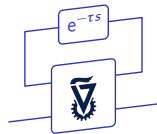


Time Delays in Control and Estimation (038806)



lecture no. 6

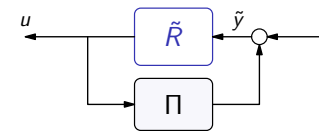
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Previously on TDCE...

Common structure:



where

- $\tilde{R}$ is finite-dimensional, designed for a finite-dimensional plant
- Π is infinite-dimensional, fixed structure

Common theme seems to be

- prediction as a rationale for Π ,

but prediction is a suspicious operation

- “prediction is difficult, especially about the future” (N. Bohr)...

Can it be justified in control-theoretic terms?

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Preliminary: partial pole placement

(A, B) controllable $\implies \exists K$ s.t. $\text{spec}(A + BK)$ arbitrarily (e.g. Ackermann).

Let $\tilde{n} < n$ and $\tilde{A} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ be such that $\text{spec}(\tilde{A}) \subset \text{spec}(A)$.

Lemma

If there is rank- $\tilde{n}$ matrix $\tilde{T} \in \mathbb{R}^{\tilde{n} \times n}$ such that $\tilde{T}A = \tilde{A}\tilde{T}$, then

$$\text{spec}(A + B\tilde{K}\tilde{T}) = \text{spec}(\tilde{A} + \tilde{B}\tilde{K}) \cup (\text{spec}(A) \setminus \text{spec}(\tilde{A})),$$

where $\tilde{B} := \tilde{T}B$ is such that $(\tilde{A}, \tilde{B})$ controllable.

Proof: Based on

$$A = T^{-1} \begin{bmatrix} \tilde{A} & 0 \\ \times & \tilde{A} \end{bmatrix} T \quad \text{and} \quad B = T^{-1} \begin{bmatrix} \tilde{B} \\ \tilde{B} \end{bmatrix}$$

and use $K = [\tilde{K} \ 0] T$.

□

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Preliminary: reduced-order observer (simple special case)

If

$$\begin{cases} \begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u[t] \\ \begin{bmatrix} y_1[t] \\ y_2[t] \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix} \end{cases}$$

then observe only x_1 . The first state:

$$\begin{cases} x_1[t+1] = A_{11}x_1[t] + \overbrace{A_{12}y_2[t]}^{\text{“known input”}} + B_1u[t] \\ y_1[t] = C_{11}x_1[t] + C_{12}y_2[t] \end{cases}$$

Its observer

$$\hat{x}_1[t+1] = A_{11}\hat{x}_1[t] + A_{12}y_2[t] + B_1u[t] - L(y_1[t] - C_{11}\hat{x}_1[t] - C_{12}y_2[t]).$$

has autonomous

$$\epsilon_1[t+1] = (A_{11} + LC_{11})\epsilon_1[t],$$

where $\epsilon_1 := x_1 - \hat{x}_1$.

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Reduced-order observer-based feedback

Combine plant, reduced-order observer, and **observer**-based control law

$$u[t] = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1[t] \\ x_2[t] \end{bmatrix}.$$

The closed-loop dynamics

$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \\ \hat{x}_1[t+1] \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} + B_1 K_2 & B_1 K_1 \\ 0 & A_{22} + B_2 K_2 & B_2 K_1 \\ -LC_{11} & A_{12} + B_1 K_2 & A_{11} + B_1 K_1 + LC_{11} \end{bmatrix} \begin{bmatrix} x_1[t] \\ x_2[t] \\ \hat{x}_1[t] \end{bmatrix}$$

or

$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \\ \epsilon_1[t+1] \end{bmatrix} = \begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 & B_1 K_1 \\ B_2 K_1 & A_{22} + B_2 K_2 & B_2 K_1 \\ \hline 0 & 0 & A_{11} + LC_{11} \end{bmatrix} \begin{bmatrix} x_1[t] \\ x_2[t] \\ \epsilon_1[t] \end{bmatrix},$$

stable whenever $\begin{bmatrix} K_1 & K_2 \end{bmatrix}$ and L are stabilizing.

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Outline

Discrete insight: state feedback

Discrete insight: output feedback

(A paraphrase on) Fiagbedzi–Pearson reduction

Loop shifting

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Problem

Given

$$x[t+1] = Ax[t] + Bu[t - \tau],$$

with measurable x and controllable (A, B) .

State equation:

$$\underbrace{\begin{bmatrix} x[t+1] \\ u[t - \tau + 1] \\ \vdots \\ u[t - 1] \\ u[t] \end{bmatrix}}_{x_\tau[t+1]} = \underbrace{\begin{bmatrix} A & B & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_\tau} \underbrace{\begin{bmatrix} x[t] \\ u[t - \tau] \\ \vdots \\ u[t - 2] \\ u[t - 1] \end{bmatrix}}_{x_\tau[t]} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}}_{B_\tau} u[t].$$

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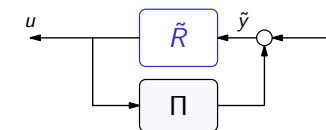
State feedback: structure

Naturally,

$$u[t] = \begin{bmatrix} K_x & K_\tau & \cdots & K_2 & K_1 \end{bmatrix} x_\tau[t] = \begin{bmatrix} K_x & K_\tau & \cdots & K_2 & K_1 \end{bmatrix} \begin{bmatrix} x[t] \\ u[t - \tau] \\ \vdots \\ u[t - 2] \\ u[t - 1] \end{bmatrix}$$

$$= K_x x[t] + \sum_{i=1}^{\tau} K_i u[t - i].$$

is dynamic as $x \mapsto u$, of the very form



for

$$\tilde{R} = I, \quad y = K_x x, \quad \text{and} \quad \Pi(z) = \sum_{i=1}^{\tau} K_i z^{-i}.$$

But how to choose K_x and K_i ?

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Pole placement

PBH:

$$\text{rank} \begin{bmatrix} A - \lambda I & B & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda I & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda I & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\lambda I & I \end{bmatrix} = \text{rank} [A - \lambda I \ B] + \tau m$$

hence, controllable as well.

- arbitrary pole placement is possible
- but does it make sense to move all open-loop poles?

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Partial pole placement: equation to solve

In this case,

$$\underbrace{\begin{bmatrix} T_x & T_\tau & \cdots & T_2 & T_1 \end{bmatrix}}_{\tilde{T}} \begin{bmatrix} A & B & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} T_x A & T_x B & T_\tau & \cdots & T_2 \end{bmatrix}$$

$$= A \underbrace{\begin{bmatrix} T_x & T_\tau & T_{\tau-1} & \cdots & T_1 \end{bmatrix}}_{\tilde{T}}$$

or

$$T_x A = A T_x, \quad T_x B = A T_\tau, \quad T_\tau = A T_{\tau-1} \quad \cdots \quad T_2 = A T_1.$$

So

$$T_\tau = A^{\tau-1} T_1 \implies T_x B = A^\tau T_1 \xrightarrow{\text{guess}} T_1 = B \wedge T_x = A^\tau$$

and

$$\begin{bmatrix} T_x & T_\tau & \cdots & T_2 & T_1 \end{bmatrix} = \begin{bmatrix} A^\tau & A^{\tau-1} B & \cdots & A B & B \end{bmatrix}$$

has full rank iff (A, B) has no unreachable modes at $z = 0$.

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Control law

Static state feedback moving only modes of A :

$$u[t] = K \left(A^\tau x[t] + \sum_{i=1}^{\tau} A^{i-1} B u[t-i] \right)$$

It is prediction-based, cf.

$$x[t+1] = A x[t] + B u[t-\tau]$$

$\Downarrow$

$$x[t+\tau] = A^\tau x[t] + \sum_{s=t}^{t+\tau-1} A^{t+\tau-1-s} B u[s-\tau] = A^\tau x[t] + \sum_{i=1}^{\tau} A^{i-1} B u[t-i]$$

for $s = t + \tau - i$.

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Outline

Discrete insight: state feedback

Discrete insight: output feedback

(A paraphrase on) Fiagbedzi–Pearson reduction

Loop shifting

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Problem

Given

$$\begin{cases} x[t+1] = Ax[t] + Bu[t-\tau] \\ y[t] = Cx[t] + Du[t-\tau] \end{cases}$$

with measurable x and stabilizable and detectable (A, B) and (C, A) .

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State space

Complement

$$\underbrace{\begin{bmatrix} x[t+1] \\ u[t-\tau+1] \\ \vdots \\ u[t-1] \\ u[t] \end{bmatrix}}_{x_\tau[t+1]} = \underbrace{\begin{bmatrix} A & B & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_\tau} \underbrace{\begin{bmatrix} x[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix}}_{x_\tau[t]} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}}_{B_\tau} u[t].$$

with the measurement equation

$$\underbrace{\begin{bmatrix} y[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix}}_{y_\tau[t]} = \underbrace{\begin{bmatrix} C & D & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix}}_{C_\tau} \underbrace{\begin{bmatrix} x[t] \\ u[t-\tau] \\ \vdots \\ u[t-2] \\ u[t-1] \end{bmatrix}}_{x_\tau[t]} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{D_\tau} u[t]$$

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Reduced-order observer

Observer:

$$\hat{x}[t+1] = A\hat{x}[t] + Bu[t-\tau] - L(y[t] - C\hat{x}[t] - Du[t-\tau])$$

yields the error equation

$$\epsilon[t+1] = (A + LC)\epsilon[t]$$

for $\epsilon[t] := x[t] - \hat{x}[t]$.

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Observer-based controller

The final observer-based controller,

$$\begin{cases} \hat{x}[t+1] = (A + LC)\hat{x}[t] + (B + LD)u[t-\tau] - Ly[t] \\ u[t] = K \left(A^\tau \hat{x}[t] + \sum_{i=1}^{\tau} A^{i-1} Bu[t-i] \right) \end{cases}$$

is

- discrete version of **observer-predictor** of Furukawa & Shimemura and yields
- closed-loop spectrum in $\text{spec}(A + BK) \cup \text{spec}(A + LC) \cup \{0\}^{m\tau}$.

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Outline

Discrete insight: state feedback

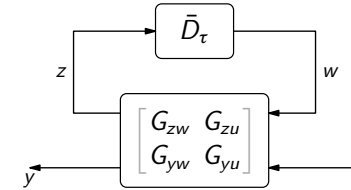
Discrete insight: output feedback

(A paraphrase on) Fiagbedzi–Pearson reduction

Loop shifting

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Setup



with state $(x(t), \tilde{z}_t)$. State equation (DDE)

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t - \tau) \end{bmatrix} + \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix} u(t).$$

with $\rho(D_{zw}) < 1$. Characteristic function

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix}$$

has an infinite number of roots $\implies$ no hope for FSA.

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Workaround

To move only $\tilde{n}$ roots, introduce the $\tilde{n}$ -dimensional

$$\tilde{x}(t) := Qx(t) + \int_{t-\tau}^t e^{\tilde{A}(t-r)} Rz(r) dr$$

for $\tilde{A} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$, $Q \in \mathbb{R}^{\tilde{n} \times n}$, and $R \in \mathbb{R}^{\tilde{n} \times m_\tau}$ to be determined.

$$\begin{aligned} \dot{\tilde{x}}(t) &= Q\dot{x}(t) + \tilde{A} \int_{t-\tau}^t e^{\tilde{A}(t-r)} Rz(r) dr + Rz(t) - e^{\tilde{A}\tau} Rz(t - \tau) \\ &= Q(Ax(t) + B_w z(t - \tau) + B_u u(t)) + \tilde{A} \int_{t-\tau}^t e^{\tilde{A}(t-r)} Rz(r) dr \\ &\quad + R(C_z x(t) + D_{zw} z(t - \tau) + D_{zu} u(t)) - e^{\tilde{A}\tau} Rz(t - \tau) \\ &= \tilde{A}\tilde{x}(t) + (QB_u + RD_{zu})u(t) \\ &\quad + (QA + RC_z - \tilde{A}Q)x(t) + (QB_w + RD_{zw} - e^{\tilde{A}\tau}R)z(t - \tau). \end{aligned}$$

- How to eliminate parts depending on $x(t)$ and $z(t - \tau)$?

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LCME

If

$$\begin{bmatrix} Q & R \end{bmatrix} \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} = \begin{bmatrix} \tilde{A}Q & e^{\tilde{A}\tau}R \end{bmatrix},$$

then

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t),$$

where $\tilde{B} := QB_u + RD_{zu}$, finite dimensional.

If η'_i is a left eigenvector of $\tilde{A}$, corresponding to $\lambda_i \in \text{spec}(\tilde{A})$, then

$$\eta'_i \begin{bmatrix} Q & R \end{bmatrix} \begin{bmatrix} \lambda_i I - A & -B_w e^{-\lambda_i \tau} \\ -C_z & I - D_{zw} e^{-\lambda_i \tau} \end{bmatrix} = 0.$$

If $\eta'_i \begin{bmatrix} Q & R \end{bmatrix} \neq 0$, then $\chi_\tau(\lambda_i) = 0$ for every such λ_i . Hence, the term

– left characteristic matrix equation (LCME),

with $\text{spec}(\tilde{A}) \in$ set of roots of $\chi_\tau(s)$.

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Closed-loop characteristic function

If

$$u(t) = \tilde{K}\tilde{x}(t) = \tilde{K}\left(Qx(t) + \int_{t-\tau}^t e^{\tilde{A}(t-r)} R z(r) dr\right),$$

then

$$\begin{bmatrix} sI - A & -B_w e^{-\tau s} & -B_u \\ -C_z & I - D_{zw} e^{-\tau s} & -D_{zu} \\ -\tilde{K}Q & -\tilde{K} \int_0^\tau e^{-(sI - \tilde{A})t} dt R & I \end{bmatrix} \begin{bmatrix} X(s) \\ Z(s) \\ U(s) \end{bmatrix} = 0$$

and its characteristic function

$$\begin{aligned} \chi_{\tau,cl}(s) &= \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} & -B_u \\ -C_z & I - D_{zw} e^{-\tau s} & -D_{zu} \\ -\tilde{K}Q & -\tilde{K} \int_0^\tau e^{-(sI - \tilde{A})t} dt R & I \end{bmatrix} \\ &= \det \left(\begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix} - \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix} \tilde{K} \left[Q \int_0^\tau e^{-(sI - \tilde{A})t} dt R \right] \right). \end{aligned}$$

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Closed-loop characteristic function (contd)

By LCME,

$$\begin{bmatrix} Q & \int_0^\tau e^{-(sI - \tilde{A})t} dt R \end{bmatrix} = (sI - \tilde{A})^{-1} \begin{bmatrix} Q & R \end{bmatrix} \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix},$$

so

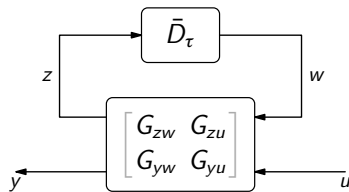
$$\begin{aligned} \chi_{\tau,cl}(s) &= \det \left(I - \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix} \tilde{K} (sI - \tilde{A})^{-1} \begin{bmatrix} Q & R \end{bmatrix} \right) \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix} \\ &= \frac{1}{\det(sI - \tilde{A})} \det \begin{bmatrix} I & 0 & B_u \tilde{K} \\ 0 & I & D_{zu} \tilde{K} \\ Q & R & sI - \tilde{A} \end{bmatrix} \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix} \\ &= \det(sI - \tilde{A} - \tilde{B} \tilde{K}) \frac{\chi_\tau(s)}{\det(sI - \tilde{A})} \end{aligned}$$

and closed-loop characteristic roots are in

$$\text{spec}(\tilde{A} + \tilde{B} \tilde{K}) \cup (\text{roots of } \chi_\tau(s) \setminus \text{spec}(\tilde{A})).$$

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Design steps (with many detail devils)



1. Determine what $\tilde{n}$ roots of $\chi_\tau(s)$ are selected for shifting by feedback
2. Solve LCME

$$\begin{bmatrix} Q & R \end{bmatrix} \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} = \begin{bmatrix} \tilde{A}Q & e^{\tilde{A}\tau} R \end{bmatrix}$$

for given $\tilde{n}$ and $\text{spec}(\tilde{A})$

3. Design $\tilde{K}$ for $\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + (QB_u + RD_{zu})u(t)$
4. implement

$$u(t) = \tilde{K}\left(Qx(t) + \int_{t-\tau}^t e^{\tilde{A}(t-r)} R z(r) dr\right)$$

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Special cases

Consider the LCME

$$\begin{bmatrix} Q & R \end{bmatrix} \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} = \begin{bmatrix} \tilde{A}Q & e^{\tilde{A}\tau} R \end{bmatrix}.$$

$$- D_{zw} = 0 \implies QB_w = e^{\tilde{A}\tau} R \implies R = e^{-\tilde{A}\tau} QB_w \implies$$

$$QA + e^{-\tilde{A}\tau} QB_w C_z = \tilde{A}Q.$$

$$- \begin{bmatrix} C_z & D_{zw} \end{bmatrix} = 0 \implies$$

$$R = e^{-\tilde{A}\tau} QB_w \quad \text{and} \quad QA = \tilde{A}Q.$$

- $\tilde{n} = n \implies$ solvable by $Q = I$ and $\tilde{A} = A$, results in FSA
- $\tilde{n} < n \implies$ solvable by a partial pole placement version of FSA

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Example 1

Let

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t-\tau) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

or $\dot{x}(t) = -x(t) + x(t-\tau) + u(t)$. Its characteristic function

$$\chi_\tau(s) = \det \begin{bmatrix} s+1 & -e^{-\tau s} \\ -1 & 1 \end{bmatrix} = s+1-e^{-\tau s}.$$

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Example 1: “bad” open-loop roots

Let $s = \sigma + j\omega$. The characteristic equation, $s+1-e^{-\tau s} = 0$, implies that

$$|1 + \sigma + j\omega| = \sqrt{(1+\sigma)^2 + \omega^2} = e^{-\sigma\tau},$$

which is

- unsolvable if $\sigma > 0$
- must have $\omega = 0$ if $\sigma = 0$

And indeed, $\chi_\tau(0) = 0$, so

- $s = 0$ is the only unstable (i.e. in $\bar{\mathbb{C}}_0$) root of the retarded $\chi_\tau(s)$.

As such, we may

- endeavor to move only the root at the origin by feedback

i.e. we have $\tilde{n} = 1$.

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Example 1: LCME

Now, the retarded

$$QA + e^{-\tilde{A}\tau}QB_wC_z = \tilde{A}Q \quad \text{and} \quad R = e^{-\tilde{A}\tau}QB_w$$

read

$$-Q + e^{-\tilde{A}\tau}Q = \tilde{A}Q \quad \text{and} \quad R = e^{-\tilde{A}\tau}Q.$$

As $\tilde{A}$ is a scalar, we choose $\tilde{A} = 0$, yielding

$$-Q + Q = 0 \quad \text{and} \quad R = Q,$$

solvable by any $Q \neq 0$. Choose $Q = R = 1$ (simple).

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Example 1: design of $\tilde{K}$ and u

Hence,

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + (QB_u + RD_{zu})u(t) = u(t)$$

is a plain integrator, can be stabilized by $u(t) = -k_p\tilde{x}(t)$, which assigns its closed-loop pole to $s = -k_p$. Thus,

$$u(t) = \tilde{K} \left(Qx(t) + \int_{t-\tau}^t e^{\tilde{A}(t-r)} R z(r) dr \right)$$

reads

$$u(t) = -k_p \left(x(t) + \int_{t-\tau}^t x(r) dr \right),$$

(because $z = x$ in this case) and the closed-loop characteristic function is

$$\chi_{\tau,cl}(s) = (s + k_p) \frac{s + 1 - e^{-\tau s}}{s},$$

whose all roots are in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$.

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Very special case

Let $D_{zw} = 0$,

$$A = \begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \end{bmatrix}, \quad \text{and} \quad A_\tau := B_w C_z = \begin{bmatrix} 0 & * & \cdots & * & * \\ 0 & 0 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

In this case

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I \end{bmatrix} = \det(sI - A - A_\tau e^{-\tau s}) = \det(sI - A)$$

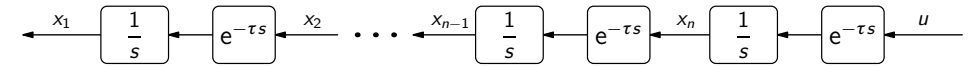
is a standard polynomial. If $\tilde{n} = n$, we can take $Q = I$ and solve LCME

$$\tilde{A} - A = e^{-\tilde{A}\tau} A_\tau$$

iteratively.

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Example 2



With

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_{n-1}(t) \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n \quad \text{and} \quad z(t) = \begin{bmatrix} x_2(t) \\ \vdots \\ x_n(t) \\ u(t) \end{bmatrix} \in \mathbb{R}^n$$

we get

$$G = \begin{bmatrix} J_n/s & e_n \\ I_n/s & 0 \end{bmatrix} = \begin{bmatrix} 0 & I_n & 0 \\ J_n & 0 & e_n \\ I_n & 0 & 0 \end{bmatrix},$$

where

$$J_n := \begin{bmatrix} 0 & e_1 & \cdots & e_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence, $\chi_\tau(s) = s^n$.

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Example 2: LCME

With $\tilde{n} = n$,

$$\tilde{A} = e^{-\tilde{A}\tau} J_n.$$

Because $J_n e_1 = 0$ and $J_n e_i = e_{i-1}$ for all $i = 2, \dots, n$,

— i th column of $\tilde{A}$ can be determined from columns $1, \dots, i-1$.

Let $n = 3$:

$$\begin{aligned} 1. \quad \tilde{A}e_1 = 0 &\implies \tilde{A} = \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \implies e^{-\tilde{A}\tau} = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \\ 2. \quad \tilde{A}e_2 = e^{-\tilde{A}\tau}e_1 &\implies \tilde{A} = \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} \implies e^{-\tilde{A}\tau} = \begin{bmatrix} 1 & -\tau & * \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix} \\ 3. \quad \tilde{A}e_3 = e^{-\tilde{A}\tau}e_2 &\implies \tilde{A} = \begin{bmatrix} 0 & 1 & -\tau \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

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Example 2: design of $\tilde{K}$

Thus,

$$R = e^{-\tilde{A}\tau} Q = \exp\left(-\begin{bmatrix} 0 & 1 & -\tau \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tau\right) = \begin{bmatrix} 1 & -\tau & 3\tau^2/2 \\ 0 & 1 & -\tau \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\tilde{B} = QB_u + RD_{zu} = \begin{bmatrix} 1 & -\tau & 3\tau^2/2 \\ 0 & 1 & -\tau \\ 0 & 0 & 1 \end{bmatrix} e_n = \begin{bmatrix} 3\tau^2/2 \\ -\tau \\ 1 \end{bmatrix}$$

so

$$\dot{\tilde{x}}(t) = \begin{bmatrix} 0 & 1 & -\tau \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 3\tau^2/2 \\ -\tau \\ 1 \end{bmatrix} u(t)$$

If $\chi_{\tau,cl}(s) = (s + \alpha)^3$, then

$$\tilde{K} = -\begin{bmatrix} \alpha^3 & \alpha^2(2\tau\alpha + 3) & \alpha((\tau\alpha + 3)^2 - 3)/2 \end{bmatrix}.$$

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Outline

Discrete insight: state feedback

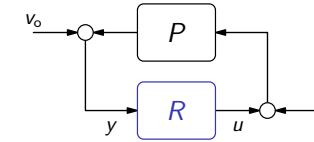
Discrete insight: output feedback

(A paraphrase on) Fiagbedzi–Pearson reduction

Loop shifting

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Internal stability



where

$$\begin{bmatrix} I & -P \\ -R & I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} v_o \\ v_i \end{bmatrix}.$$

The closed-loop system is well-posed if $\exists \alpha > 0$ such that

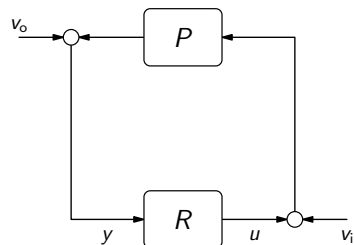
$$\det \begin{bmatrix} I & -P(s) \\ -R(s) & I \end{bmatrix} = \det(I - P(s)R(s)) \neq 0 \quad \forall s \in \mathbb{C}_\alpha$$

and internally stable if

$$\begin{bmatrix} I & -P \\ -R & I \end{bmatrix}^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I & P \end{bmatrix} = \begin{bmatrix} I \\ R \end{bmatrix} (I - PR)^{-1} \begin{bmatrix} I & P \end{bmatrix} \in H_\infty.$$

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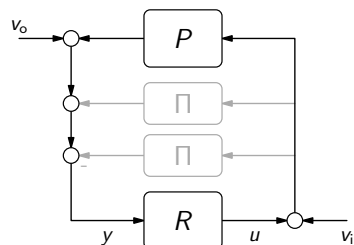
Loop transformations



internally stable

$\Uparrow$

4 systems $(v_o, v_i) \mapsto (y, u)$ are stable



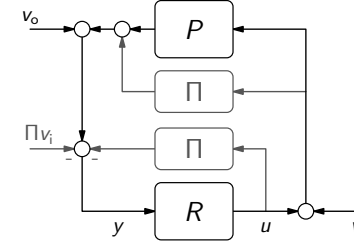
internally stable

$\Uparrow$

4 systems $(v_o, v_i) \mapsto (y, u)$ are stable

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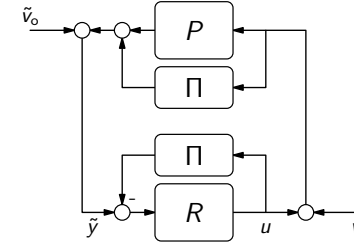
Loop transformations



internally stable

$\Uparrow$

4 systems $(v_o, v_i) \mapsto (y, u)$ are stable



internally stable

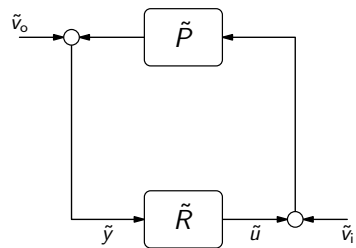
$\Uparrow$

4 systems $(v_o, v_i) \mapsto (y, u)$ are stable

where $\tilde{v}_o = v_o - \Pi v_i$ and $\tilde{y} = y + \Pi u$.

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Loop transformations



internally stable

$\Uparrow$

4 systems $(\tilde{v}_o, \tilde{v}_i) \mapsto (\tilde{u}, \tilde{y})$ are stable

where

$$\begin{bmatrix} \tilde{v}_o \\ \tilde{v}_i \end{bmatrix} = \begin{bmatrix} I & -\Pi \\ 0 & I \end{bmatrix} \begin{bmatrix} v_o \\ v_i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} I & \Pi \\ 0 & I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

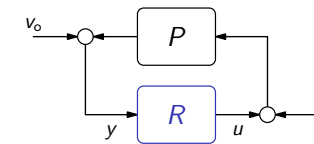
and

$$\tilde{P} = P + \Pi \quad \text{and} \quad \tilde{R} = R(I + \Pi R)^{-1}.$$

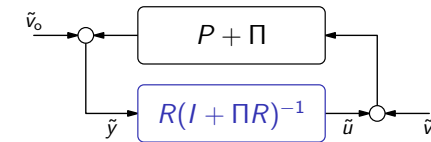
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Loop shifting

If $\Pi \in H_\infty$ and the loop of R and Π is well posed, then



is well posed and internally stable iff



is well posed and internally stable.

This transformation makes sense if the

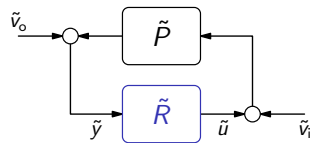
- transformed system has more favorable properties than the original one.

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Loop shifting: design

Two stages:

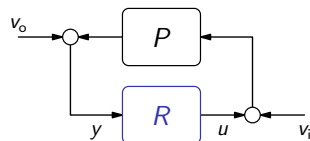
1. design $\tilde{R}$ for $\tilde{P} := P + \Pi$ in



2. implement

$$R = (I - \tilde{R}\Pi)^{-1}\tilde{R} \iff \tilde{R} = R(I + \Pi R)^{-1}$$

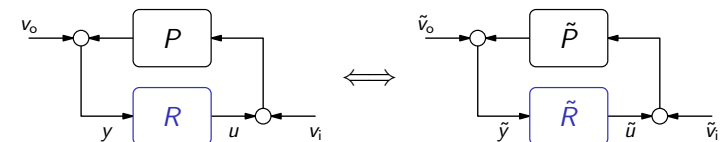
for



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Loop shifting: to remember

The equivalence



involves signal transformations

$$\begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} I & \Pi \\ 0 & I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{v}_o \\ \tilde{v}_i \end{bmatrix} = \begin{bmatrix} I & -\Pi \\ 0 & I \end{bmatrix} \begin{bmatrix} v_o \\ v_i \end{bmatrix}.$$

Not always

- “good” $\tilde{y} \implies$ “good” y
- “good” response to $\tilde{v}_o \implies$ “good” response to v_o

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