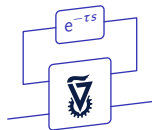


Time Delays in Control and Estimation (038806)

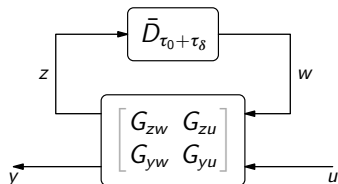


lecture no. 8

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Setup

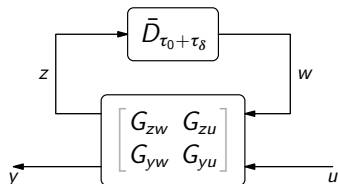


where

- $\tau_\delta \geq -\tau_0$ is deviation from the nominal delay $\tau_0 \geq 0$
- the system is stable for $\tau_\delta = 0$

Equivalently,

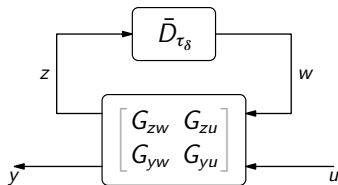
Setup



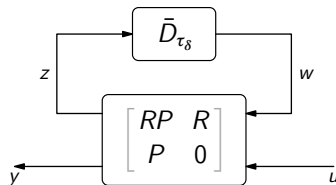
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- $\tau_\delta \geq -\tau_0$ is deviation from the nominal delay $\tau_0 \geq 0$
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Equivalently,



or



Outline

Delay margin

Delay margin for DTC-based loops

Embedding

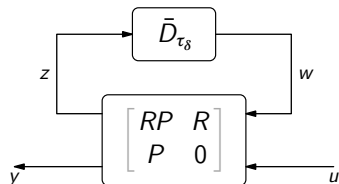
Outline

Delay margin

Delay margin for DTC-based loops

Embedding

Definition



Assuming the system is stable for $\tau_\delta = 0$ (i.e. $\bar{D}_{\tau_\delta} = 1$),

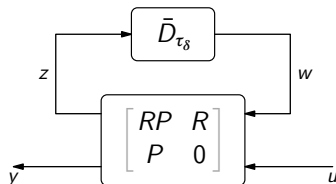
- delay margin μ_d is the minimum destabilizing τ_δ .

Sometimes calculated in terms of the loop $L(s) = R(s)P(s)$ as

$$\mu_d = \frac{\mu_{ph}}{\omega_c}$$

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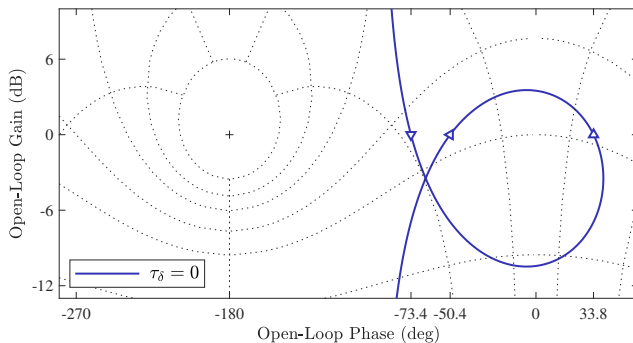
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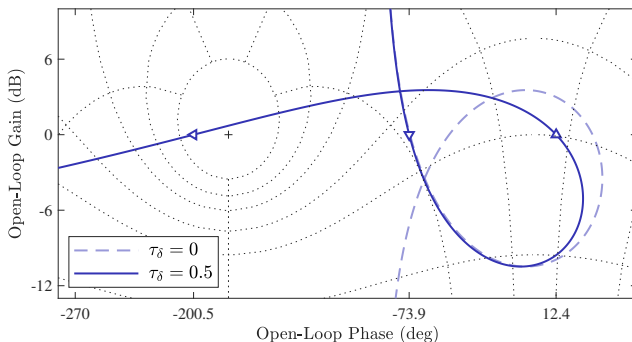
Example 1



Crossover:

- $\omega_c \approx 0.015$ with $\mu_{ph} \approx 1.86$ [rad], so $\mu_d \approx 121.16$
- $\omega_{c,2} \approx 0.746$ with $\mu_{ph,2} \approx 3.73$, so $\mu_{d,2} \approx 5$
- $\omega_{c,3} \approx 5.239$ with $\mu_{ph,3} \approx 2.26$, so $\mu_{d,3} \approx 0.432$

Example 1



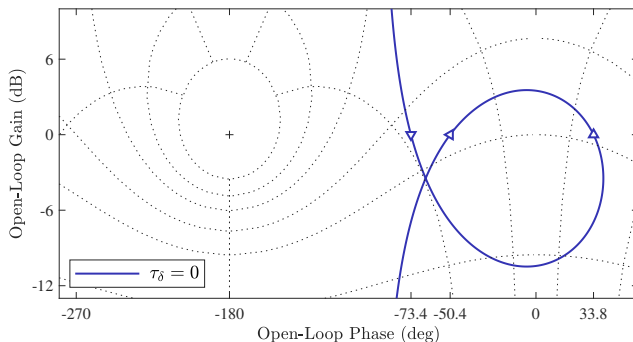
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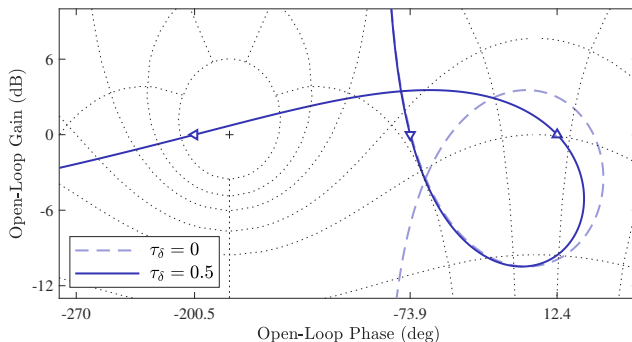
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Crossovers:

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- $\omega_{c,2} \approx 0.746$ with $\mu_{ph,2} \approx 3.73$, so $\mu_{d,2} \approx 5$
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Accurate calculation

In the positive direction ($\tau_d > 0$):

$$\mu_d = \begin{cases} 0 & \text{if } \limsup_{\omega \rightarrow \infty} |P(j\omega)R(j\omega)| \geq 1 \\ \min_i \frac{\mu_{ph,i}}{\omega_{c,i}} & \text{otherwise} \end{cases}$$

In the negative direction ($\tau_d < 0$):

$$\bar{\mu}_d = \min \left\{ -\tau_0, \max_i \frac{\bar{\mu}_{ph,i}}{\omega_{c,i}} \right\}$$

Accurate calculation

In the positive direction ($\tau_\delta > 0$):

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In the negative direction ($\tau_\delta < 0$):

$$\tilde{\mu}_d = \min \left\{ -\tau_0, \max_i \frac{\tilde{\mu}_{\text{ph},i}}{\omega_{\text{c},i}} \right\}$$

Bounds on μ_d : no poles in $\bar{\mathbb{C}}_0 \setminus \{0\}$

Proposition

If $P(s)$ has no poles in $\bar{\mathbb{C}}_0 \setminus \{0\}$, then any μ_d is attainable by a stabilizing R .

Bounds on μ_d : real poles in $\bar{\mathbb{C}}_0$

Proposition

If $P(s)$ has a pole at $s = a > 0$, then

$$\mu_d < \frac{2}{a}.$$

This bound is tight if this pole is the only pole of $P(s)$ in $\bar{\mathbb{C}}_0$ and $P(s)$ is minimum-phase.

Proof (idea): Use continuity arguments for Rekašius transformation,

$$\tilde{R}_\alpha(s) := \frac{-s + \alpha}{s + \alpha} \text{ with } \tau = \frac{2}{\omega} \arctan \frac{\omega}{\alpha} < \frac{2}{\alpha}$$

If $\alpha \downarrow$ from $+\infty$, the loop is unstable for $\alpha = a$ (unstable cancellations) and has pure imaginary pole for some $\alpha > a$. $\square$

Bounds on μ_d : real poles in $\bar{\mathbb{C}}_0$

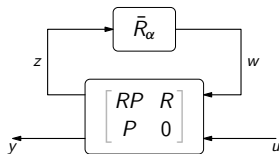
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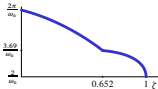
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Bounds on μ_d : complex poles in $\bar{\mathbb{C}}_0$

Proposition

If $P(s)$ has poles at $s = (\zeta \pm j\sqrt{1-\zeta^2})\omega_n$ for $0 \leq \zeta < 1$ and $\omega_n > 0$, then

$$\mu_d < \frac{\sqrt{1-\zeta^2}}{\omega_n} \left(\pi + 2 \max \left\{ \frac{\zeta}{\sqrt{1-\zeta^2}}, \arctan \frac{\sqrt{1-\zeta^2}}{\zeta} \right\} \right) =$$


This bound is tight if these poles are the only poles of $P(s)$ in $\bar{\mathbb{C}}_0$ and $P(s)$ is minimum-phase.

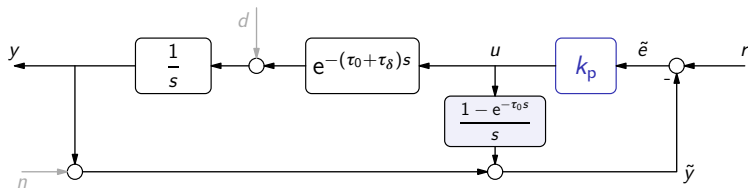
Outline

Delay margin

Delay margin for DTC-based loops

Embedding

Setup



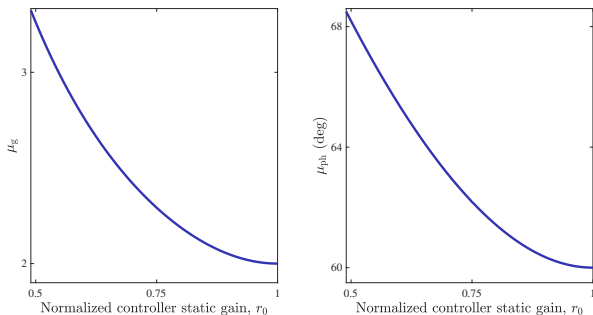
Delay-free design for $P(s) = 1/s$ has $\tilde{\chi}(s) = s + k_p$, so $k_p > 0$. Hence

$$T(s) = \frac{k_p}{s + k_p} e^{-\tau_0 s},$$

whose bandwidth $\omega_b = k_p$ increases with k_p . The static gain of R is

$$R(0) = \lim_{s \rightarrow 0} \frac{k_p s}{s + k_p (1 - e^{-\tau_0 s})} = \frac{k_p}{1 + \tau_0 k_p} = \frac{1}{\tau_0} \underbrace{\frac{\tau_0 k_p}{1 + \tau_0 k_p}}_{:= r_0 \in (0,1)}$$

Stability margins



Expectably decrease

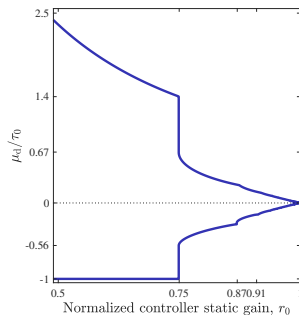
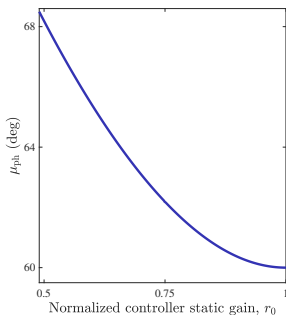
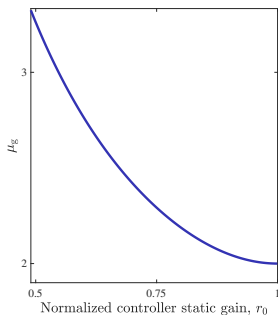
μ_g : from ∞ to 2 (not bad)

μ_{ph} : from 90° to 60° (not bad)

Unexpectedly decreases

μ_d : from ∞ to 0 (hmm), discontinuous, with dramatic drop at $k_p \approx 0.75$

Stability margins



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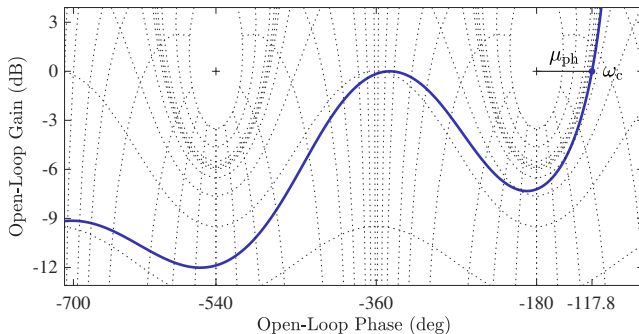
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Loops



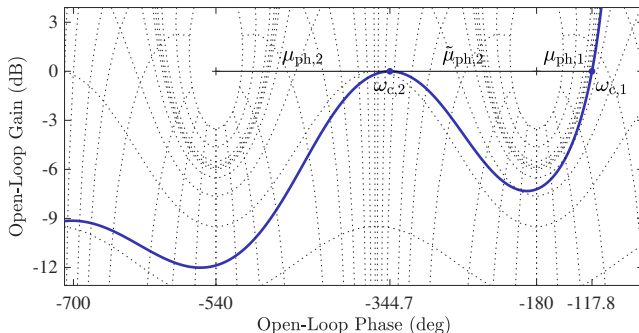
For

$$r_0 = 0.749: \mu_d \approx \frac{1.086}{0.774/\tau_0} \approx 1.4\tau_0$$

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$$\mu_d = \min \left\{ \frac{\mu_{ph,1}}{\omega_{c,1}}, \frac{\mu_{ph,2}}{\omega_{c,2}} \right\} \approx \min \left\{ \frac{1.086}{0.774/\tau_0}, \frac{3.409}{5.1075/\tau_0} \right\} \approx 0.667\tau_0$$

Loops



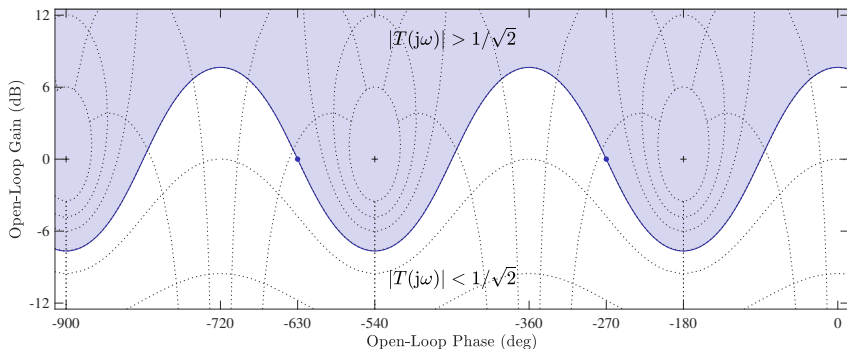
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General observations

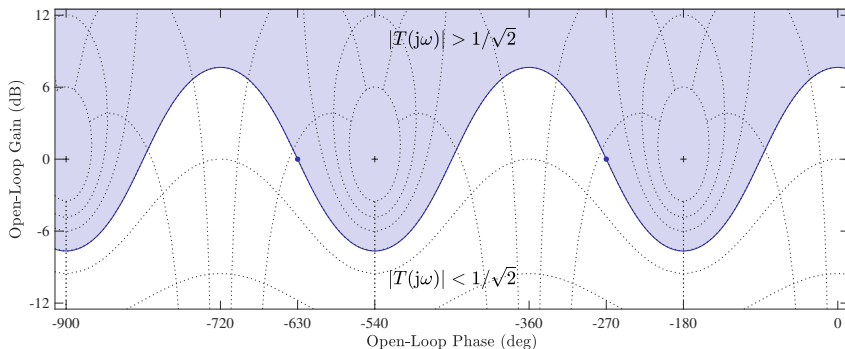


Crossover

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— via $|S(j\omega)| < 1/\sqrt{2} \ll 1$ may help to resist the temptation

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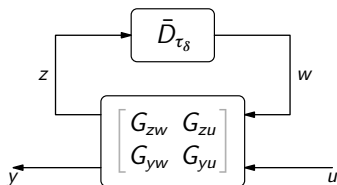
Outline

Delay margin

Delay margin for DTC-based loops

Embedding

Idea



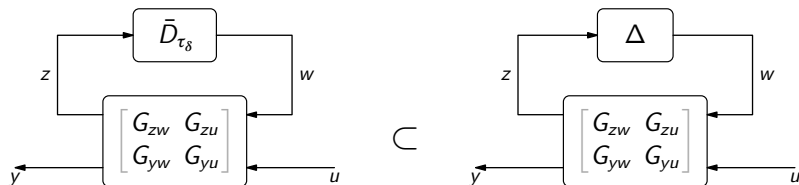
Might be hard to deal with but easier via

$$\{\bar{D}_{\tau_\delta} \mid \tau_\delta \in [0, \bar{\tau}]\} \subset \{\Delta \in H_\infty \mid \Delta \in \Delta_{\bar{\tau}}\} \text{ for a wider } \Delta_{\bar{\tau}}$$

Key to success is to find $\Delta_{\bar{\tau}}$, which

- covers $\{\bar{D}_{\tau_\delta} \mid \tau_\delta \in [0, \bar{\tau}]\}$ tightly
- is easier to analyze, can be incorporated into design problems

Idea



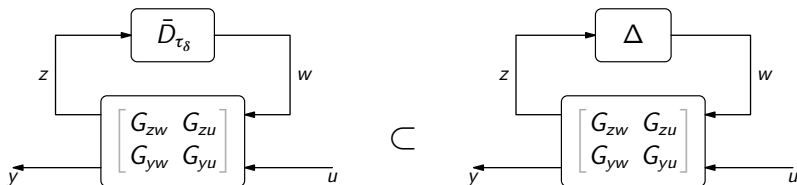
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Unit ball $\mathcal{B}_\infty$

Handiest class:

$$\mathcal{B}_\infty^{m \times m} := \{\Delta \in H_\infty^{m \times m} \mid \|\Delta\|_\infty \leq 1\}$$

Theorem (Robust Stability Theorem)

The feedback interconnection

is internally stable for all $\Delta \in \mathcal{B}_\infty^{m \times m} \iff T_{rob} \in H_\infty$ and $\|T_{rob}\|_\infty < 1$.

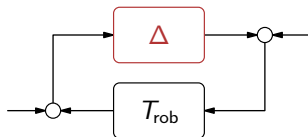
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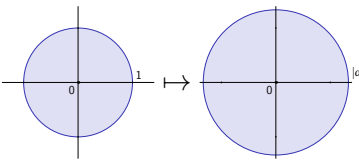
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Adding structure via LFTs

Scaling with $\alpha \in \mathbb{C}$

$$\mathcal{B}_\infty \mapsto \mathcal{F}_u \left(\begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}, \mathcal{B}_\infty \right) \iff$$


Shift with $\kappa \in \mathbb{C}$

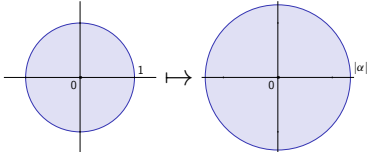
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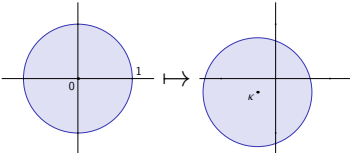
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The diagram illustrates the scaling operation in the complex plane. On the left, a unit circle is centered at the origin (0) with a radius of 1. An arrow points to the right, where a larger circle is shown, also centered at the origin (0), with a radius labeled $|\alpha|$. This represents the mapping of the unit disk to a scaled disk.

Shift with $\kappa \in \mathbb{C}$

$$\mathcal{B}_\infty \mapsto \mathcal{F}_u \left(\begin{bmatrix} 0 & 1 \\ 1 & \kappa \end{bmatrix}, \mathcal{B}_\infty \right) \iff$$


The diagram illustrates the shifting operation in the complex plane. On the left, a unit circle is centered at the origin (0). An arrow points to the right, where a circle of the same radius is shown, but its center is shifted to a point labeled κ .

Shifted scaling

$$\mathcal{B}_\infty \mapsto \mathcal{F}_u \left(\begin{bmatrix} 0 & \alpha \\ 1 & \kappa \end{bmatrix}, \mathcal{B}_\infty \right) \iff$$

Adding structure via LFTs

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$$\mathcal{B}_\infty \mapsto \mathcal{F}_u \left(\begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix}, \mathcal{B}_\infty \right) \iff \text{Diagram showing scaling of the unit disk by } |\alpha|$$

Shift with $\kappa \in \mathbb{C}$

$$\mathcal{B}_\infty \mapsto \mathcal{F}_u \left(\begin{bmatrix} 0 & 1 \\ 1 & \kappa \end{bmatrix}, \mathcal{B}_\infty \right) \iff \text{Diagram showing shifting of the unit disk to center } \kappa$$

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$$\mathcal{B}_\infty \mapsto \mathcal{F}_u \left(\begin{bmatrix} 0 & \alpha \\ 1 & \kappa \end{bmatrix}, \mathcal{B}_\infty \right) \iff \text{Diagram showing shifted scaling of the unit disk}$$

Adding structure via LFTs

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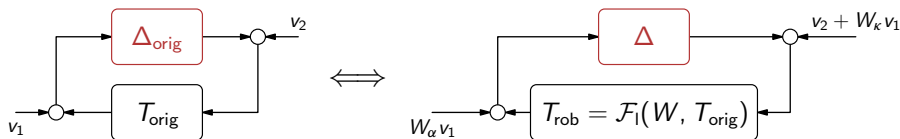
All this can be **frequency-dependent** and in all **stability** must be preserved.

Back to $\mathcal{B}_\infty$

If

$$\Delta_{\text{orig}} \in \mathcal{F}_u \left(\underbrace{\begin{bmatrix} 0 & W_\alpha \\ I & W_\kappa \end{bmatrix}}_W, \mathcal{B}_\infty \right),$$

then



for $\Delta \in \mathcal{B}_\infty$ ($W_\kappa, W_\alpha, W_\alpha^{-1} \in H_\infty$). Hence,

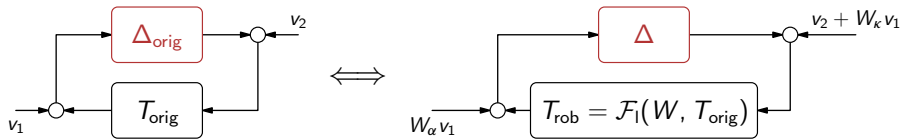
- W_κ is a stable system such that $W_\kappa(j\omega) \approx x(\omega)$
- W_α is a bi-stable system such that $|W_\alpha(j\omega)| \geq \alpha(\omega)$
- W_α with $W_\alpha(0) = 0$ and/or non-proper $W_\alpha(s)$ are often used, but this is sloppy

Back to $\mathcal{B}_\infty$

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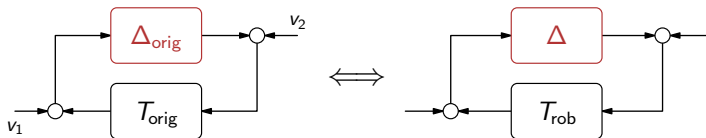
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for $\Delta \in \mathcal{B}_\infty (W_\kappa, W_\alpha, W_\alpha^{-1} \in H_\infty)$. Hence,

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Back to $\mathcal{B}_\infty$ (contd)



If the (dead-time) system to analyze is

$$\mathcal{F}_u \left(\begin{bmatrix} RP & R \\ P & 0 \end{bmatrix}, \bar{D}_{\tau_\delta} \right) \implies T_{\text{orig}} = RP,$$

then

$$\begin{aligned} T_{\text{rob}} &= \mathcal{F}_l \left(\begin{bmatrix} 0 & W_\alpha \\ I & W_\kappa \end{bmatrix}, RP \right) \\ &= \mathcal{F}_l \left(\begin{bmatrix} 0 & W_\alpha \\ P & PW_\kappa \end{bmatrix}, R \right) = W_\alpha R (I - PW_\kappa R)^{-1} P \end{aligned}$$

Example

Consider the delay margin for

$$P(s) = \frac{1}{s - a}, \quad a > 0$$

i.e. the supremal $\bar{\tau}$ such that it can be stabilized for all $\tau \in [0, \bar{\tau})$. If

$$\bar{D}_{\tau_\delta} \subset \mathcal{F}_u \left(\begin{bmatrix} 0 & W_\alpha \\ 1 & W_\kappa \end{bmatrix}, \mathcal{B}_\infty \right)$$

for stable weights W_α and W_κ , then by the Robust Stability Theorem the system stable *if*

$$\|W_\alpha R(I - PW_\kappa R)^{-1}P\|_\infty < 1$$

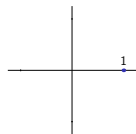
or T_{rob} stable and

$$\left| \frac{P(j\omega)R(j\omega)W_\alpha(j\omega)}{1 - P(j\omega)R(j\omega)W_\kappa(j\omega)} \right| < 1, \quad \forall \omega$$

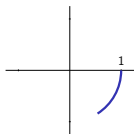
Uncertain delays in $\mathbb{C}$

If $\tau_\delta \in [0, \bar{\tau}]$, then at a **fixed** ω the delay $e^{-\tau_\delta \omega}$ is

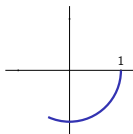
$$\bar{\tau}\omega = 0$$



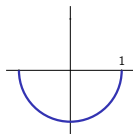
$$\bar{\tau}\omega = 1$$



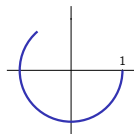
$$\bar{\tau}\omega = 2$$



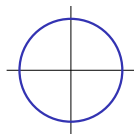
$$\bar{\tau}\omega = \pi$$



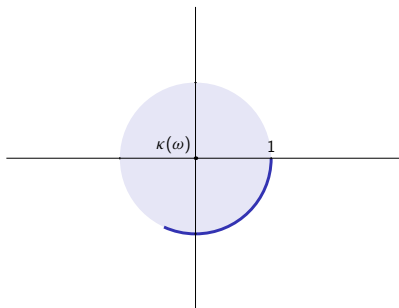
$$\bar{\tau}\omega = 4$$



$$\bar{\tau}\omega \geq 2\pi$$



Covering with $\kappa(\omega) = 0$



in this case

$$\alpha(\omega) = 1$$

and

$$T_{\text{rob}}(s) = P(s)R(s)$$

Because

- neither α nor κ depends on $\bar{\tau} \implies$ delay-independent conditions.

Covering with $\kappa(\omega) = 0$: example

Sufficient stability condition

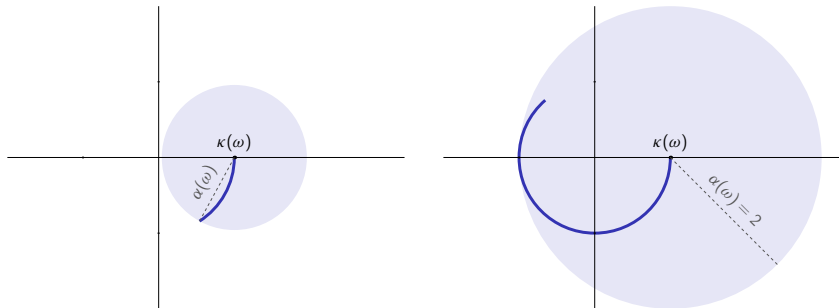
$$\|PR\|_{\infty} < 1$$

obviously

- is feasible for no R ,

as no R can internally stabilize the unstable $P(s) = 1/(s - a)$ in open loop.

Covering with $\kappa(\omega) = 1$



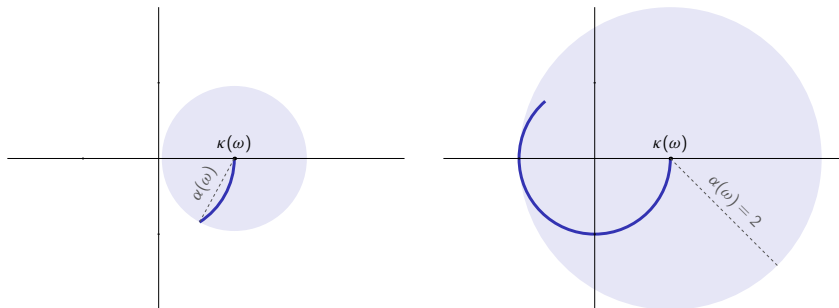
in this case

$$\alpha(\omega) = \max_{\tau_\delta \in [0, \bar{\tau}]} \{|1 - e^{-j\omega\tau_\delta}|\} = \begin{cases} |1 - e^{-j\bar{\tau}\omega}| & \text{if } \bar{\tau}\omega \in [0, \pi] \\ |1 - e^{-j\pi}| = 2 & \text{otherwise.} \end{cases}$$

Hence, $\alpha(\omega) = \alpha_1(\bar{\tau}\omega)$, where

$$\alpha_1(\omega) := \begin{cases} 2\sin(\omega/2) & \text{if } \omega \in [0, \pi] \\ 2 & \text{otherwise} \end{cases}$$

Covering with $\kappa(\omega) = 1$



in this case

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Hence, $\alpha(\omega) = \alpha_1(\bar{\tau}\omega)$, where

$$\alpha_1(\omega) := \begin{cases} 2 \sin(\omega/2) & \text{if } \omega \in [0, \pi] \\ 2 & \text{otherwise} \end{cases} = \begin{array}{c} 6 \text{ dB} \\ \text{---} \end{array}$$

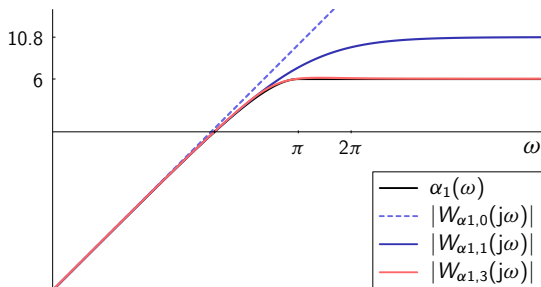
Rational covering of α_1

Approximations

$$W_{\alpha 1,0}(s) := s, \quad W_{\alpha 1,1}(s) := \frac{2\sqrt{3}s}{s + 2\sqrt{3}},$$

$$W_{\alpha 1,3}(s) := \frac{2.007s}{s + 2} \frac{s^2 + 3.695s + 5.56}{s^2 + 3.026s + 5.56},$$

yield



Covering with $\kappa(\omega) = 1$: example

The H_∞ problem to solve:

$$\left\| \mathcal{F}_l \left(\begin{bmatrix} 0 & W_\alpha \\ P & P \end{bmatrix}, R \right) \right\|_\infty = \left\| \frac{PR}{1 - PR} W_\alpha \right\|_\infty < 1.$$

Because R must not cancel the plant pole,

$$\frac{P(s)R(s)}{1 - P(s)R(s)} W_\alpha(s) \Big|_{s=a} = -W_\alpha(a) \quad \text{for every stabilizing } R,$$

so that $\|T_{\text{rob}}\|_\infty \geq W_\alpha(1)$. This bound is **approachable** (single constraint in $\mathbb{C}_0$).

$\ell = 0$: $|W_{\alpha,1,0}(\bar{r}a)| < 1 \iff \bar{r} < 1/a$,

$\ell = 1$: $|W_{\alpha,1,1}(\bar{r}a)| < 1 \iff \bar{r} < 1.406/a$,

$\ell = 3$: $|W_{\alpha,1,3}(\bar{r}a)| < 1 \iff \bar{r} < 1.702/a$

more conservative than the actual bound, $2/a$.

Covering with $\kappa(\omega) = 1$: example

The H_∞ problem to solve:

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so that $\|T_{\text{rob}}\|_\infty \geq W_\alpha(1)$. This bound is approachable (single constraint in $\mathbb{C}_0$). Hence, the robustly stability problem is feasible iff $|W_{\alpha 1, i}(\bar{\tau}a)| < 1$.

$$i = 0: |W_{\alpha 1, 0}(\bar{\tau}a)| < 1 \iff \bar{\tau} < 1/a,$$

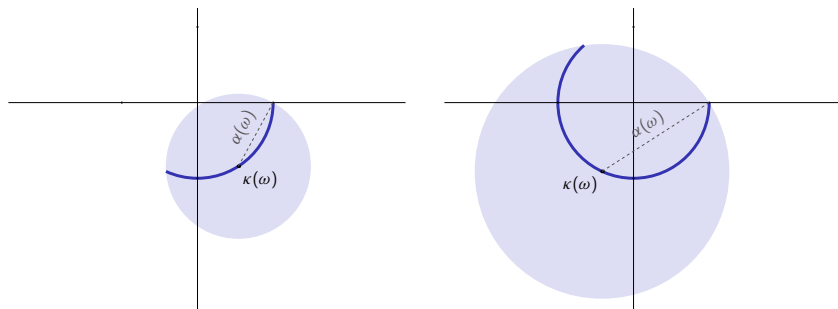
$$i = 1: |W_{\alpha 1, 1}(\bar{\tau}a)| < 1 \iff \bar{\tau} < 1.406/a,$$

$$i = 3: |W_{\alpha 1, 3}(\bar{\tau}a)| < 1 \iff \bar{\tau} < 1.702/a$$

$$i = \infty: \alpha_1(\bar{\tau}a) < 1 \iff \bar{\tau} < 1.7257788469179/a$$

more conservative than the actual bound, $2/a$.

Covering with $\kappa(\omega) = e^{-\bar{\tau}s/2}$



$$\begin{aligned}
 \alpha(\omega) &= \max_{\tau_\delta \in [0, \bar{\tau}]} \{ |e^{-j\bar{\tau}\omega/2} - e^{-j\omega\tau_\delta}| \} = \max_{\tau_\delta \in [0, \bar{\tau}]} \{ |1 - e^{-j\omega(\tau_\delta - \bar{\tau}/2)}| \} \\
 &= \begin{cases} |1 - e^{-j\bar{\tau}\omega/2}| & \text{if } \bar{\tau}\omega \in [0, 2\pi] \\ |1 - e^{-j\pi}| & \text{otherwise} \end{cases} \\
 &= \alpha_1(\bar{\tau}\omega/2) < \alpha_1(\bar{\tau}\omega), \quad \forall \omega > 0
 \end{aligned}$$

Covering with $\kappa(\omega) = e^{-\bar{\tau}s/2}$: example

The H_∞ problem to solve:

$$\left\| \mathcal{F}_l \left(\begin{bmatrix} 0 & W_\alpha \\ P & P\bar{D}_{\bar{\tau}/2} \end{bmatrix}, R \right) \right\|_\infty = \left\| \frac{PR}{1 - PR\bar{D}_{\bar{\tau}/2}} W_\alpha \right\|_\infty < 1.$$

Now

$$\frac{P(s)R(s)}{1 - P(s)R(s)e^{-\bar{\tau}s/2}} W_\alpha(s) \Big|_{s=a} = -e^{a\bar{\tau}/2} W_\alpha(a) \quad \text{for every stabilizing } R$$

and the problem is feasible iff $|W_{\alpha 2,i}(\bar{\tau}a)| = |W_{\alpha 1,i}(\bar{\tau}a/2)| < e^{-a\bar{\tau}/2}$.

$$i=0: |W_{\alpha 2,0}(\bar{\tau}a)| < e^{-a\bar{\tau}/2} \iff \bar{\tau} < 1.134/a,$$

$$i=1: |W_{\alpha 2,1}(\bar{\tau}a)| < e^{-a\bar{\tau}/2} \iff \bar{\tau} < 1.259/a,$$

$$i=2: |W_{\alpha 2,2}(\bar{\tau}a)| < e^{-a\bar{\tau}/2} \iff \bar{\tau} < 1.305/a.$$

$$i=3: |W_{\alpha 2,3}(\bar{\tau}a)| < e^{-a\bar{\tau}/2} \iff \bar{\tau} < 1.342/a.$$

If $i \neq 0$, more conservative than with $\kappa(\omega) = 1$. Why?

Covering with $\kappa(\omega) = e^{-\bar{\tau}s/2}$: example

The H_∞ problem to solve:

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Now

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$$i = 0: |W_{\alpha 2,0}(\bar{\tau}a)| < e^{-a\bar{\tau}/2} \iff \bar{\tau} < 1.134/a,$$

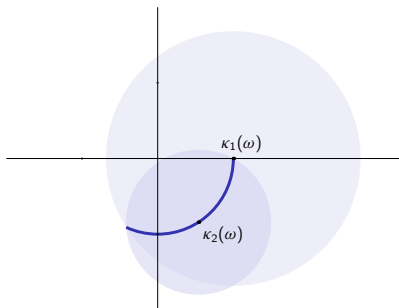
$$i = 1: |W_{\alpha 2,1}(\bar{\tau}a)| < e^{-a\bar{\tau}/2} \iff \bar{\tau} < 1.259/a,$$

$$i = 3: |W_{\alpha 2,3}(\bar{\tau}a)| < e^{-a\bar{\tau}/2} \iff \bar{\tau} < 1.305/a.$$

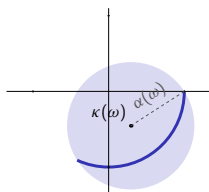
$$i = \infty: \alpha_1(\bar{\tau}a/2) < e^{-a\bar{\tau}/2} \iff \bar{\tau} < 1.3098794227775/a$$

If $i \neq 0$, **more conservative** than with $\kappa(\omega) = 1$. Why?

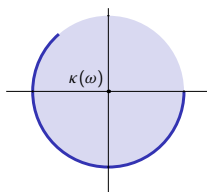
Conservatism vs. radii



Tightest covering



if $\omega \leq \pi$



if $\omega \geq \pi$

so

$$\kappa_3(\omega) := \frac{1 + e^{-j\omega}}{2} \mathbb{1}_{[0, \pi]}(\omega) = \begin{cases} (1 + e^{-j\omega})/2 & \text{if } \omega \in [0, \pi] \\ 0 & \text{otherwise} \end{cases}$$

In this case

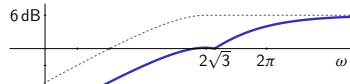
$$\alpha(\omega) = \max_{\tau_\delta \in [0, \bar{\tau}]} \left\{ \left| \frac{1 + e^{-j\bar{\tau}\omega}}{2} - e^{-j\omega\tau_\delta} \right| \right\} = \frac{|1 - e^{-j\bar{\tau}\omega}|}{2} = \sin \frac{\bar{\tau}\omega}{2} = \frac{\alpha_1(\bar{\tau}\omega)}{2}$$

Tightest covering: approximation

$[2, 2]$ -Padé of $(1 + e^{-s})/2$:

$$W_{k3}(s) = \frac{s^2 + 12}{s^2 + 6s + 12},$$

for which $\alpha(\omega) = \alpha_3(\bar{\tau}\omega)$, where

$$\alpha_3(\omega) = \sqrt{2 - 2 \frac{18\omega^2 + (12 - \omega^2)\beta(\omega)}{\omega^4 + 12\omega^2 + 144}} =$$


where

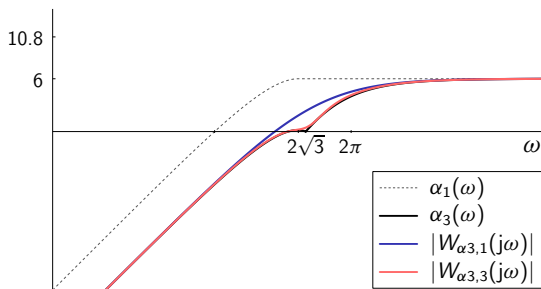
$$\beta(\omega) = \begin{cases} (12 - \omega^2) \cos \omega + 6\omega \sin \omega & \text{if } \omega \in [0, 2\sqrt{3}] \\ \sqrt{\omega^4 + 12\omega^2 + 144} & \text{otherwise} \end{cases}$$

Rational approximations of α_3

Approximations

$$W_{\alpha_3,1}(s) := \frac{2s}{s+4} \quad \text{and} \quad W_{\alpha_3,3}(s) := W_{\alpha_3,1}(s) \frac{s^2 + 1.92s + 12.781}{s^2 + 2.393s + 12.781},$$

yield



and

$$- W_{\kappa_3}(j\omega) + |W_{\alpha_3,1}(j\omega)|\bar{\mathbb{D}} \subset 1 + |W_{\alpha_1,1}(j\omega)|\bar{\mathbb{D}} \quad \text{hence, less conservative}$$

Covering with $W_{k3}(s)$: example

The H_∞ problem to solve now:

$$\left\| \mathcal{F}_l \left(\begin{bmatrix} 0 & W_{\alpha 3} \\ P & PW_{k3} \end{bmatrix}, R \right) \right\|_\infty = \left\| \frac{PR}{1 - PRW_{k3}} W_{\alpha 3} \right\|_\infty < 1.$$

Now

$$\frac{P(s)R(s)}{1 - P(s)R(s)W_{k3}(s)} W_{\alpha 3}(s) \Big|_{s=a} = -\frac{W_{\alpha 3}(a)}{W_{k3}(a)} \quad \text{for every stabilizing } R$$

and the robustly stability problem is feasible iff $|W_{\alpha 3,i}(\bar{\tau}a)| < |W_{k3}(\bar{\tau}a)|$.

$$i=1 \quad |W_{\alpha 3,1}(\bar{\tau}a)| < |W_{k3}(\bar{\tau}a)| \iff \bar{\tau} < 1.691/a$$

$$i=2 \quad |W_{\alpha 3,1}(\bar{\tau}a)| < |W_{k3}(\bar{\tau}a)| \iff \bar{\tau} < 1.769/a$$

Covering with $W_{k3}(s)$: example

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μ_d for DTC-based loops: covering

In this case

$$T_{\text{rob}}(s) = -\frac{W_\alpha(s)e^{-\tau_0 s}}{s/k_p + 1 - e^{-\tau_0 s} + W_\kappa(s)e^{-\tau_0 s}}$$

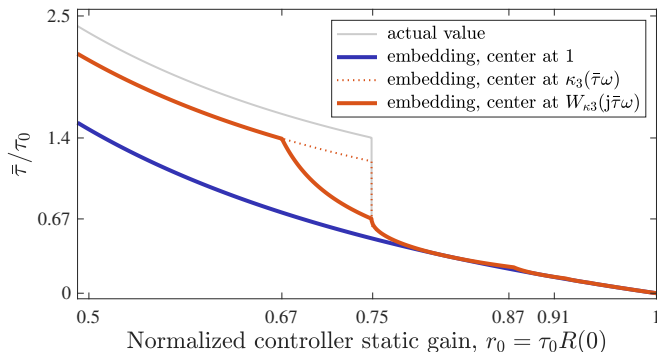
(especially simple if $W_\kappa = 1$) and stability condition is $\|T_{\text{rob}}\|_\infty < 1$. Also, in the analysis we use $\alpha(\omega)$ for the radius, not W_α .

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Time-varying delays

Lemma (Lemma 4.8)

If $\tau(t) \in [0, \bar{\tau}]$ and $|\dot{\tau}(t)| \leq \nu < 1$, then $\bar{D}_{\tau(t)}$ is a bounded operator on $L_2(\mathbb{R}_+)$, with

$$\|\bar{D}_{\tau(t)}\|_{L_2(\mathbb{R}_+) \rightarrow L_2(\mathbb{R}_+)} = \frac{1}{\sqrt{1-\nu}} > 1.$$

Lemma (Lemma 4.9)

If $\tau(t) \in [0, \bar{\tau}]$, then the system $\Delta_{\bar{\tau}} : z \mapsto w$ such that

$$w = \int_{t-\tau(t)}^t z(s) ds = \int_{-\tau(t)}^0 z(t+s) ds$$

is a bounded operator on $L_2(\mathbb{R}_+)$ with

$$\|\Delta_{\bar{\tau}}\|_{L_2(\mathbb{R}_+) \rightarrow L_2(\mathbb{R}_+)} = \bar{\tau}.$$

Time-varying delays

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