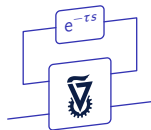


Time Delays in Control and Estimation (038806)



lecture no. 7

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Outline

Loop shifting applied to dead-time systems

Delay as a constraint: extraction

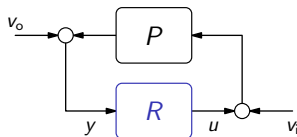
Outline

Loop shifting applied to dead-time systems

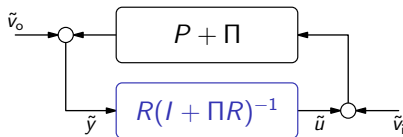
Delay as a constraint: extraction

Loop shifting (contd)

If $\Pi \in H_\infty$ and the loop of R and Π is well posed, then



is well posed and internally stable iff



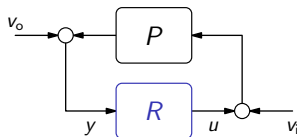
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This transformation makes sense if the

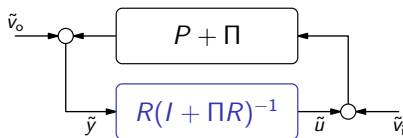
transformed system has more favorable properties than the original one.

Loop shifting (contd)

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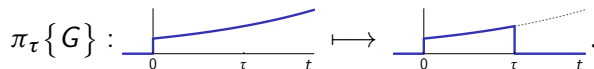
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This transformation makes sense if the

- transformed system has more favorable properties than the original one.

Truncation operator

Let G be LTI. The associated **FIR truncation operator** is



If $G(s) = D + C(sI - A)^{-1}B$, then $g(t) = D\delta(t) + Ce^{At}B1(t)$ and the impulse response of $\pi_\tau\{G\}$ is

$$\begin{aligned} g(t)1_{[0,\tau]}(t) &= g(t)(1(t) - 1(t - \tau)) = g(t) - Ce^{At}B1(t - \tau) \\ &= g(t) - Ce^{A\tau}e^{A(t-\tau)}B1(t - \tau). \end{aligned}$$

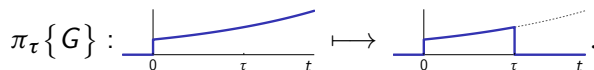
The last term is the impulse response of $\hat{G}\tilde{D}_\tau$, where

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline Ce^{A\tau} & 0 \end{array} \right] = \left[\begin{array}{c|c} A & e^{A\tau}B \\ \hline C & 0 \end{array} \right]$$

(the latter can be derived by, e.g. the similarity transformation $T = e^{A\tau}$).

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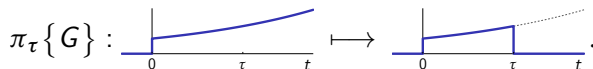
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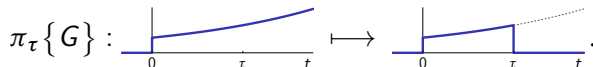
The **last term** is the impulse response of $\hat{G}\bar{D}_\tau$, where

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(the latter can be derived by, e.g. the similarity transformation $T = e^{A\tau}$).

Truncation: transfer function

Thus



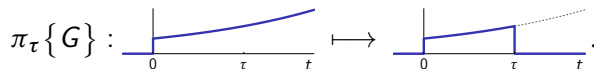
yields

$$\begin{aligned}
 \pi_{\tau} \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \right\} &= G(s) - \hat{G}(s)e^{-\tau s} \\
 &= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] - \left[\begin{array}{c|c} A & B \\ \hline C e^{A\tau} & 0 \end{array} \right] e^{-\tau s} \\
 &= D + C(sI - A)^{-1}B - C e^{A\tau} (sI - A)^{-1} B e^{-\tau s} \\
 &= D + C(I - e^{-(\tau I - A)\tau}) (sI - A)^{-1} B,
 \end{aligned}$$

where all **eigenvalues of A** are **removable singularities**.

Truncation: transfer function (contd)

Alternatively, with the impulse response of



being $g(t)\mathbb{1}_{[0,\tau]}(t) = D\delta(t) + Ce^{At}B\mathbb{1}_{[0,\tau]}(t)$, we have

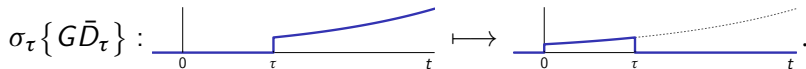
$$\pi_{\tau} \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \right\} = \mathfrak{L}\{D\delta(t) + Ce^{At}B\mathbb{1}_{[0,\tau]}(t)\} = D + C \int_0^{\tau} e^{-(sI-A)t} dt B,$$

which is already familiar entire function of s .

Completion operator

Let G be LTI. The **FIR completion operator** of $G\bar{D}_\tau$ is defined as the

- truncation of a finite-dimensional $\tilde{G}$ with a strictly proper $\tilde{G}(s)$, whose impulse response matches that of $G\bar{D}_\tau$ in (τ, ∞) , i.e.



If $g(t)$ is the impulse response of $G(s) = D + C(sI - A)^{-1}B$, then we are looking for $\tilde{g}(t)$ such that

$$\tilde{g}(t)\mathbb{1}_{(\tau, \infty)}(t) = g(t - \tau)\mathbb{1}_{(\tau, \infty)}(t) = Ce^{A(t-\tau)}B\mathbb{1}(t - \tau).$$

Hence,

$$\tilde{g}(t) = Ce^{-A\tau}e^{At}B\mathbb{1}(t)$$

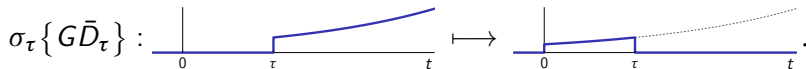
and

$$\tilde{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline Ce^{-A\tau} & 0 \end{array} \right] \rightarrow \left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array} \right].$$

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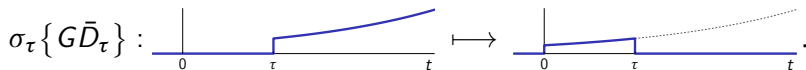
and

$$\tilde{G}(s) = \begin{bmatrix} A & B \\ Ce^{-A\tau} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Completion operator

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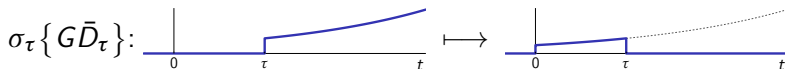
$$\tilde{g}(t) = Ce^{-A\tau}e^{At}B\mathbb{1}(t)$$

and

$$\tilde{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline Ce^{-A\tau} & 0 \end{array} \right] = \left[\begin{array}{c|c} A & e^{-A\tau}B \\ \hline C & 0 \end{array} \right].$$

Completion: transfer function

Thus



yields

$$\sigma_\tau \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] e^{-\tau s} \right\} = \tilde{G}(s) - G(s)e^{-\tau s} = \left[\begin{array}{c|c} A & B \\ \hline Ce^{-A\tau} & 0 \end{array} \right] - \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] e^{-\tau s},$$

where all **eigenvalues of A** are again **removable singularities**.

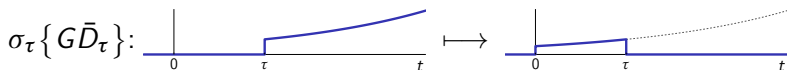
Alternatively,

$$\sigma_\tau \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] e^{-\tau s} \right\} = \mathcal{L}\{Ce^{A(t-\tau)}B1_{[0,\tau]}(t)\} = Ce^{-A\tau} \int_0^\tau e^{-(s-A)t} dt B,$$

which is already familiar entire function of s .

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yields

$$\sigma_\tau \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] e^{-\tau s} \right\} = \tilde{G}(s) - G(s)e^{-\tau s} = \left[\begin{array}{c|c} A & B \\ \hline C e^{-A\tau} & 0 \end{array} \right] - \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] e^{-\tau s},$$

where all eigenvalues of A are again removable singularities.

Alternatively,

$$\sigma_\tau \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] e^{-\tau s} \right\} = \mathcal{L}\{C e^{A(t-\tau)} B \mathbb{1}_{[0,\tau]}(t)\} = C e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B,$$

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Completion: example

Let

$$G(s) = \frac{1}{s(s-1)} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 1 & 0 & 0 \end{array} \right].$$

In this case

$$\exp\left(-\left[\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right]\tau\right) = \left[\begin{array}{cc} 1 & e^{-\tau} - 1 \\ 0 & e^{-\tau} \end{array}\right]$$

and

$$\tilde{G}(s) = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 1 & e^{-\tau} - 1 & 0 \end{array} \right] = \frac{-(1 - e^{-\tau})s + 1}{s(s-1)}$$

(nonminimum-phase) and

$$\sigma_{\tau}\{G(s)e^{-\tau s}\} = \frac{-(1 - e^{-\tau})s + 1 - e^{-\tau s}}{s(s-1)} = -\int_0^{\tau} (1 - e^{t-\tau})e^{-st} dt.$$

Truncation and completion: key properties

- $\pi_\tau\{G\} \in H_\infty$ for all finite-dimensional G
- $\sigma_\tau\{G\bar{D}_\tau\} \in H_\infty$ for all finite-dimensional G
- given any finite-dimensional G , there is finite-dimensional $\hat{G}$ such that

$$G = \pi_\tau\{G\} + \hat{G}\bar{D}_\tau = \begin{array}{c} \text{graph of } G \\ \text{graph of } \pi_\tau\{G\} \\ \text{graph of } \hat{G}\bar{D}_\tau \end{array}$$

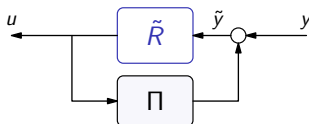
The diagram illustrates the decomposition of a signal G into its truncated part $\pi_\tau\{G\}$ and a completion term $\hat{G}\bar{D}_\tau$. The first graph shows G as a solid blue line starting at 0 and increasing. The second graph shows $\pi_\tau\{G\}$ as a solid blue line that follows G until $t = \tau$ and then drops to zero. The third graph shows $\hat{G}\bar{D}_\tau$ as a solid blue line that is zero until $t = \tau$ and then follows the original G curve. A dashed line in the second and third graphs represents the continuation of G from $t = \tau$ onwards.

- given any finite-dimensional G , there is finite-dimensional $\tilde{G}$ such that

$$G\bar{D}_\tau = \tilde{G} - \pi_\tau\{G\} = \begin{array}{c} \text{graph of } G\bar{D}_\tau \\ \text{graph of } \tilde{G} \\ \text{graph of } \pi_\tau\{G\} \end{array}$$

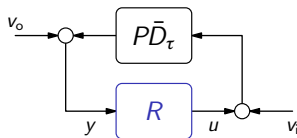
The diagram illustrates the decomposition of the truncated signal $G\bar{D}_\tau$ into a completion term $\tilde{G}$ minus the truncated part $\pi_\tau\{G\}$. The first graph shows $G\bar{D}_\tau$ as a solid blue line that is zero until $t = \tau$ and then follows the original G curve. The second graph shows $\tilde{G}$ as a solid blue line that follows G until $t = \tau$ and then continues as a dashed line. The third graph shows $\pi_\tau\{G\}$ as a solid blue line that follows G until $t = \tau$ and then drops to zero. A dashed line in the second and third graphs represents the continuation of G from $t = \tau$ onwards.

Completion: connections



- MSP: $\Pi(s) = -C e^{-A\tau} \int_0^\tau e^{-(sl-A)t} dt B = -\sigma_\tau \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] e^{-\tau s} \right\}$
- FSA: $\Pi(s) = e^{-A\tau} \int_0^\tau e^{-(sl-A)t} dt B = \sigma_\tau \left\{ \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array} \right] e^{-\tau s} \right\}$
- Reduction: $\Pi(s) = e^{-A\tau} \int_0^\tau e^{-(sl-A)t} dt B = \sigma_\tau \left\{ \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array} \right] e^{-\tau s} \right\}$

Loop shifting for dead-time systems



We know that if $P(s) = C(sI - A)^{-1}B$, then

$$\sigma_\tau\{P\bar{D}_\tau\} = \tilde{P} - P\bar{D}_\tau \quad \Longleftrightarrow \quad P\bar{D}_\tau = \tilde{P} - \sigma_\tau\{P\bar{D}_\tau\},$$

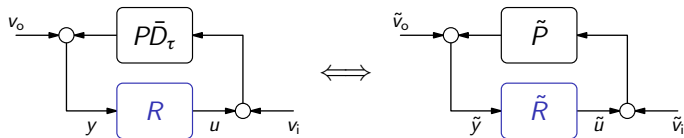
where

$$\tilde{P}(s) = Ce^{-A\tau}(sI - A)^{-1}B \quad \text{and} \quad \sigma_\tau\{P\bar{D}_\tau\} \in H_\infty.$$

Hence

- loop shifting with $\Pi = \sigma_\tau\{P\bar{D}_\tau\}$ (MSP) is
 - legitimate
 - yields a delay-free stabilization problem for $\tilde{P} = P\bar{D}_\tau + \Pi$

Loop shifting for dead-time systems (contd)



Theorem

R internally stabilizes $P\bar{D}_\tau$ iff

$$R = \tilde{R}(I - \sigma_\tau\{P\bar{D}_\tau\}\tilde{R})^{-1} =$$

The block diagram shows a feedback system where the output y is summed with the input u to produce $\tilde{u}$. $\tilde{u}$ enters the controller $\tilde{R}$. The output of $\tilde{R}$ is summed with the output of a block $\sigma_\tau\{P\bar{D}_\tau\}$ to produce u . The block $\sigma_\tau\{P\bar{D}_\tau\}$ receives $\tilde{u}$ as input.

for some $\tilde{R}$ internally stabilizing the delay-free $\tilde{P} = P\bar{D}_\tau + \sigma_\tau\{P\bar{D}_\tau\}$. Here

$$\tilde{P}(s) = \left[\begin{array}{c|c} A & B \\ \hline C e^{-A\tau} & 0 \end{array} \right] \quad \text{and} \quad \sigma_\tau\{P(s)e^{-\tau s}\} = C e^{-A\tau} \int_0^\tau e^{-(sl-A)t} dt B.$$

Moreover, R causal $\iff \tilde{R}$ causal.

Outline

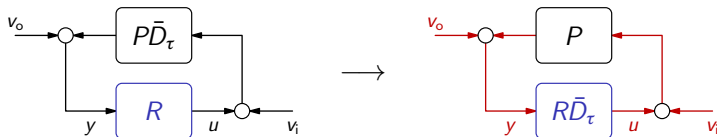
Loop shifting applied to dead-time systems

Delay as a constraint: extraction

Idea

Two steps:

1. replace



2. treat $R\bar{D}_\tau$ as a constrained version of R , because

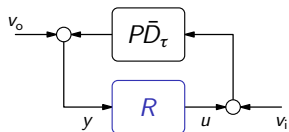
$$\{y \mapsto u : u(t) \text{ may depend on } y(s) \text{ for } s \leq t\}$$

$$\cap$$

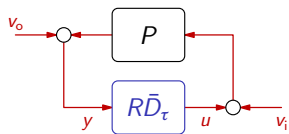
$$\{y \mapsto u : u(t) \text{ may depend on } y(s) \text{ for } s \leq t - \tau\}$$

Moving delay

Because both P and R are LTI,



$$\Rightarrow \begin{bmatrix} S_o & T_d \\ T_c & T_i \end{bmatrix} = \begin{bmatrix} I \\ R \end{bmatrix} (I - PR\bar{D}_\tau)^{-1} \begin{bmatrix} I & P\bar{D}_\tau \end{bmatrix}$$



$$\Rightarrow \begin{bmatrix} S_o & T_d \\ T_c & T_i \end{bmatrix} = \begin{bmatrix} I \\ R\bar{D}_\tau \end{bmatrix} (I - PR\bar{D}_\tau)^{-1} \begin{bmatrix} I & P \end{bmatrix}$$

Hence,

$$\begin{bmatrix} S_o & T_d \\ T_c\bar{D}_\tau & T_i \end{bmatrix} = \begin{bmatrix} S_o & T_d\bar{D}_\tau \\ T_c & T_i \end{bmatrix}$$

When can we have $G \in H_\infty \iff G\bar{D}_\tau \in H_\infty$?

Technical result

$G(s)$ is said to be proper if $\exists \alpha > 0$ such that $\|G(s)\| < \infty$ for all $s \in \mathbb{C}_\alpha$.

Lemma

If the transfer function of G is proper, then

$$G \in H_\infty \iff G_\tau := G\bar{D}_\tau \in H_\infty.$$

Proof: Clearly $G \in H_\infty \implies G_\tau \in H_\infty$. If $G_\tau \in H_\infty$, then $G(s) = G_\tau(s)e^{s\tau}$ is holomorphic in $\mathbb{C}_0$. By assumption, $G(s)$ is bounded in $\mathbb{C}_\alpha$. So $G \notin H_\infty$ only if $G(s)$ is unbounded in $\mathbb{C}_0 \setminus \mathbb{C}_\alpha$. But

$$\|G(s)\| = |e^{s\tau}| \|G_\tau(s)\| \leq e^{s\alpha} \|G_\tau(s)\| < \infty$$

there



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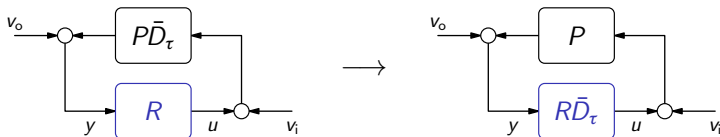
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$$\|G(s)\| = |e^{\tau s}| \|G_\tau(s)\| \leq e^{\tau \alpha} \|G_\tau(s)\| < \infty$$

there. □

Constraining controllers



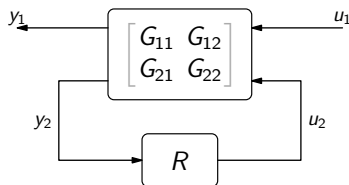
As

$$\{y \mapsto u : u(t) \text{ may depend on } y(s) \text{ for } s \leq t\} \\ \cap \\ \{y \mapsto u : u(t) \text{ may depend on } y(s) \text{ for } s \leq t - \tau\}$$

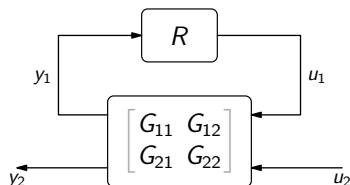
we may

- start with a family of stabilizing controllers
- find a sub-family of stabilizing delayed controllers

Preliminary: linear fractional transformation



lower, $\mathcal{F}_l(G, H)$



upper, $\mathcal{F}_u(G, H)$

The lower LFT $\mathcal{F}_l(G, R) : u_1 \mapsto y_1$ reads

$$\mathcal{F}_l(G, R) = G_{11} + G_{12}R(I - G_{22}R)^{-1}G_{21} = G_{11} + G_{12}(I - RG_{22})^{-1}RG_{21}$$

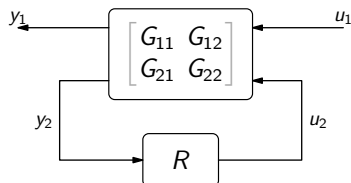
The upper LFT $\mathcal{F}_u(G, R) : u_2 \mapsto y_2$ reads

$$\mathcal{F}_u(G, R) = G_{22} + G_{21}(I - RG_{11})^{-1}RG_{12} = G_{22} + G_{21}R(I - G_{11}R)^{-1}G_{12}$$

Moreover,

$$\mathcal{F}_u(G, R) = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} G \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, R\right)$$

Preliminary: linear fractional transformation (contd)



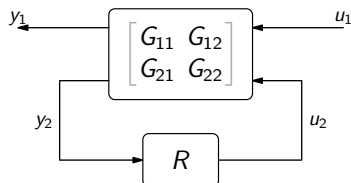
Well posed if

- $I - G_{22}R$ is invertible equivalently, $I - RG_{22}$ is invertible
- $\exists \alpha \in \mathbb{R}$ such that $\inf_{s \in \mathbb{C}_\alpha} \|I - G_{22}(s)R(s)\| > 0$

If $\mathcal{F}_l(G, R)$ is well posed then

- G and R causal $\implies \mathcal{F}_l(G, R)$ is causal

Preliminary: linear fractional transformation (contd)



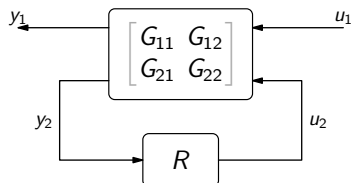
Well posed if

- $I - G_{22}R$ is invertible equivalently, $I - RG_{22}$ is invertible
- $\exists \alpha \in \mathbb{R}$ such that $\inf_{s \in \mathbb{C}_\alpha} \|I - G_{22}(s)R(s)\| > 0$

If $\mathcal{F}_I(G, R)$ is well posed then

- G and R causal $\implies \mathcal{F}_I(G, R)$ is causal

Preliminary: linear fractional transformation (contd)



If G is invertible, with square and invertible G_{12} and G_{21} , then

$$T = \mathcal{F}_l(G, R) \iff R = \mathcal{F}_u(G^{-1}, T) = \mathcal{F}_l\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} G^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, T\right).$$

and $\mathcal{F}_l(G, R)$ is well posed iff $\mathcal{F}_u(G^{-1}, T)$ is well posed.

Proof (outline) Follows by

$$T : u_1 \mapsto y_1 \text{ for } \begin{cases} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = G \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ u_2 = R y_2 \end{cases} \quad \text{vs} \quad R : u_1 \mapsto y_1 \text{ for } \begin{cases} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = G^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ y_1 = T u_1 \end{cases}$$

The invertibility of G_{12} and G_{21} required for the equivalence of well-posedness.



Preliminary: Youla–Kučera parametrization

Given

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

all causal internally stabilizing controllers are $R = \mathcal{F}_1(J, Q)$, where

- given any K and L such that $A + BK$ and $A + LC$ are Hurwitz,

$$J(s) = \left[\begin{array}{cc} J_{11}(s) & J_{12}(s) \\ J_{21}(s) & J_{22}(s) \end{array} \right] = \left[\begin{array}{c|cc} A + BK + LC + LDK & -L & B + LD \\ \hline K & 0 & I \\ -C - DK & I & -D \end{array} \right]$$

- $Q \in H_\infty$ but otherwise arbitrary

J_{11} , J_{12} , and J_{21} are invertible,

$$J^{-1}(s) = \left[\begin{array}{c|cc} A & B & -L \\ \hline C & D & I \\ -K & I & 0 \end{array} \right] \quad \text{and} \quad \begin{cases} J_{12}^{-1}(s) = \left[\begin{array}{c|c} A + LC & B + LD \\ \hline K & I \end{array} \right] \\ J_{21}^{-1}(s) = \left[\begin{array}{c|c} A + BK & L \\ \hline C + DK & I \end{array} \right] \end{cases}$$

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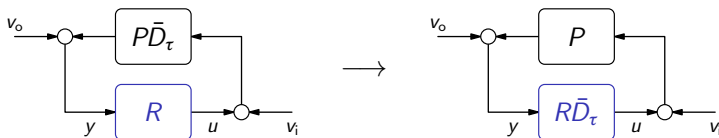
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Constraining controllers



As

$$\{y \mapsto u : u(t) \text{ may depend on } y(s) \text{ for } s \leq t\} \\ \cap \\ \{y \mapsto u : u(t) \text{ may depend on } y(s) \text{ for } s \leq t - \tau\}$$

we may

- start with the YK parametrization of stabilizing controllers
- find constraints on Q ensuring $\mathcal{F}_l(J, Q)$ has a delay

Constraints on Q : preliminaries

Let

$$H(s) := J^{-1}(s) = \left[\begin{array}{cc} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{array} \right] = \left[\begin{array}{c|cc} A & B & -L \\ \hline C & D & I \\ -K & I & 0 \end{array} \right]$$

(with $H_{11} = P$). Decompose

$$H_{22} = \pi_{\tau}\{H_{22}\} + \hat{H}_{22}\bar{D}_{\tau} \quad \text{and} \quad H_{11}\bar{D}_{\tau} = \tilde{H}_{11} - \sigma_{\tau}\{H_{11}\bar{D}_{\tau}\},$$

where

$$\hat{H}_{22}(s) = \left[\begin{array}{c|c} A & -e^{A\tau}L \\ \hline -K & 0 \end{array} \right] \quad \text{and} \quad \tilde{H}_{11}(s) = \left[\begin{array}{c|c} A & B \\ \hline Ce^{-A\tau} & 0 \end{array} \right]$$

Note that

$$\left[\begin{array}{cc} \hat{H}_{11}(s) & \hat{H}_{12}(s) \\ \hat{H}_{21}(s) & \hat{H}_{22}(s) \end{array} \right] = \left[\begin{array}{c|cc} A & B & -e^{A\tau}L \\ \hline Ce^{-A\tau} & 0 & I \\ -K & I & 0 \end{array} \right].$$

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Constraints on Q

Lemma

$\mathcal{F}_I(J, Q) = R\bar{D}_\tau$ for a causal $R \iff Q = \pi_\tau\{H_{22}\} + \tilde{Q}\bar{D}_\tau$ for a causal $\tilde{Q}$.

Proof: Because J_{12} and J_{21} invertible, $R\bar{D}_\tau = \mathcal{F}_I(J, Q)$ iff

$$\begin{aligned} Q &= \mathcal{F}_u(H, R\bar{D}_\tau) = H_{22} + H_{21}(I - R\bar{D}_\tau H_{11})^{-1}R\bar{D}_\tau H_{12} \\ &= \pi_\tau\{H_{22}\} + \hat{H}_{22}\bar{D}_\tau + H_{21}(I - R\bar{D}_\tau H_{11})^{-1}R\bar{D}_\tau H_{12} \\ &= \pi_\tau\{H_{22}\} + (\hat{H}_{22} + H_{21}(I - RH_{11}\bar{D}_\tau)^{-1}RH_{12})\bar{D}_\tau \end{aligned}$$

Now,

$$\begin{aligned} (I - RH_{11}\bar{D}_\tau)^{-1}R &= (I - R(\hat{H}_{11} - \sigma_\tau\{H_{11}\bar{D}_\tau\}))^{-1}R \\ &= (I - R\sigma_\tau\{H_{11}\bar{D}_\tau\} - R\hat{H}_{11})^{-1}R \end{aligned}$$

where $\tilde{R} := (I - R\sigma_\tau\{H_{11}\bar{D}_\tau\})^{-1}R$ (cf. loop shifting) and we know that
 R causal $\iff \tilde{R}$ causal.

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where $\tilde{R} := (I - R\sigma_\tau\{H_{11}\bar{D}_\tau\})^{-1}R$ (cf. loop shifting) and we know that

— R causal $\iff \tilde{R}$ causal.

Constraints on Q (contd)

Proof (contd): Thus,

$$\begin{aligned}\tilde{Q} &= \hat{H}_{22} + H_{21}(I - RH_{11}\bar{D}_\tau)^{-1}RH_{12} \\ &= \hat{H}_{22} + H_{21}(I - \tilde{R}\tilde{H}_{11})^{-1}\tilde{R}H_{12} \\ &= \mathcal{F}_u\left(\begin{bmatrix} \tilde{H}_{11} & H_{12} \\ H_{21} & \hat{H}_{22} \end{bmatrix}, \tilde{R}\right)\end{aligned}$$

The last LFT

- well posed
- has H_{12} and H_{21} invertible

Hence,

$$\tilde{Q} \text{ causal} \iff \tilde{R} \text{ causal} \iff R \text{ causal}$$



Constraints on Q (contd)

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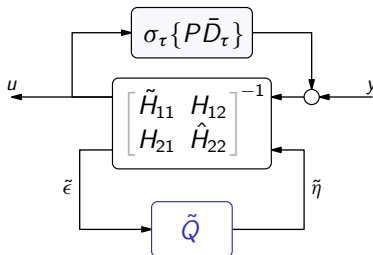


All stabilizing R

Important:

- $\tilde{Q} = \mathcal{F}_u \left(\begin{bmatrix} \tilde{H}_{11} & H_{12} \\ H_{21} & \hat{H}_{22} \end{bmatrix}, \tilde{R} \right) \iff \tilde{R} = \mathcal{F}_l \left(\begin{bmatrix} \tilde{H}_{11} & H_{12} \\ H_{21} & \hat{H}_{22} \end{bmatrix}^{-1}, \tilde{Q} \right)$
- $\tilde{R} = (I - R\sigma_\tau\{P\bar{D}_\tau\})^{-1}R \iff R = \tilde{R}(I + \sigma_\tau\{P\bar{D}_\tau\}\tilde{R})^{-1}$
- $\pi_\tau\{H_{22}\} + \tilde{Q}\bar{D}_\tau \in H_\infty \iff \tilde{Q} \in H_\infty$

Hence, all R 's such that $R\bar{D}_\tau$ is stabilizing are



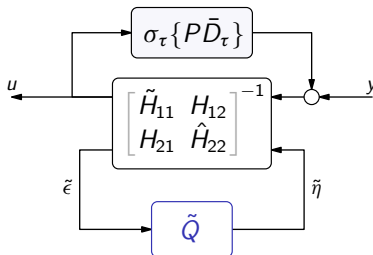
for $\tilde{Q} \in H_\infty$ but otherwise arbitrary

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