac{-L}{0} \right].$$

Strangely enough, it is

- finite dimensional.

Cost of delay

Riccati reads $AY + YA' + B_w B_w' - LL' = 0$, so

$$\begin{aligned}
 \gamma_{\text{opt},\tau}^2 &= \int_0^\tau \text{tr}(C_v e^{At} B_w B_w' e^{A't} C_v') dt + \text{tr}(C_v e^{A\tau} Y e^{A'\tau} C_v') \\
 &= \int_0^\tau \text{tr}(C_v e^{At} (LL' - AY - YA') e^{A't} C_v') dt + \text{tr}(C_v e^{A\tau} Y e^{A'\tau} C_v') \\
 &= \int_0^\tau \text{tr}(C_v e^{At} LL' e^{A't} C_v') dt \\
 &\quad - \int_0^\tau \frac{d}{dt} \text{tr}(C_v e^{At} Y e^{A't} C_v') dt + \text{tr}(C_v e^{A\tau} Y e^{A'\tau} C_v') \\
 &= \int_0^\tau \|C_v e^{At} L\|^2 dt + \text{tr}(C_v Y C_v')
 \end{aligned}$$

cost of delay, γ_τ^2 filtering cost

with

γ_τ monotonically increasing and $\gamma_\infty = 0$ iff $F_{\text{opt}} = 0$.

Cost of delay

Riccati reads $AY + YA' + B_w B_w' - LL' = 0$, so

$$\begin{aligned}
 \gamma_{\text{opt},\tau}^2 &= \int_0^\tau \text{tr}(C_v e^{At} B_w B_w' e^{A't} C_v') dt + \text{tr}(C_v e^{A\tau} Y e^{A'\tau} C_v') \\
 &= \int_0^\tau \text{tr}(C_v e^{At} (LL' - AY - YA') e^{A't} C_v') dt + \text{tr}(C_v e^{A\tau} Y e^{A'\tau} C_v') \\
 &= \int_0^\tau \text{tr}(C_v e^{At} LL' e^{A't} C_v') dt \\
 &\quad - \int_0^\tau \frac{d}{dt} \text{tr}(C_v e^{At} Y e^{A't} C_v') dt + \text{tr}(C_v e^{A\tau} Y e^{A'\tau} C_v') \\
 &= \underbrace{\int_0^\tau \|C_v e^{At} L\|_F^2 dt}_{\text{cost of delay, } \gamma_\tau^2} + \underbrace{\text{tr}(C_v Y C_v')}_{\text{filtering cost}}
 \end{aligned}$$

with

- γ_τ monotonically increasing and $\gamma_\tau = 0$ iff $F_{\text{opt}} = 0$.

Outline

Optimization-based design: introduction

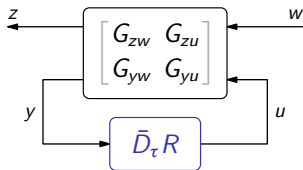
H_2 prediction: gaining insight via a simple problem

H_2 standard problem: single-delay case

H_2 standard problem: multiple-delay extensions

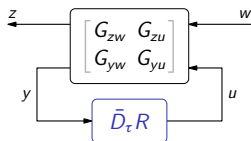
H_∞ standard problem

Setup



for a given $\tau \geq 0$ the rest is as before.

Loop shifting with performance



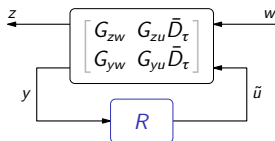
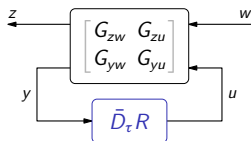
where

$$- \quad G_{yu} \bar{D}_\tau = \tilde{G}_{yu} - \Pi \text{ for FIR } \Pi = \sigma_x \{ G_{yu} \bar{D}_\tau \}$$

$$- \quad \bar{R} = (I + R\Pi)^{-1} R$$

$$- \quad G_{zw} = \Delta + \bar{D}_\tau \tilde{G}_{zw} \text{ for FIR } \Delta = \pi_\tau \{ G_{zw} \}$$

Loop shifting with performance



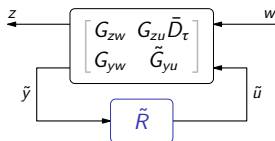
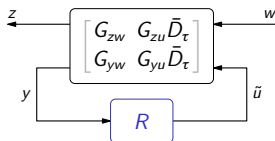
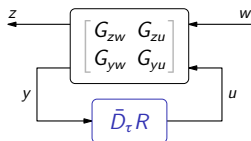
where

$$G_{yu} \bar{D}_\tau = \hat{G}_{yu} - \Pi \text{ for FIR } \Pi = \sigma_x \{ G_{yu} \bar{D}_\tau \}$$

$$\bar{R} = (I + R\Pi)^{-1} R$$

$$G_{zw} = \Delta + \bar{D}_\tau \hat{G}_{zw} \text{ for FIR } \Delta = \pi_\tau \{ G_{zw} \}$$

Loop shifting with performance

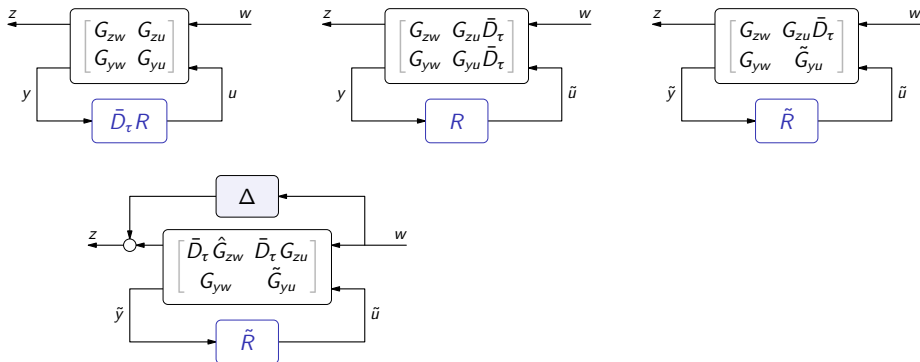


where

- $G_{yu}\bar{D}_\tau = \tilde{G}_{yu} - \Pi$ for FIR $\Pi = \sigma_\tau \{G_{yu}\bar{D}_\tau\}$
- $\tilde{R} = (I + R\Pi)^{-1}R$

– $G_{zw} = \Delta + \bar{D}_\tau \tilde{G}_{zw}$ for FIR $\Delta = \pi_\tau \{G_{zw}\}$

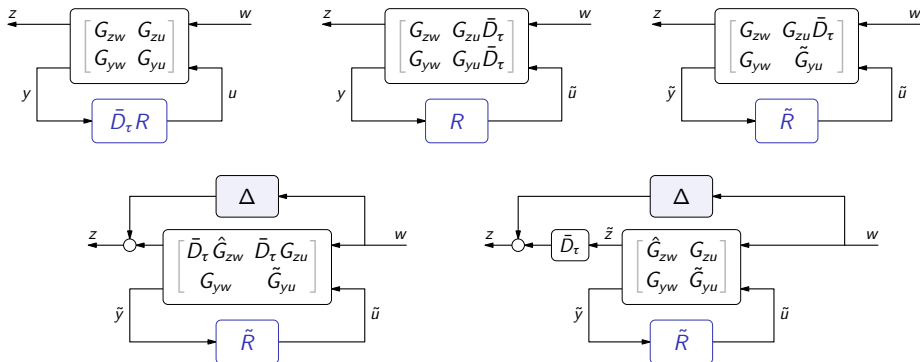
Loop shifting with performance



where

- $G_{yu} \bar{D}_\tau = \tilde{G}_{yu} - \Pi$ for FIR $\Pi = \sigma_\tau \{ G_{yu} \bar{D}_\tau \}$
- $\tilde{R} = (I + R\Pi)^{-1} R$
- $G_{zw} = \Delta + \bar{D}_\tau \hat{G}_{zw}$ for FIR $\Delta = \pi_\tau \{ G_{zw} \}$

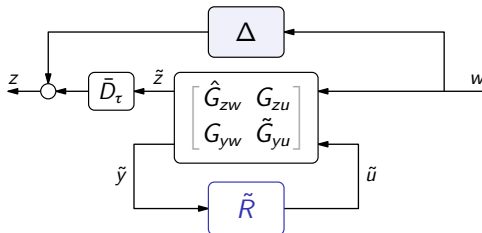
Loop shifting with performance



where

- $G_{yu} \bar{D}_\tau = \tilde{G}_{yu} - \Pi$ for FIR $\Pi = \sigma_\tau \{ G_{yu} \bar{D}_\tau \}$
- $\tilde{R} = (I + R\Pi)^{-1} R$
- $G_{zw} = \Delta + \bar{D}_\tau \hat{G}_{zw}$ for FIR $\Delta = \pi_\tau \{ G_{zw} \}$

Loop shifting with performance (contd)



- $R = \tilde{R}(I - \Pi\tilde{R})^{-1}$ is the standard DTC (MSP)
- $\Delta \perp \bar{D}_\tau \mathcal{F}_I \left(\begin{bmatrix} \hat{G}_{zw} & G_{zu} \\ G_{yw} & \tilde{G}_{yu} \end{bmatrix}, \tilde{R} \right)$ whenever the latter is in H_2
- optimal $\tilde{R}$ for

$$\begin{bmatrix} \hat{G}_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & \tilde{G}_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & e^{-A\tau} B_u \\ \hline C_z e^{A\tau} & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array} \right]$$

yields optimal $\bar{D}_\tau R = \bar{D}_\tau \tilde{R}(I - \Pi\tilde{R})^{-1}$ for G

Some details

Control Riccati (call it X_τ)

$$\begin{aligned}
 0 &= X_\tau A + A' X_\tau + e^{A' \tau} C_z' C_z e^{A \tau} \\
 &\quad - (X_\tau e^{-A \tau} B_u + e^{A' \tau} C_z' D_{zu}) (B_u' e^{-A' \tau} X_\tau + D_{zu}' C_z e^{A \tau}) \\
 &= e^{A' \tau} (e^{-A' \tau} X_\tau e^{-A \tau} A + A' e^{-A' \tau} X_\tau e^{-A \tau} + C_z' C_z \\
 &\quad - (e^{-A' \tau} X_\tau e^{-A \tau} B_u + C_z' D_{zu}) (B_u' e^{-A' \tau} X_\tau e^{-A \tau} + D_{zu}' C_z)) e^{A \tau}
 \end{aligned}$$

Hence,

$$e^{-A' \tau} X_\tau e^{-A \tau} = X \implies X_\tau = e^{A' \tau} X e^{A \tau}$$

and

$$K_\tau = K e^{A \tau}$$

Thus,

— solution based on delay-free Riccati equations.

Some details

Control Riccati (call it X_τ)

$$\begin{aligned}
 0 &= X_\tau A + A' X_\tau + e^{A'\tau} C_z' C_z e^{A\tau} \\
 &\quad - (X_\tau e^{-A\tau} B_u + e^{A'\tau} C_z' D_{zu}) (B_u' e^{-A'\tau} X_\tau + D_{zu}' C_z e^{A\tau}) \\
 &= e^{A'\tau} (e^{-A'\tau} X_\tau e^{-A\tau} A + A' e^{-A'\tau} X_\tau e^{-A\tau} + C_z' C_z \\
 &\quad - (e^{-A'\tau} X_\tau e^{-A\tau} B_u + C_z' D_{zu}) (B_u' e^{-A'\tau} X_\tau e^{-A\tau} + D_{zu}' C_z)) e^{A\tau}
 \end{aligned}$$

Hence,

$$e^{-A'\tau} X_\tau e^{-A\tau} = X \implies X_\tau = e^{A'\tau} X e^{A\tau}$$

and

$$K_\tau = K e^{A\tau}$$

Thus,

→ solution based on delay-free Riccati equations.

Some details

Control Riccati (call it X_τ)

$$\begin{aligned}
 0 &= X_\tau A + A' X_\tau + e^{A'\tau} C_z' C_z e^{A\tau} \\
 &\quad - (X_\tau e^{-A\tau} B_u + e^{A'\tau} C_z' D_{zu}) (B_u' e^{-A'\tau} X_\tau + D_{zu}' C_z e^{A\tau}) \\
 &= e^{A'\tau} (e^{-A'\tau} X_\tau e^{-A\tau} A + A' e^{-A'\tau} X_\tau e^{-A\tau} + C_z' C_z \\
 &\quad - (e^{-A'\tau} X_\tau e^{-A\tau} B_u + C_z' D_{zu}) (B_u' e^{-A'\tau} X_\tau e^{-A\tau} + D_{zu}' C_z)) e^{A\tau}
 \end{aligned}$$

Hence,

$$e^{-A'\tau} X_\tau e^{-A\tau} = X \implies X_\tau = e^{A'\tau} X e^{A\tau}$$

and

$$K_\tau = K e^{A\tau}.$$

Thus,

- solution based on delay-free Riccati equations.

Extraction

We know (Lect. 7) that

$$\mathcal{F}_1(J_2, Q) = \bar{D}_\tau R \iff Q = \pi_\tau\{H_{22}\} + \bar{D}_\tau \bar{Q},$$

where

$$H_{22}(s) = \left[\begin{array}{c|c} A & L \\ \hline K & 0 \end{array} \right].$$

We now also know that

$$\|T_{zw}\|_2^2 = \gamma_{\text{opt}}^2 + \|Q\|_2^2, \quad \pi_\tau\{H_{22}\} \perp \bar{D}_\tau \bar{Q}, \quad \text{and} \quad e^{-\tau s} \text{ is inner.}$$

Hence,

$$\|T_{zw}\|_2^2 = \gamma_{\text{opt}}^2 + \|\pi_\tau\{H_{22}\} + \bar{D}_\tau \bar{Q}\|_2^2 = \gamma_{\text{opt}}^2 + \|\pi_\tau\{H_{22}\}\|_2^2 + \|\bar{Q}\|_2^2$$

and

- optimal $\bar{Q} = 0$
- cost of delay $\gamma_\tau^2 = \int_0^\tau \|K e^{s\tau} L\|_2^2 dt$, monotonically increasing again

Extraction

We know (Lect. 7) that

$$\mathcal{F}_1(J_2, Q) = \bar{D}_\tau R \iff Q = \pi_\tau\{H_{22}\} + \bar{D}_\tau \bar{Q},$$

where

$$H_{22}(s) = \left[\begin{array}{c|c} A & L \\ \hline K & 0 \end{array} \right].$$

We now also know that

$$- \quad \|T_{zw}\|_2^2 = \gamma_{\text{opt}}^2 + \|Q\|_2^2, \quad \pi_\tau\{H_{22}\} \perp \bar{D}_\tau \bar{Q}, \quad \text{and} \quad e^{-\tau s} \text{ is inner.}$$

Hence,

$$\|T_{zw}\|_2^2 = \gamma_{\text{opt}}^2 + \|\pi_\tau\{H_{22}\} + \bar{D}_\tau \bar{Q}\|_2^2 = \gamma_{\text{opt}}^2 + \|\pi_\tau\{H_{22}\}\|_2^2 + \|\bar{Q}\|_2^2$$

and

$$\begin{aligned} & - \text{optimal } \bar{Q} = 0 \\ & - \text{cost of delay } \gamma_\tau^2 = \int_0^\tau \|K e^{sA} L\|_2^2 ds, \text{ monotonically increasing again} \end{aligned}$$

Extraction

We know (Lect. 7) that

$$\mathcal{F}_1(J_2, Q) = \bar{D}_\tau R \iff Q = \pi_\tau\{H_{22}\} + \bar{D}_\tau \bar{Q},$$

where

$$H_{22}(s) = \left[\begin{array}{c|c} A & L \\ \hline K & 0 \end{array} \right].$$

We now also know that

$$- \quad \|T_{zw}\|_2^2 = \gamma_{\text{opt}}^2 + \|Q\|_2^2, \quad \pi_\tau\{H_{22}\} \perp \bar{D}_\tau \bar{Q}, \quad \text{and} \quad e^{-\tau s} \text{ is inner.}$$

Hence,

$$\|T_{zw}\|_2^2 = \gamma_{\text{opt}}^2 + \|\pi_\tau\{H_{22}\} + \bar{D}_\tau \bar{Q}\|_2^2 = \gamma_{\text{opt}}^2 + \|\pi_\tau\{H_{22}\}\|_2^2 + \|\bar{Q}\|_2^2$$

and

- optimal $\bar{Q} = 0$
- cost of delay $\gamma_\tau^2 = \int_0^\tau \|K e^{At} L\|_F^2 dt$, monotonically increasing again

Outline

Optimization-based design: introduction

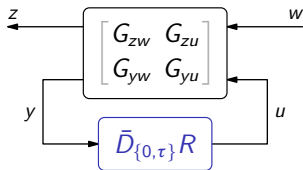
H_2 prediction: gaining insight via a simple problem

H_2 standard problem: single-delay case

H_2 standard problem: multiple-delay extensions

H_∞ standard problem

Setup



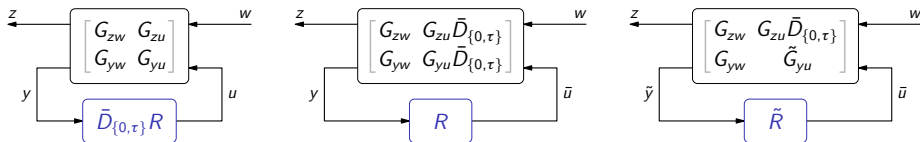
where

$$\bar{D}_{\{0,\tau\}} := \begin{bmatrix} I & 0 \\ 0 & \bar{D}_\tau \end{bmatrix}$$

(known as **input adobe delay**), i.e.

$$u = \begin{bmatrix} u_0 \\ u_\tau \end{bmatrix} = \begin{bmatrix} u_0 \\ \bar{D}_\tau \bar{u}_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \bar{D}_\tau \end{bmatrix} R y = \begin{bmatrix} I & 0 \\ 0 & \bar{D}_\tau \end{bmatrix} \begin{bmatrix} R_0 \\ R_\tau \end{bmatrix} y.$$

Loop shifting: first try



with

$$G_{yu} \bar{D}_{\{0,\tau\}} = \begin{bmatrix} G_{yu,0} & G_{yu,\tau} \bar{D}_\tau \end{bmatrix} = \overbrace{\begin{bmatrix} G_{yu,0} & \tilde{G}_{yu,\tau} \end{bmatrix}}^{\tilde{G}_{yu}} + \overbrace{\begin{bmatrix} 0 & \sigma_\tau \{ G_{yu,\tau} \bar{D}_\tau \} \end{bmatrix}}^{\Pi}$$

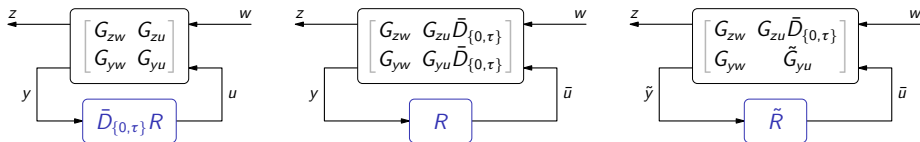
and

$$\tilde{R} = R(I + \Pi R)^{-1}.$$

But

G_{zu} and $\bar{D}_{\{0,\tau\}}$ do not commute in general

Loop shifting: first try



with

$$G_{yu}\bar{D}_{\{0,\tau\}} = \begin{bmatrix} G_{yu,0} & G_{yu,\tau}\bar{D}_\tau \end{bmatrix} = \begin{bmatrix} G_{yu,0} & \tilde{G}_{yu,\tau} \end{bmatrix} + \begin{bmatrix} 0 & \sigma_\tau \{ G_{yu,\tau}\bar{D}_\tau \} \end{bmatrix}$$

and

$$\tilde{R} = R(I + \Pi R)^{-1}.$$

But

☹ G_{zu} and $\bar{D}_{\{0,\tau\}}$ do not commute in general

Loop shifting: simple particular case

Assume that

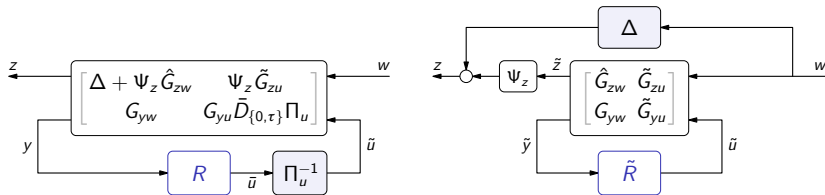
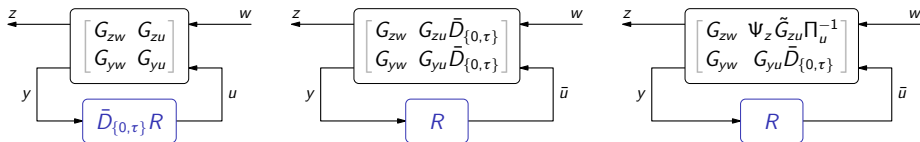
$$G_{zu} = \begin{bmatrix} G_{zu,0} & G_{zu,0\tau} \\ 0 & G_{zu,\tau} \end{bmatrix} \quad \text{for square and invertible } G_{zu,0}.$$

In this case

$$\begin{aligned} G_{zu} \bar{D}_{\{0,\tau\}} &= \begin{bmatrix} G_{zu,0} & G_{zu,0\tau} \bar{D}_\tau \\ 0 & G_{zu,\tau} \bar{D}_\tau \end{bmatrix} = \begin{bmatrix} G_{zu,0} & G_{zu,0} (G_{zu,0}^{-1} G_{zu,0\tau} \bar{D}_\tau) \\ 0 & \bar{D}_\tau G_{zu,\tau} \end{bmatrix} \\ &= \begin{bmatrix} G_{zu,0} & G_{zu,0} (\tilde{G}_a - \sigma_\tau \{ G_{zu,0}^{-1} G_{zu,0\tau} \bar{D}_\tau \}) \\ 0 & \bar{D}_\tau G_{zu,\tau} \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} I & 0 \\ 0 & \bar{D}_\tau \end{bmatrix}}_{\Psi_z} \underbrace{\begin{bmatrix} G_{zu,0} & \tilde{G}_{zu,0\tau} \\ 0 & G_{zu,\tau} \end{bmatrix}}_{\tilde{G}_{zu}} \underbrace{\begin{bmatrix} I & -\Pi_{u,0\tau} \\ 0 & I \end{bmatrix}}_{\Pi_u^{-1}}, \end{aligned}$$

where $\tilde{G}_{zu,0\tau} := G_{zu,0} \tilde{G}_a$ and $\Pi_{u,0\tau} := \sigma_\tau \{ G_{zu,0}^{-1} G_{zu,0\tau} \bar{D}_\tau \}$.

Loop shifting: simple particular case (contd)



for

$$\tilde{R} := \Pi_u^{-1} R (I + \Pi_y \Pi_u^{-1} R)^{-1} \quad \text{and} \quad \Pi_y = \sigma_\tau \{ G_{yu} \bar{D}_{\{0,\tau\}} \Pi_u \}.$$

Moreover, $\Delta \perp \Psi_z T_{\tilde{z}w}$ for all $T_{\tilde{z}w} \in H_2$ (nonoverlapping impulse responses).

Loop shifting: general case

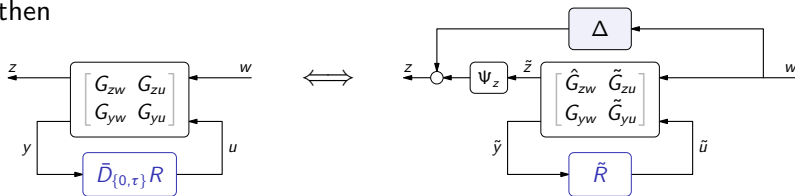
Key is to factorize

$$G_{zu} \bar{D}_{\{0,\tau\}} = \Psi_z \tilde{G}_{zu} \Pi_u^{-1}$$

for

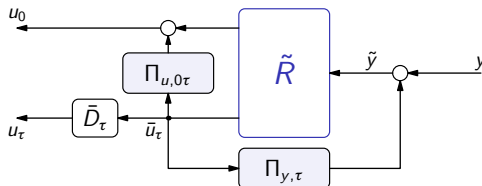
- finite-dimensional $\tilde{G}_{zu}$
- inner Ψ_z (i.e. stable and energy preserving)
- bi-stable Π_u (i.e. $\Pi_u, \Pi_u^{-1} \in H_\infty$)

and then



with $\Delta \perp \Psi_z T_{\tilde{z}w}$ for all $T_{\tilde{z}w} \in H_2$.

Controller architecture: FASP



Two FIR DTC elements:

- almost conventional feedback $\Pi_{y,\tau}$
compensates delay in the feedback loop
- unorthodox feedforward $\Pi_{u,0\tau}$
compensates delay in the performance channel

Outline

Optimization-based design: introduction

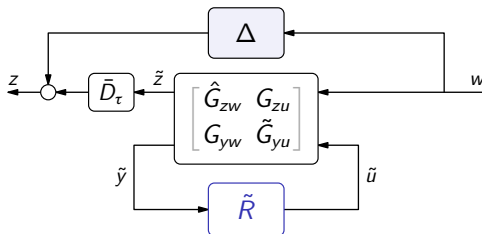
H_2 prediction: gaining insight via a simple problem

H_2 standard problem: single-delay case

H_2 standard problem: multiple-delay extensions

H_∞ standard problem

Loop shifting with performance (contd)



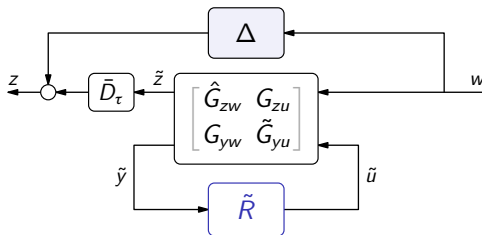
Important:

- H_∞ is not a Hilbert space $\implies$ orthogonality might not be useful

Still,

- if $G_{zu} = 0$, then $\Delta = 0$ and $\|T_{zw}\|_\infty = \|\bar{D}_\tau T_{zw}\|_\infty = \|T_{zw}\|_\infty$

Loop shifting with performance (contd)



Important:

- H_∞ is not a Hilbert space $\implies$ orthogonality might not be useful

Still,

- if $G_{zw} = 0$, then $\Delta = 0$ and $\|T_{zw}\|_\infty = \|\bar{D}_\tau T_{\tilde{z}w}\|_\infty = \|T_{\tilde{z}w}\|_\infty$

example:

- multiplicative uncertainty $P(1 + W\Delta) \implies G = \begin{bmatrix} 0 & 1 \\ PW & P \end{bmatrix}$, optimally robust controller is the MSP!

Example

Let

$$T_{zw}(s) = \pi_\tau \left\{ \frac{1}{s} \right\} + e^{-\tau s} Q = \frac{1 - e^{-\tau s}}{s} + e^{-\tau s} Q.$$

For (H_2 -optimal) $Q = 0$

$$\|T_{zw}\|_\infty = \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \left| \frac{1 - e^{-j\omega\tau}}{\omega} \right| = \max_{\omega \in \mathbb{R}} \tau \left| \operatorname{sinc} \frac{\omega\tau}{2} \right| = \tau = |T_{zw}(0)|.$$

Now, let

$$Q(s) = Q_\infty(s) := \frac{1}{s} - \frac{(2\tau)^2 s^2 + \pi^2}{\pi s(2\tau s + \pi e^{-\tau s})} \in H_\infty$$

so that

$$T_{zw}(s) = \frac{2\tau \pi - 2\tau s e^{-\tau s}}{\pi (2\tau s + \pi e^{-\tau s})} = \frac{2\tau}{\pi} e^{-\tau s} \frac{(2\tau s + \pi e^{-\tau s})^{\text{inner}}}{2\tau s + \pi e^{-\tau s}}$$

and $\|T_{zw}\|_\infty = (2/\pi)\tau$, $\approx 64\%$ of what we had with $Q = 0$. As a matter of fact, $Q = Q_\infty$ is the optimal solution.

Example

Let

$$T_{zw}(s) = \pi_\tau \left\{ \frac{1}{s} \right\} + e^{-\tau s} Q = \frac{1 - e^{-\tau s}}{s} + e^{-\tau s} Q.$$

For (H_2 -optimal) $Q = 0$

$$\|T_{zw}\|_\infty = \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \left| \frac{1 - e^{-j\omega\tau}}{\omega} \right| = \max_{\omega \in \mathbb{R}} \tau \left| \operatorname{sinc} \frac{\omega\tau}{2} \right| = \tau = |T_{zw}(0)|.$$

Now, let

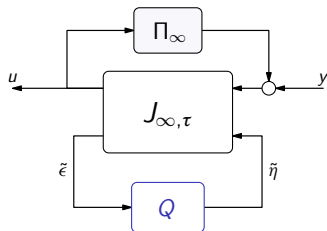
$$Q(s) = Q_\infty(s) := \frac{1}{s} - \frac{(2\tau)^2 s^2 + \pi^2}{\pi s(2\tau s + \pi e^{-\tau s})} \in H_\infty$$

so that

$$T_{zw}(s) = \frac{2\tau}{\pi} \frac{\pi - 2\tau s e^{-\tau s}}{2\tau s + \pi e^{-\tau s}} = \frac{2\tau}{\pi} e^{-\tau s} \overbrace{\frac{(2\tau s + \pi e^{-\tau s})^\sim}{2\tau s + \pi e^{-\tau s}}}^{\text{inner}}$$

and $\|T_{zw}\|_\infty = (2/\pi)\tau$, $\approx 64\%$ of what we had with $Q = 0$. As a matter of fact, $Q = Q_\infty$ is the optimal solution.

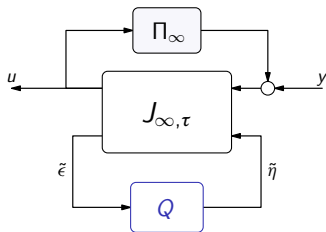
All solutions of H_∞ problem with loop delay



for

- finite-dimensional $J_{\infty, \tau}$
 - the same state dimension as that of $J_{\infty, 0} = J_\infty$
 - requires the same CAREs as $J_{\infty, 0} = J_\infty$
- any $Q \in H_\infty$ such that $\|Q\|_\infty < \gamma$
- FIR $\Pi_\infty = \sigma_\tau \{ \mathcal{F}_u(G, \gamma^{-2} G_{zw}^\sim) \bar{D}_\tau \}$

All solutions of H_∞ problem with loop delay



for

- finite-dimensional $J_{\infty, \tau}$
 - the same state dimension as that of $J_{\infty, 0} = J_\infty$
 - requires the same CAREs as $J_{\infty, 0} = J_\infty$
- any $Q \in H_\infty$ such that $\|Q\|_\infty < \gamma$
- FIR $\Pi_\infty = \sigma_\tau \{ \mathcal{F}_u(G, \gamma^{-2} G_{zw}^\sim) \bar{D}_\tau \}$

H_∞ dead-time compensation

Because $y = G_{yu}\bar{D}_\tau u + G_{yw}w$, the signal

$$\mathcal{F}_u(G, \gamma^{-2}G_{zw}^\sim)\bar{D}_\tau u = G_{yu}\bar{D}_\tau u + G_{yw}(\gamma^2 I - G_{zw}^\sim G_{zw})^{-1}G_{zw}^\sim G_{zu}\bar{D}_\tau u$$

can be viewed as

- response of the system to $w = w_* := (\gamma^2 I - G_{zw}^\sim G_{zw})^{-1}G_{zw}^\sim G_{zu}\bar{D}_\tau u$.

It can be shown that

- w_* is the worst-case disturbance for z in case when u plays open-loop (it maximizes $\|z\|_2^2 - \gamma^2\|w\|_2^2$ for every given u).

In other words,

- H_∞ DTC aims at compensating delay in the presence of the worst-case disturbance while the loop is open (in $[0, \tau]$)
- thus, coping for the best and preparing for the worst.

H_∞ dead-time compensation

Because $y = G_{yu}\bar{D}_\tau u + G_{yw}w$, the signal

$$\mathcal{F}_u(G, \gamma^{-2}G_{zw}^\sim)\bar{D}_\tau u = G_{yu}\bar{D}_\tau u + G_{yw}(\gamma^2 I - G_{zw}^\sim G_{zw})^{-1}G_{zw}^\sim G_{zu}\bar{D}_\tau u$$

can be viewed as

- response of the system to $w = w_* := (\gamma^2 I - G_{zw}^\sim G_{zw})^{-1}G_{zw}^\sim G_{zu}\bar{D}_\tau u$.

It can be shown that

- w_* is the **worst-case** disturbance for z in case when u plays open-loop (it maximizes $\|z\|_2^2 - \gamma^2\|w\|_2^2$ for every given u).

In other words,

H_∞ DTC aims at compensating delay in the presence of the worst-case disturbance while the loop is open (in $[0, \tau]$)

thus **compensating the delay by preparing for the worst.**

H_∞ dead-time compensation

Because $y = G_{yu}\bar{D}_\tau u + G_{yw}w$, the signal

$$\mathcal{F}_u(G, \gamma^{-2}G_{zw}^\sim)\bar{D}_\tau u = G_{yu}\bar{D}_\tau u + G_{yw}(\gamma^2 I - G_{zw}^\sim G_{zw})^{-1}G_{zw}^\sim G_{zu}\bar{D}_\tau u$$

can be viewed as

- response of the system to $w = w_* := (\gamma^2 I - G_{zw}^\sim G_{zw})^{-1}G_{zw}^\sim G_{zu}\bar{D}_\tau u$.

It can be shown that

- w_* is the **worst-case** disturbance for z in case when u plays open-loop (it maximizes $\|z\|_2^2 - \gamma^2\|w\|_2^2$ for every given u).

In other words,

- H_∞ DTC aims at compensating delay in the presence of the worst-case disturbance while the loop is open (in $[0, \tau]$)

thus (hoping for the best and) preparing for the worst.

H_∞ cost of delay

If delay-free H_∞ problem is solvable, then conditions below are equivalent:

- the delayed problem is solvable
- $\begin{bmatrix} -X & I \end{bmatrix} \exp \left(\begin{bmatrix} A & \gamma^{-2} B_w B'_w \\ -C_z C'_z & -A' \end{bmatrix} t \right) \begin{bmatrix} \gamma^{-2} Y \\ I \end{bmatrix}$ invertible $\forall t \in [0, \tau]$
- $\Upsilon(t)$ satisfying

$$\dot{\Upsilon} + \Upsilon A + A' \Upsilon + C'_z C_z + \gamma^{-2} \Upsilon B_w B'_w \Upsilon = 0, \quad \Upsilon(\tau) = X$$

exists $\forall t \in [0, \tau]$ and is such that $\rho(\Upsilon(0)Y) < \gamma^2$

- $\Omega(t)$ satisfying

$$-\dot{\Omega} + \Omega A + A' \Omega + C'_z C_z + \gamma^{-2} \Omega B_w B'_w \Omega = 0, \quad \Omega(0) = Y$$

exists $\forall t \in [0, \tau]$ and is such that $\rho(X\Omega(\tau)) < \gamma^2$