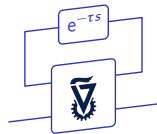


Time Delays in Control and Estimation (038806)



lecture no. 5

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

1/46

Preliminary: Leibniz integration rule

$$\begin{aligned} \frac{d}{dt} \int_{a(t)}^{b(t)} f(s, t) ds \\ = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(s, t) ds + \frac{db(t)}{dt} f(b(t), t) - \frac{da(t)}{dt} f(a(t), t) \end{aligned}$$

2/46

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe-Itō)

Finite spectrum assignment (Manitius-Olbrot)

Reduction (Kwon-Pearson, Artstein)

Controller structures

3/46

ODEs: Lyapunov stability

Let

$$\dot{x} = f(x)$$

Its equilibrium $x_{eq} \in \mathbb{R}^n$ is

- **stable** if $\forall \epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0) - x_{eq}\| < \delta \implies \|x(t) - x_{eq}\| < \epsilon, \quad \forall t \in \mathbb{R}_+$$

- **asymptotically stable** if stable and $\exists \delta > 0$ such that

$$\|x(0) - x_{eq}\| < \delta \implies \lim_{t \rightarrow \infty} \|x(t) - x_{eq}\| = 0$$

Linear case: $f(x) = Ax$ and $x_{eq} = 0$ (unless $\det A = 0$).

4/46

ODEs: Lyapunov's direct method

Assuming $x_{\text{eq}} = 0$, Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$

- continuously differentiable
- satisfies $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$

Its derivative along trajectories of the system

$$\dot{V}(x) := \frac{dV(x)}{dt} = \frac{\partial V(x)}{\partial x} \frac{dx}{dt} = \frac{\partial V(x)}{\partial x} f(x).$$

and

- $\dot{V}(x) \leq 0$ for all $x \implies$ stable
- $\dot{V}(x) < 0$ for all $x \neq 0 \implies$ stable asymptotically
- $\dot{V}(x) \leq 0$ for all x & $\dot{V}(x) \equiv 0 \implies x \equiv 0 \implies$ stable asymptotically

5/46

Retarded equations: Lyapunov stability

Let

$$\dot{x}(t) = Ax(t) + A_\tau x(t - \tau)$$

Its state $\check{x}_t \in C^n([-\tau, 0])$ and $\check{x}_{\text{eq}} = 0$ (unless $\det(A + A_\tau) = 0$), so

- **stable** if $\forall \epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\|\check{x}_0\| < \delta \implies \|\check{x}_t\| < \epsilon, \quad \forall t \in \mathbb{R}_+$$

- **asymptotically stable** if stable and $\exists \delta > 0$ such that

$$\|\check{x}_0\| < \delta \implies \lim_{t \rightarrow \infty} \|\check{x}_t\| = 0$$

for whatever norm on $C^n([-\tau, 0])$.

Lyapunov function

- shall be based on $\check{x}_t$, with complications of the meaning of $\dot{V} < 0$, etc
- Popular relaxations proposed by Razumikhin and by Krasovskii in '50s.

6/46

Retarded equations: Razumikhin

Theorem

If there is a continuous $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $V(0) = 0$,

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|),$$

and

$$\dot{V}(x(t)) \leq -\gamma(\|x(t)\|) \text{ whenever } V(x(t+\theta)) \leq \rho(V(x(t))), \quad \forall \theta \in [-\tau, 0]$$

for continuous non-decreasing $\alpha, \beta, \gamma, \rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\lim_{\|x\| \rightarrow \infty} \alpha(\|x\|) = \infty \quad \text{and} \quad \rho(V) \geq V,$$

then the system is Lyapunov stable. If in addition $\rho(V) > V$ for all $V > 0$, then the system is stable asymptotically.

7/46

Retarded equations: Razumikhin (contd)

Choose $V = x'Px$, $P > 0$. Then

$$\begin{aligned} \dot{V}(x(t)) &= 2x'(t)P(Ax(t) + A_\tau x(t - \tau)) \\ &= \begin{bmatrix} x'(t) & x'(t - \tau) \end{bmatrix} \begin{bmatrix} A'P + PA & PA_\tau \\ A'_\tau P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}. \end{aligned}$$

Now, given $a > 0$ and $\eta > 1$

$$W(t) := \dot{V}(x(t)) + a(\eta x'(t)Px(t) - x'(t - \tau)Px(t - \tau)) \leq -\zeta x'(t)Px(t)$$

$\Downarrow$

$$\dot{V}(x(t)) \leq -\zeta x'(t)Px(t) \quad \text{whenever } x'(t + \theta)Px(t + \theta) \leq \eta x'(t)Px(t)$$

for all $\theta \in [-\tau, 0]$. Thus, if $\exists P = P' > 0$ and $a, \epsilon > 0$ such that

$$\begin{bmatrix} A'P + PA + (a\eta + \zeta)P & PA_\tau \\ A'_\tau P & -aP \end{bmatrix} \leq 0 \iff \begin{bmatrix} A'P + PA + aP & PA_\tau \\ A'_\tau P & -aP \end{bmatrix} \leq \begin{bmatrix} -\epsilon P & 0 \\ 0 & 0 \end{bmatrix}$$

($\epsilon := a(\eta - 1) + \zeta > 0$, arbitrary), then the system **asymptotically stable** $\forall \tau > 0$.

8/46

Retarded equations: Krasovskii

Theorem

If there is a continuous $V(\check{x}) : C^n([-\tau, 0]) \rightarrow \mathbb{R}$ such that $V(0) = 0$,

$$V(\check{x}) \geq \alpha(\|x\|), \quad \text{and} \quad \dot{V}(\check{x}) \leq -\beta(\|x\|)$$

for $\alpha, \beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\lim_{\|x\| \rightarrow \infty} \alpha(\|x\|) = \infty,$$

then the system is Lyapunov stable. If in addition $\beta(\|x\|) > 0, \forall \|x\| > 0$, then the system is stable asymptotically.

9/46

Retarded equations: Krasovskii (contd)

Now let

$$V(\check{x}_t) = x'(t)P_0x(t) + \int_{-\tau}^0 x'(t+s)P_\tau x(t+s)ds = x'(t)P_0x(t) + \int_{t-\tau}^t x'(s)P_\tau x(s)ds$$

for some $P_0 = P'_0 > 0$ and $P_\tau = P'_\tau > 0$. By the Leibniz integral rule

$$\begin{aligned} \dot{V}(\check{x}_t) &= 2x'(t)P_0\dot{x}(t) + x'(t)P_\tau x(t) - x'(t-\tau)P_\tau x(t-\tau) \\ &= 2x'(t)P_0Ax(t) + 2x'(t)P_0A_\tau x(t-\tau) + x'(t)P_\tau x(t) - x'(t-\tau)P_\tau x(t-\tau) \\ &= \begin{bmatrix} x'(t) & x'(t-\tau) \end{bmatrix} \begin{bmatrix} A'P_0 + P_0A + P_\tau & P_0A_\tau \\ A'_\tau P_0 & -P_\tau \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \end{aligned}$$

Hence, if $\exists P_0 = P'_0 > 0$ and $P_\tau = P'_\tau > 0$ such that

$$\begin{bmatrix} A'P_0 + P_0A + P_\tau & P_0A_\tau \\ A'_\tau P_0 & -P_\tau \end{bmatrix} < 0,$$

then the system is asymptotically stable $\forall \tau > 0$.

10/46

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe-Itō)

Finite spectrum assignment (Manitius-Olbrot)

Reduction (Kwon-Pearson, Artstein)

Controller structures

11/46

Problem

FOPTD plant

$$P_\tau(s) = \frac{e^{-\tau s}}{s-1}$$

(if $P(s) = e^{-\tau s}/(Ts-1)$ for $T > 0$, then $s \rightarrow s/T \implies \tau \rightarrow \tau/T$).

PI controller:

$$R(s) = k_p \left(1 + \frac{1}{T_i s} \right)$$

PD controller:

$$R(s) = k_p(1 + T_d s)$$

Goal:

- conditions on controller parameters for which $\chi_\tau(s)$ is Hurwitz.

12/46

PI: stability conditions

Characteristic function: $\chi_\tau(s) = T_i s(s-1) + k_p(T_i s + 1)e^{-\tau s}$

Crossing (crossover) frequency:

$$\omega_c^2 = \frac{k_p^2 - 1}{2} + \frac{1}{T_i} \sqrt{k_p^2 + \frac{T_i^2}{2}(k_p^2 - 1)^2}$$

switch, hence, need delay-free stability.

Delay-free stability:

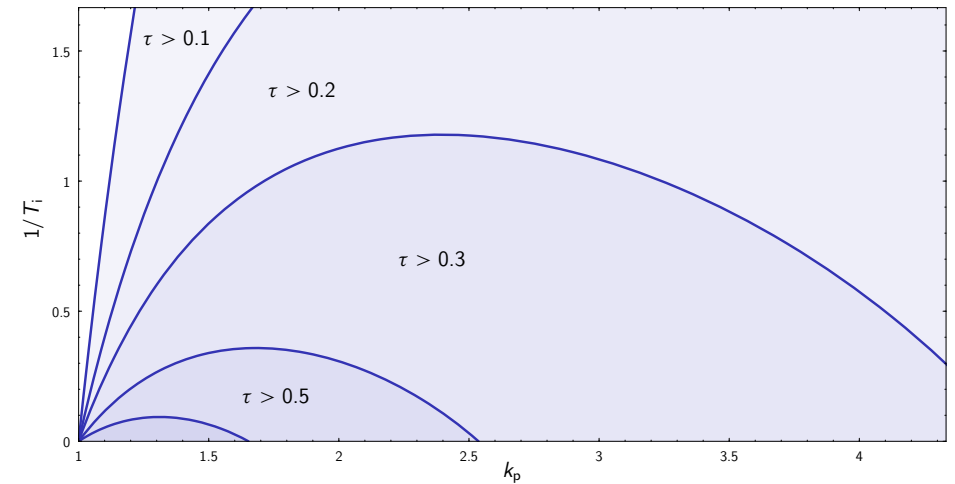
$$k_p > 1 \quad \text{and} \quad T_i > 0 \quad \implies \quad T_i \omega_c^2 > k_p > 1.$$

Stabilizing delays range:

$$0 \leq \tau < \frac{1}{\omega_c} \arctan \frac{T_i \omega_c^2 - 1}{(T_i + 1) \omega_c} < 1$$

13/46

PI: stability map in $(k_p, 1/T_i)$ -plane



14/46

PI: ω_c as alternative design parameter (to k_p)

$k_p \leftrightarrow \omega_c$ is one-to-one (bijective):

$$\omega_c^2 = \frac{1}{2} \left(k_p^2 - 1 + \sqrt{(k_p^2 - 1)^2 + \frac{4k_p^2}{T_i^2}} \right) \iff k_p = T_i \omega_c \sqrt{\frac{1 + \omega_c^2}{1 + T_i^2 \omega_c^2}}$$

and $k_p > 1 \iff \omega_c > 1/\sqrt{T_i} > 0$, i.e. is always the case.

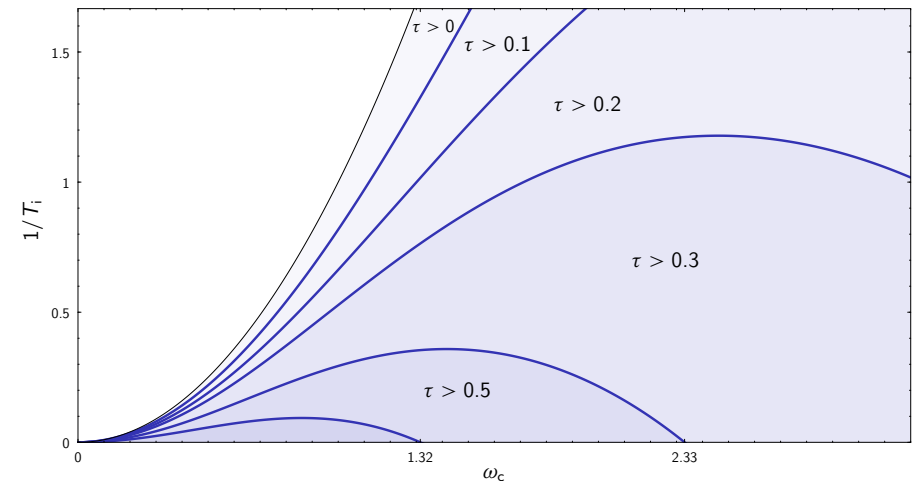
Given $\tau \in (0, 1)$, the closed-loop system is stable iff

$$\frac{1}{T_i} < \frac{\omega_c(\omega_c - \tan(\tau\omega_c))}{1 + \omega_c \tan(\tau\omega_c)},$$

and $T_i > 0$ exists iff $\omega_c < \omega_\tau$, where $\omega_\tau = \tan(\tau\omega_\tau)$ with $\tau\omega_\tau < \pi/2$.

15/46

PI: stability map in $(\omega_c, 1/T_i)$ -plane



Thus,

- larger delay tolerance imposes more severe constraints on ω_c .

16/46

PD: stability conditions

Characteristic function: $\chi_\tau(s) = s - 1 + k_p(T_d s + 1)e^{-\tau s}$ with $|k_p T_d| < 1$

Crossing (crossover) frequency (exists iff $k_p > 1$):

$$\omega_c^2 = \frac{k_p^2 - 1}{1 - k_p^2 T_d^2}$$

switch, hence, need delay-free stability.

Delay-free stability:

$$k_p > 1$$

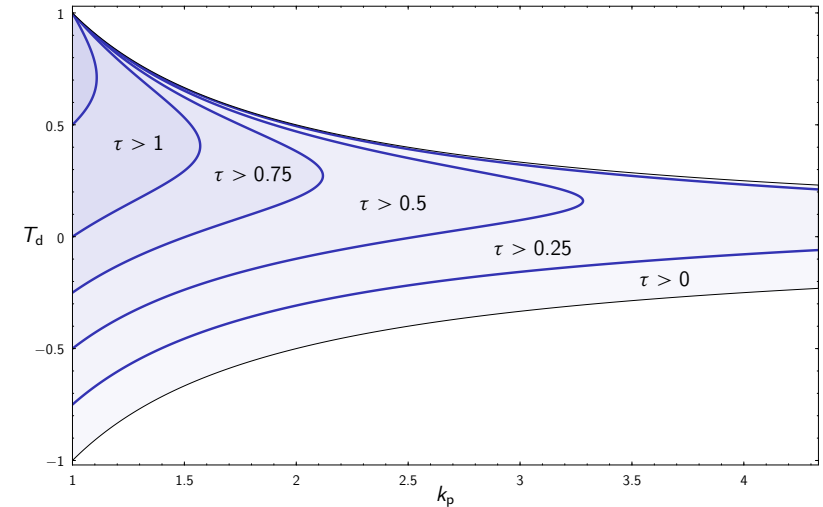
Stabilizing delays range:

$$0 \leq \tau < \frac{1}{\omega_c} (\arctan(T_d \omega_c) + \arctan \omega_c) \leq 2$$

provided $|T_d| < 1/k_p < 1$.

17/46

PD: stability map in (k_p, T_d) -plane



18/46

PD: ω_c as alternative design parameter (to k_p)

$k_p \leftrightarrow \omega_c$ is one-to-one (bijective):

$$\omega_c^2 = \frac{k_p^2 - 1}{1 - k_p^2 T_d^2} \iff k_p = \sqrt{\frac{1 + \omega_c^2}{1 + T_d^2 \omega_c^2}}$$

for all $k_p > 1$ and $\omega_c > 0$ (provided $|T_d| < 1$).

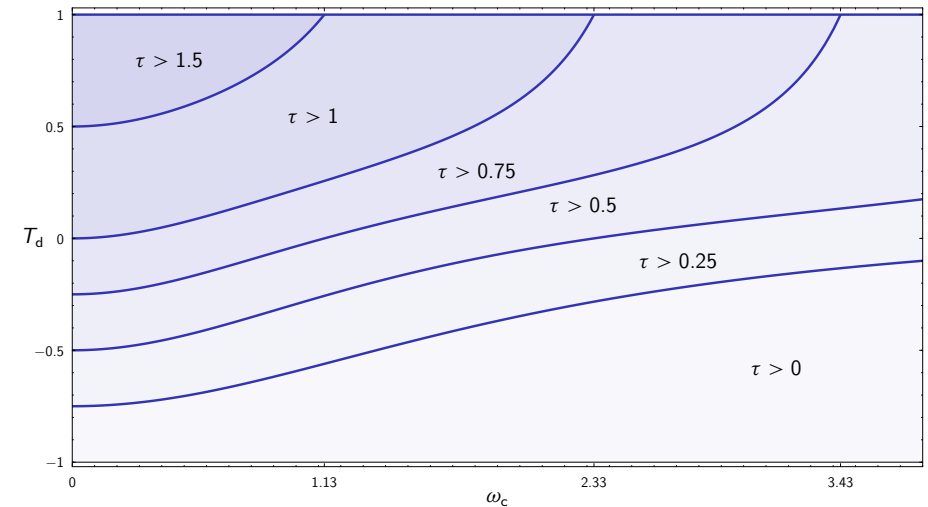
Given $\tau \in (0, 2)$, the closed-loop system is stable iff

$$-1 < \frac{\tan(\tau \omega_c - \arctan \omega_c)}{\omega_c} < T_d < 1,$$

admissible iff $0 < \tau \omega_c < \arctan \omega_c + \pi/2 < \pi$.

19/46

PD: stability map in (ω_c, T_d) -plane



Thus,

- larger delay tolerance imposes more severe constraints on ω_c .

20/46

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe-Itō)

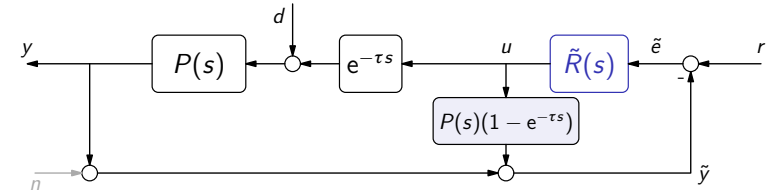
Finite spectrum assignment (Manitius-Olbrot)

Reduction (Kwon-Pearson, Artstein)

Controller structures

21/46

Smith controller



Comprises

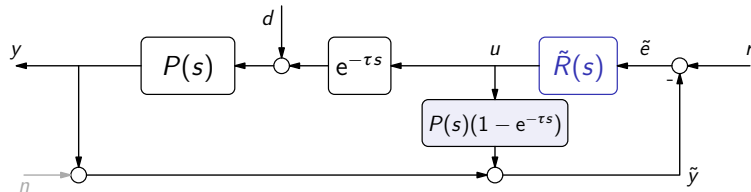
- primary controller $\tilde{R}$
- Smith predictor $P(1 - \bar{D}_\tau)$

The overall controller $R : y \mapsto u$ has

$$R(s) = \frac{\tilde{R}(s)}{1 + \tilde{R}(s)P(s)(1 - e^{-\tau s})}.$$

22/46

Smith controller: rationale



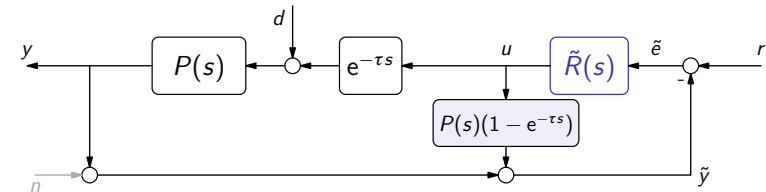
Signal

$$\tilde{y} = y + P(1 - \bar{D}_\tau)u = P(\bar{D}_\tau u + d) + P(1 - \bar{D}_\tau)u = P(u + d)$$

may be seen as a **predicted** version of y .

23/46

Smith controller: the magic



Closed-loop transfer functions:

- control sensitivity

$$T_c(s) = \frac{\tilde{R}(s)}{1 + P(s)\tilde{R}(s)}$$

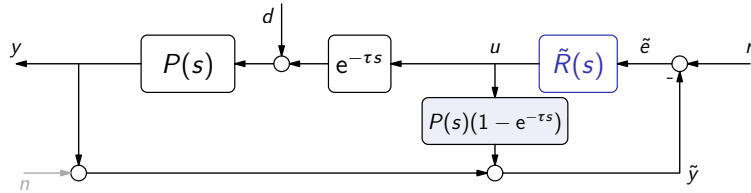
- complementary sensitivity

$$T(s) = \frac{P(s)\tilde{R}(s)}{1 + P(s)\tilde{R}(s)} e^{-\tau s}$$

have **polynomial** denominators.

24/46

Smith controller: the catch



The disturbance sensitivity transfer function:

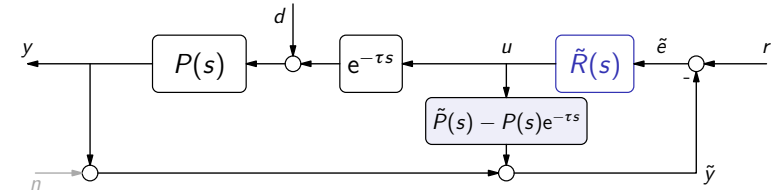
$$T_d(s) = \frac{P(s)}{1 + P(s)\tilde{R}(s)} + \frac{P(s)\tilde{R}(s)}{1 + P(s)\tilde{R}(s)}P(s)(1 - e^{-\tau s})$$

is unstable, unless $P(s)(1 - e^{-\tau s})$ is stable. The reason: unstable pole-zero cancellations, cf.

$$R(s) = \frac{\tilde{R}(s)}{1 + \tilde{R}(s)P(s)(1 - e^{-\tau s})} = \frac{M_P(s)N_{\tilde{R}}(s)}{M_P(s)M_{\tilde{R}}(s) + N_P(s)N_{\tilde{R}}(s)(1 - e^{-\tau s})}.$$

25/46

Modified Smith predictor of Watanabe & Ito



for a finite-dimensional $\tilde{P}$. Now,

$$\tilde{y} = \tilde{P}u + Pd$$

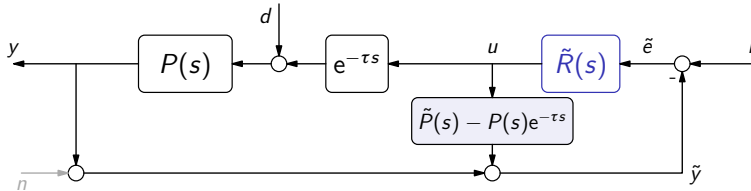
no longer predicts y . Still, it is delay free, so the term

— **dead-time compensator** (DTC)

for this class of controllers.

26/46

Modified Smith predictor: closed-loop transfer functions



If $\Pi(s) := \tilde{P}(s) - P(s)e^{-\tau s}$, then

$$\begin{bmatrix} S(s) & T_c(s) \\ T_d(s) & T(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\Pi(s) & 1 \end{bmatrix} \begin{bmatrix} \tilde{S}(s) & \tilde{T}_c(s) \\ \tilde{T}_d(s) & \tilde{T}(s) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Pi(s) & 1 \end{bmatrix},$$

where

$$\begin{bmatrix} \tilde{S}(s) & \tilde{T}_c(s) \\ \tilde{T}_d(s) & \tilde{T}(s) \end{bmatrix} := \frac{1}{1 + \tilde{P}(s)\tilde{R}(s)} \begin{bmatrix} 1 \\ \tilde{P}(s) \end{bmatrix} \begin{bmatrix} 1 & \tilde{R}(s) \end{bmatrix}.$$

27/46

Modified Smith predictor: choice of $\tilde{P}$

Assuming $P(s) = C(sI - A)^{-1}B$, Watanabe & Ito suggested

$$\tilde{P}(s) = Ce^{-A\tau}(sI - A)^{-1}B$$

for which

$$\begin{aligned} \Pi(s) &= \tilde{P}(s) - P(s)e^{-\tau s} = C(e^{-A\tau} - e^{-\tau s}I)(sI - A)^{-1}B \\ &= Ce^{-A\tau} \int_0^\tau e^{-(sI - A)t} dt B \end{aligned}$$

is an entire function, bounded in $\mathbb{C}_0 \implies$ stable.

28/46

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe-Itō)

Finite spectrum assignment (Manitius-Olbrot)

Reduction (Kwon-Pearson, Artstein)

Controller structures

29/46

Idea

Let

$$\dot{x}(t) = Ax(t) + Bu(t - \tau),$$

with measurable x . Ideally,

$$u(t) = Kx(t + \tau) + K_v v(t)$$

and K such that $A + BK$ is Hurwitz. Because we do not measure $x(t + \tau)$, try to use its predicted value

$$x(t + \tau) = e^{A\tau}x(t) + \int_{t-\tau}^t e^{A(t-r)}Bu(r)dr,$$

which is **causally implementable**.

30/46

Predictor: impulse response and transfer function

The relation

$$u(t) \mapsto \int_{t-\tau}^t e^{A(t-r)}Bu(r)dr = \int_{-\infty}^{\infty} e^{A(t-r)}B\mathbb{1}_{[0,\tau]}(t-r)u(r)dr$$

is the convolution of $u(t)$ and $e^{At}B\mathbb{1}_{[0,\tau]}(t)$. Hence, it's LTI and its transfer function is

$$\int_0^{\infty} e^{At}B\mathbb{1}_{[0,\tau]}(t)e^{-st}dt = \int_0^{\tau} e^{-(sI-A)t}dtB = (sI - A)^{-1}(I - e^{-(sI-A)\tau})B$$

which is an entire function, bounded on $\mathbb{C}_0$, hence, belongs to H_{∞} .

31/46

Characteristic function

With $v = 0$

$$U(s) = Ke^{A\tau}X(s) + K \int_0^{\tau} e^{-(sI-A)t}dtBU(s)$$

and the closed-loop relation in the s -domain:

$$\begin{bmatrix} sI - A & -Be^{-\tau s} \\ -Ke^{A\tau} & I - K \int_0^{\tau} e^{-(sI-A)t}dtB \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = 0.$$

Its characteristic function, corresponding to the nontrivial free response, is

$$\chi_{\tau}(s) = \det \begin{bmatrix} sI - A & -Be^{-\tau s} \\ -Ke^{A\tau} & I - K \int_0^{\tau} e^{-(sI-A)t}dtB \end{bmatrix}.$$

Hmm, looks complicated...

32/46

Characteristic function (contd)

Because

$$(sI - A) \int_0^\tau e^{-(sI-A)t} dt B = (I - e^{-(sI-A)\tau}) B = e^{A\tau} (e^{-A\tau} B - e^{-\tau s} B),$$

we have

$$\begin{bmatrix} sI - A & -Be^{-\tau s} \\ -Ke^{A\tau} & I - K \int_0^\tau e^{-(sI-A)t} dt B \end{bmatrix} = \begin{bmatrix} sI - A & -e^{-A\tau} B \\ -Ke^{A\tau} & I \end{bmatrix} \begin{bmatrix} I & e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B \\ 0 & I \end{bmatrix}.$$

Hence,

$$\begin{aligned} \chi_\tau(s) &= \det \begin{bmatrix} sI - A & -e^{-A\tau} B \\ -Ke^{A\tau} & I \end{bmatrix} = \det(sI - A - e^{-A\tau} B K e^{A\tau}) \\ &= \det(sI - (A + BK)) \end{aligned}$$

is a plain n -degree polynomial.

33/46

Prediction logic is not always obvious

Let

$$\dot{x}(t) = Ax(t) + \sum_i B_i u(t - \tau_i).$$

The control law

$$u(t) = K \left(x(t) + \sum_i \int_{t-\tau_i}^t e^{A(t-\tau_i-r)} B u(r) dr \right) + K_v v(t)$$

renders

$$\chi_\tau(s) = \det \left(sI - A - \sum_i e^{-A\tau_i} B_i K \right)$$

finite polynomial again. But this is not an obvious predictor...

34/46

Observer-predictor

If $y(t) = Cx(t)$ is measurable, then

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t - \tau) - L(y(t) - C\hat{x}(t)) \\ u(t) &= K \left(e^{A\tau} \hat{x}(t) + \int_{t-\tau}^t e^{A(t-r)} B u(r) dr \right) + K_v v(t), \end{aligned}$$

for some L such that $A + LC$ is Hurwitz. In this case

$$\begin{bmatrix} sI - A - LC & 0 & 0 \\ 0 & sI - A & -Be^{-\tau s} \\ Ke^{A\tau} & -Ke^{A\tau} & I - K \int_0^\tau e^{-(sI-A)t} dt B \end{bmatrix} \begin{bmatrix} E(s) \\ X(s) \\ U(s) \end{bmatrix} = 0$$

and its characteristic function is again a polynomial,

$$\chi_\tau(s) = \det(sI - A - LC) \det(sI - A - BK),$$

just now having the degree $2n$.

35/46

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe-Itō)

Finite spectrum assignment (Manitius-Olbrot)

Reduction (Kwon-Pearson, Artstein)

Controller structures

36/46

Idea

Let again

$$\dot{x}(t) = Ax(t) + Bu(t - \tau),$$

with measurable x . Define

$$\tilde{x}(t) := x(t) + \int_{t-\tau}^t e^{A(t-\tau-r)} Bu(r) dr$$

(prediction of $e^{-A\tau}x(t + \tau)$). With a little help of the Leibniz integral rule,

$$\begin{aligned} \dot{\tilde{x}}(t) &= \dot{x}(t) + A \int_{t-\tau}^t e^{A(t-\tau-r)} Bu(r) dr + e^{-A\tau} Bu(t) - Bu(t - \tau) \\ &= Ax(t) + A \int_{t-\tau}^t e^{A(t-\tau-r)} Bu(r) dr + e^{-A\tau} Bu(t) \\ &= A\tilde{x}(t) + e^{-A\tau} Bu(t), \end{aligned}$$

which is finite dimensional and can be stabilized by $u(t) = \tilde{K}\tilde{x}(t)$.

37/46

Time-varying delays

Time-domain analysis $\implies$ not limited to time-invariant systems. Let

$$\dot{x}(t) = Ax(t) + Bu(\theta(t))$$

for absolutely continuous $\theta(t) \leq t$ with $\dot{\theta}(t) > 0$ (i.e. delay $\tau(t) = t - \theta(t)$).

Let $\theta^{-1}(t)$ denote the inverse of $\theta(t)$, i.e. r for which $\theta(r) = t$. Then

$$\tilde{x}(t) = x(t) + \int_{\theta(t)}^t e^{A(t-\theta^{-1}(r))} B \dot{\theta}^{-1}(r) u(r) dr$$

satisfies the delay free

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + e^{-A(\theta^{-1}(t)-t)} B \dot{\theta}^{-1}(t) u(t).$$

38/46

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe-Itō)

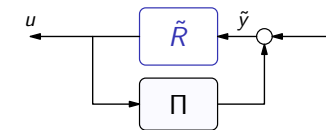
Finite spectrum assignment (Manitius-Olbrot)

Reduction (Kwon-Pearson, Artstein)

Controller structures

39/46

Dead-time compensators



Smith controller has

$$\Pi(s) = -P(s)(1 - e^{-\tau s})$$

and the primary controller as $\tilde{R}$.

Modified Smith predictor has

$$\Pi(s) = -\tilde{P}(s) + P(s)e^{-\tau s} = -C e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B$$

and the primary controller as $\tilde{R}$. In this case

$$\tilde{y}(t) = y(t) - C e^{-A\tau} \int_{t-\tau}^t e^{A(t-r)} Bu(r) dr.$$

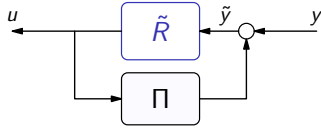
40/46

FSA

Control law

$$u(t) = \tilde{K} \left(e^{A\tau} x(t) + \int_{t-\tau}^t e^{A(t-r)} B u(r) dr \right)$$

is also of the form



with

$$\tilde{R} = K e^{A\tau} \quad \text{and} \quad \Pi(s) = e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B$$

for which

$$\tilde{y}(t) = x(t) + e^{-A\tau} \int_{t-\tau}^t e^{A(t-r)} B u(r) dr.$$

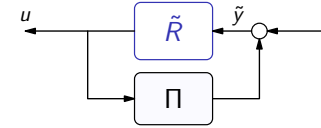
41/46

Reduction

Control law

$$u(t) = \tilde{K} \left(x(t) + \int_{t-\tau}^t e^{A(t-\tau-r)} B u(r) dr \right)$$

is also of the form



with

$$\tilde{R} = \tilde{K} \quad \text{and} \quad \Pi(s) = e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B$$

for which

$$\tilde{y}(t) = x(t) + e^{-A\tau} \int_{t-\tau}^t e^{A(t-r)} B u(r) dr,$$

again.

42/46

Observer-predictor

Let

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t-\tau) - L(y(t) - C\hat{x}(t))$$

$$u(t) = K \left(e^{A\tau} \hat{x}(t) + \int_{t-\tau}^t e^{A(t-r)} B u(r) dr \right).$$

Denote

$$\bar{x}(t) := e^{A\tau} \hat{x}(t) + \int_{t-\tau}^t e^{A(t-r)} B u(r) dr,$$

with which $u(t) = K\bar{x}(t)$. By Leibnitz integral rule

$$\begin{aligned} \dot{\bar{x}}(t) &= e^{A\tau} (A\hat{x}(t) + Bu(t-\tau) - L(y(t) - C\hat{x}(t))) \\ &\quad + A \int_{t-\tau}^t e^{A(t-r)} B u(r) dr + Bu(t) - e^{A\tau} Bu(t-\tau) \\ &= A\bar{x}(t) + Bu(t) - e^{A\tau} L(y(t) - C\hat{x}(t)). \end{aligned}$$

43/46

Observer-predictor (contd)

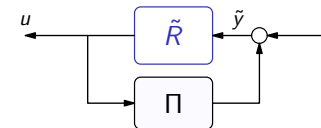
Thus, observer-predictor reads

$$\begin{aligned} \dot{\bar{x}}(t) &= (A + e^{A\tau} L C e^{-A\tau}) \bar{x}(t) + Bu(t) - e^{A\tau} L \tilde{y}(t) \\ u(t) &= K \bar{x}(t). \end{aligned}$$

where

$$\tilde{y}(t) = y(t) + C e^{-A\tau} \int_{t-\tau}^t e^{A(t-r)} B u(r) dr$$

This is



with

$$\tilde{R}(s) = \left[\frac{A + BK + e^{A\tau} L C e^{-A\tau}}{K} \mid \frac{-e^{A\tau} L}{0} \right], \quad \Pi(s) = e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B.$$

44/46

Observer-predictor vs. MSP

Controller

$$\tilde{R}(s) = \left[\frac{A + BK + e^{A\tau} L C e^{-A\tau}}{K} \mid \frac{-e^{A\tau} L}{0} \right]$$

is the observer based-controller for

$$\tilde{P}(s) = \left[\frac{A}{C e^{-A\tau}} \mid \frac{B}{0} \right]$$

with the gain $\tilde{L} = e^{A\tau} L$, for which

$$A + \tilde{L} C e^{-A\tau} = A + e^{A\tau} L C e^{-A\tau} = e^{A\tau} (A + L C) e^{-A\tau}.$$

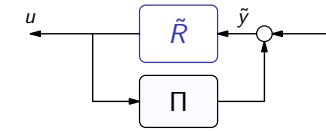
Thus,

- MSP with observer-based $\tilde{R} \iff$ observer-predictor.

45/46

Summarizing

Common structure:



where

- $\tilde{R}$ is finite-dimensional, designed for a finite-dimensional plant
- Π is infinite-dimensional, fixed structure

Common theme seems to be

- prediction as a rationale for Π ,

but prediction is a suspicious operation

- “prediction is difficult, especially about the future” (N. Bohr) . . .

Can it be justified in control-theoretic terms?

46/46