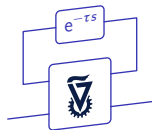


Time Delays in Control and Estimation (038806)



lecture no. 5

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe–Itō)

Finite spectrum assignment (Manitius–Olbrot)

Reduction (Kwon–Pearson, Artstein)

Controller structures

Preliminary: Leibniz integration rule

$$\begin{aligned} \frac{d}{dt} \int_{a(t)}^{b(t)} f(s, t) ds \\ = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(s, t) ds + \frac{db(t)}{dt} f(b(t), t) - \frac{da(t)}{dt} f(a(t), t) \end{aligned}$$

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe–Itō)

Finite spectrum assignment (Manitius–Olbrot)

Reduction (Kwon–Pearson, Artstein)

Controller structures

ODEs: Lyapunov stability

Let

$$\dot{x} = f(x)$$

Its equilibrium $x_{\text{eq}} \in \mathbb{R}^n$ is

- **stable** if $\forall \epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0) - x_{\text{eq}}\| < \delta \implies \|x(t) - x_{\text{eq}}\| < \epsilon, \quad \forall t \in \mathbb{R}_+$$

- **asymptotically stable** if stable and $\exists \delta > 0$ such that

$$\|x(0) - x_{\text{eq}}\| < \delta \implies \lim_{t \rightarrow \infty} \|x(t) - x_{\text{eq}}\| = 0$$

Linear case: $f(x) = Ax$ and $x_{\text{eq}} = 0$ (unless $\det A = 0$).

ODEs: Lyapunov's direct method

Assuming $x_{\text{eq}} = 0$, Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$

- continuously differentiable
- satisfies $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$

Its derivative along trajectories of the system

$$\dot{V}(x) := \frac{dV(x)}{dt} = \frac{\partial V(x)}{\partial x} \frac{dx}{dt} = \frac{\partial V(x)}{\partial x} f(x).$$

and

- $\dot{V}(x) \leq 0$ for all $x \implies$ stable
- $\dot{V}(x) < 0$ for all $x \neq 0 \implies$ stable asymptotically
- $\dot{V}(x) \leq 0$ for all x & $\dot{V}(x) \equiv 0 \implies x \equiv 0 \implies$ stable asymptotically

Retarded equations: Lyapunov stability

Let

$$\dot{x}(t) = Ax(t) + A_\tau x(t - \tau)$$

Its state $\check{x}_t \in C^n([-\tau, 0])$ and $\check{x}_{\text{eq}} = 0$ (unless $\det(A + A_\tau) = 0$), so

- **stable** if $\forall \epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\|\check{x}_0\| < \delta \implies \|\check{x}_t\| < \epsilon, \quad \forall t \in \mathbb{R}_+$$

- **asymptotically stable** if stable and $\exists \delta > 0$ such that

$$\|\check{x}_0\| < \delta \implies \lim_{t \rightarrow \infty} \|\check{x}_t\| = 0$$

for whatever norm on $C^n([-\tau, 0])$.

Lyapunov function

— shall be based on $\check{x}_t$, with complications of the meaning of $\dot{V} < 0$, etc.

Popular relaxations proposed by Razumikhin and by Krasovskii in '50s.

Retarded equations: Lyapunov stability

Let

$$\dot{x}(t) = Ax(t) + A_\tau x(t - \tau)$$

Its state $\check{x}_t \in C^n([-\tau, 0])$ and $\check{x}_{\text{eq}} = 0$ (unless $\det(A + A_\tau) = 0$), so

- **stable** if $\forall \epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\|\check{x}_0\| < \delta \implies \|\check{x}_t\| < \epsilon, \quad \forall t \in \mathbb{R}_+$$

- **asymptotically stable** if stable and $\exists \delta > 0$ such that

$$\|\check{x}_0\| < \delta \implies \lim_{t \rightarrow \infty} \|\check{x}_t\| = 0$$

for whatever norm on $C^n([-\tau, 0])$.

Lyapunov function

- shall be based on $\check{x}_t$, with complications of the meaning of $\dot{V} < 0$, etc

Popular relaxations proposed by Razumikhin and by Krasovskii in '50s.

Retarded equations: Razumikhin

Theorem

If there is a continuous $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $V(0) = 0$,

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|),$$

and

$$\dot{V}(x(t)) \leq -\gamma(\|x(t)\|) \text{ whenever } V(x(t+\theta)) \leq \rho(V(x(t))), \quad \forall \theta \in [-\tau, 0]$$

for continuous non-decreasing $\alpha, \beta, \gamma, \rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\lim_{\|x\| \rightarrow \infty} \alpha(\|x\|) = \infty \quad \text{and} \quad \rho(V) \geq V,$$

then the system is Lyapunov stable. If in addition $\rho(V) > V$ for all $V > 0$, then the system is stable asymptotically.

Retarded equations: Razumikhin (contd)

Choose $V = x'Px$, $P > 0$. Then

$$\begin{aligned}\dot{V}(x(t)) &= 2x'(t)P(Ax(t) + A_\tau x(t - \tau)) \\ &= \begin{bmatrix} x'(t) & x'(t - \tau) \end{bmatrix} \begin{bmatrix} A'P + PA & PA_\tau \\ A'_\tau P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}.\end{aligned}$$

Now, given $a > 0$ and $\eta > 1$

$$W(t) := \dot{V}(x(t)) + a(\eta x'(t)Px(t) - x'(t - \tau)Px(t - \tau)) \leq -\zeta x'(t)Px(t)$$

$$\Downarrow$$

$$\dot{V}(x(t)) \leq -\zeta x'(t)Px(t) \quad \text{whenever } x'(t + \theta)Px(t + \theta) \leq \eta x'(t)Px(t)$$

for all $\theta \in [-\tau, 0]$. Thus, if $\exists P = P' > 0$ and $a, \epsilon > 0$ such that

$$\begin{bmatrix} A'P + PA + (a\eta + \zeta)P & PA_\tau \\ A'_\tau P & -aP \end{bmatrix} \leq 0 \iff \begin{bmatrix} A'P + PA + aP & PA_\tau \\ A'_\tau P & -aP \end{bmatrix} \leq \begin{bmatrix} -\epsilon P & 0 \\ 0 & 0 \end{bmatrix}$$

($\epsilon := a(\eta - 1) + \zeta > 0$, arbitrary), then the system asymptotically stable $\forall \tau > 0$.

Retarded equations: Razumikhin (contd)

Choose $V = x'Px$, $P > 0$. Then

$$\begin{aligned}\dot{V}(x(t)) &= 2x'(t)P(Ax(t) + A_\tau x(t - \tau)) \\ &= \begin{bmatrix} x'(t) & x'(t - \tau) \end{bmatrix} \begin{bmatrix} A'P + PA & PA_\tau \\ A'_\tau P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - \tau) \end{bmatrix}.\end{aligned}$$

Now, given $a > 0$ and $\eta > 1$

$$W(t) := \dot{V}(x(t)) + a(\eta x'(t)Px(t) - x'(t - \tau)Px(t - \tau)) \leq -\zeta x'(t)Px(t)$$

$$\Downarrow$$

$$\dot{V}(x(t)) \leq -\zeta x'(t)Px(t) \quad \text{whenever } x'(t + \theta)Px(t + \theta) \leq \eta x'(t)Px(t)$$

for all $\theta \in [-\tau, 0]$. Thus, if $\exists P = P' > 0$ and $a, \epsilon > 0$ such that

$$\begin{bmatrix} A'P + PA + (a\eta + \zeta)P & PA_\tau \\ A'_\tau P & -aP \end{bmatrix} \leq 0 \iff \begin{bmatrix} A'P + PA + aP & PA_\tau \\ A'_\tau P & -aP \end{bmatrix} \leq \begin{bmatrix} -\epsilon P & 0 \\ 0 & 0 \end{bmatrix}$$

($\epsilon := a(\eta - 1) + \zeta > 0$, arbitrary), then the system **asymptotically stable** $\forall \tau > 0$.

Retarded equations: Krasovskii

Theorem

If there is a continuous $V(\check{x}) : C^n([-\tau, 0]) \rightarrow \mathbb{R}$ such that $V(0) = 0$,

$$V(\check{x}) \geq \alpha(\|x\|), \quad \text{and} \quad \dot{V}(\check{x}) \leq -\beta(\|x\|)$$

for $\alpha, \beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\lim_{\|x\| \rightarrow \infty} \alpha(\|x\|) = \infty,$$

then the system is Lyapunov stable. If in addition $\beta(\|x\|) > 0, \forall \|x\| > 0$, then the system is stable asymptotically.

Retarded equations: Krasovskii (contd)

Now let

$$V(\check{x}_t) = x'(t)P_0x(t) + \int_{-\tau}^0 x'(t+s)P_\tau x(t+s)ds = x'(t)P_0x(t) + \int_{t-\tau}^t x'(s)P_\tau x(s)ds$$

for some $P_0 = P'_0 > 0$ and $P_\tau = P'_\tau > 0$. By the Leibniz integral rule

$$\begin{aligned}\dot{V}(\check{x}_t) &= 2x'(t)P_0\dot{x}(t) + x'(t)P_\tau x(t) - x'(t-\tau)P_\tau x(t-\tau) \\ &= 2x'(t)P_0Ax(t) + 2x'(t)P_0A_\tau x(t-\tau) + x'(t)P_\tau x(t) - x'(t-\tau)P_\tau x(t-\tau) \\ &= \begin{bmatrix} x'(t) & x'(t-\tau) \end{bmatrix} \begin{bmatrix} A'P_0 + P_0A + P_\tau & P_0A_\tau \\ A'_\tau P_0 & -P_\tau \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}\end{aligned}$$

Hence, if $\exists P_0 = P'_0 > 0$ and $P_\tau = P'_\tau > 0$ such that

$$\begin{bmatrix} A'P_0 + P_0A + P_\tau & P_0A_\tau \\ A'_\tau P_0 & -P_\tau \end{bmatrix} < 0,$$

then the system **asymptotically stable** $\forall \tau > 0$.

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe–Itō)

Finite spectrum assignment (Manitius–Olbrot)

Reduction (Kwon–Pearson, Artstein)

Controller structures

Problem

FOPTD plant

$$P_{\tau}(s) = \frac{e^{-\tau s}}{s - 1}$$

(if $P(s) = e^{-\tau s}/(Ts - 1)$ for $T > 0$, then $s \rightarrow s/T \implies \tau \rightarrow \tau/T$).

PI controller:

$$R(s) = k_p \left(1 + \frac{1}{T_i s} \right)$$

PD controller:

$$R(s) = k_p(1 + T_d s)$$

Goal:

- conditions on controller parameters for which $\chi_{\tau}(s)$ is Hurwitz.

PI: stability conditions

Characteristic function: $\chi_\tau(s) = T_i s(s-1) + k_p(T_i s + 1)e^{-\tau s}$

Crossing (crossover) frequency:

$$\omega_c^2 = \frac{k_p^2 - 1}{2} + \frac{1}{T_i} \sqrt{k_p^2 + \frac{T_i^2}{2}(k_p^2 - 1)^2}$$

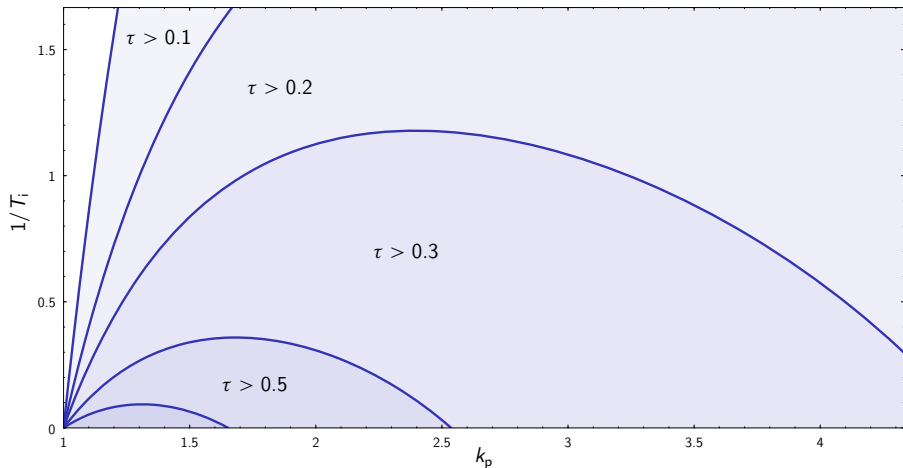
switch, hence, need delay-free stability.

Delay-free stability:

$$k_p > 1 \quad \text{and} \quad T_i > 0 \quad \implies \quad T_i \omega_c^2 > k_p > 1.$$

Stabilizing delays range:

$$0 \leq \tau < \frac{1}{\omega_c} \arctan \frac{T_i \omega_c^2 - 1}{(T_i + 1) \omega_c} < 1$$

PI: stability map in $(k_p, 1/T_i)$ -plane

PI: ω_c as alternative design parameter (to k_p)

$k_p \leftrightarrow \omega_c$ is one-to-one (bijective):

$$\omega_c^2 = \frac{1}{2} \left(k_p^2 - 1 + \sqrt{(k_p^2 - 1)^2 + \frac{4k_p^2}{T_i^2}} \right) \iff k_p = T_i \omega_c \sqrt{\frac{1 + \omega_c^2}{1 + T_i^2 \omega_c^2}}$$

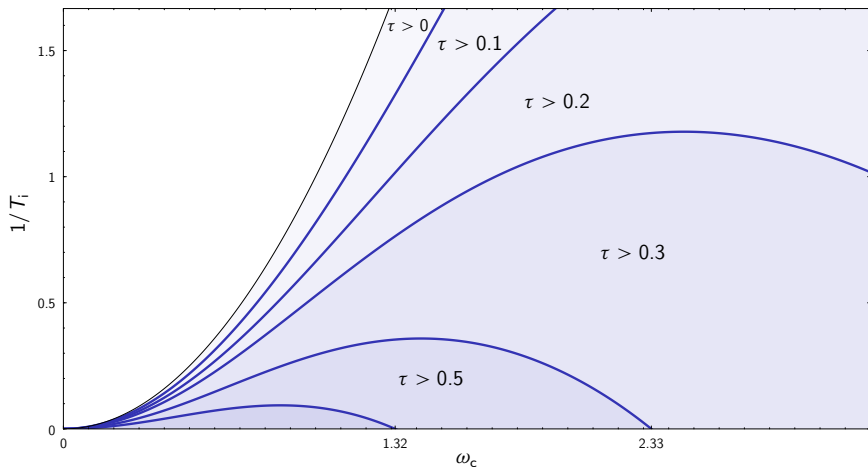
and $k_p > 1 \iff \omega_c > 1/\sqrt{T_i} > 0$, i.e. is always the case.

Given $\tau \in (0, 1)$, the closed-loop system is stable iff

$$\frac{1}{T_i} < \frac{\omega_c(\omega_c - \tan(\tau\omega_c))}{1 + \omega_c \tan(\tau\omega_c)},$$

and $T_i > 0$ exists iff $\omega_c < \omega_\tau$, where $\omega_\tau = \tan(\tau\omega_\tau)$ with $\tau\omega_\tau < \pi/2$.

PI: stability map in $(\omega_c, 1/T_i)$ -plane



Thus,

- larger delay tolerance imposes more severe constraints on ω_c .

PD: stability conditions

Characteristic function: $\chi_\tau(s) = s - 1 + k_p(T_d s + 1)e^{-\tau s}$ with $|k_p T_d| < 1$

Crossing (crossover) frequency (exists iff $k_p > 1$):

$$\omega_c^2 = \frac{k_p^2 - 1}{1 - k_p^2 T_d^2}$$

switch, hence, need delay-free stability.

Delay-free stability:

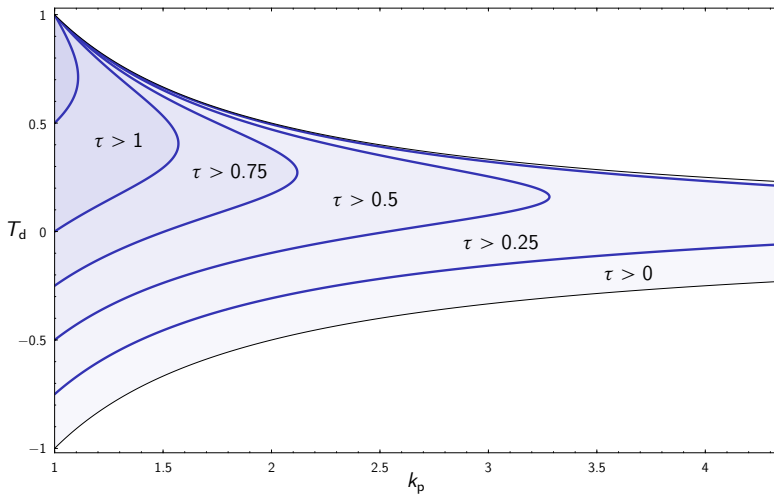
$$k_p > 1$$

Stabilizing delays range:

$$0 \leq \tau < \frac{1}{\omega_c} (\arctan(T_d \omega_c) + \arctan \omega_c) \leq 2$$

provided $|T_d| < 1/k_p < 1$.

PD: stability map in (k_p, T_d) -plane



PD: ω_c as alternative design parameter (to k_p)

$k_p \leftrightarrow \omega_c$ is one-to-one (bijective):

$$\omega_c^2 = \frac{k_p^2 - 1}{1 - k_p^2 T_d^2} \iff k_p = \sqrt{\frac{1 + \omega_c^2}{1 + T_d^2 \omega_c^2}}$$

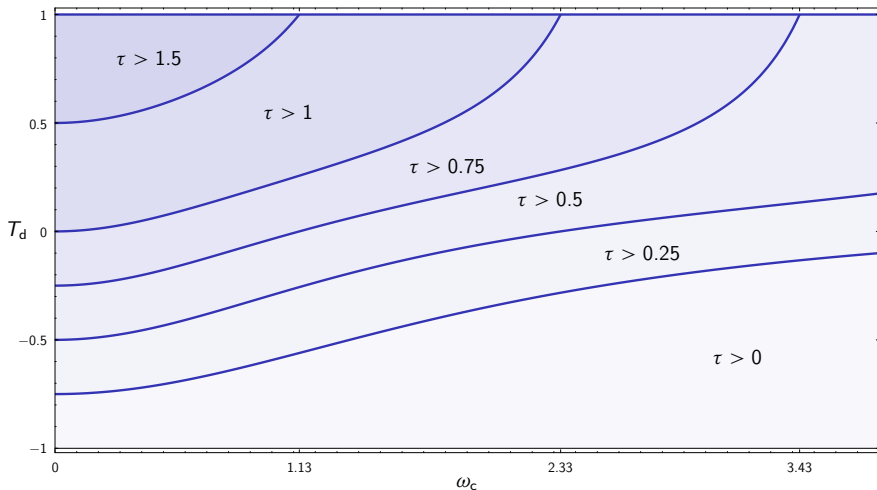
for all $k_p > 1$ and $\omega_c > 0$ (provided $|T_d| < 1$).

Given $\tau \in (0, 2)$, the closed-loop system is stable iff

$$-1 < \frac{\tan(\tau \omega_c - \arctan \omega_c)}{\omega_c} < T_d < 1,$$

admissible iff $0 < \tau \omega_c < \arctan \omega_c + \pi/2 < \pi$.

PD: stability map in (ω_c, T_d) -plane



Thus,

- larger delay tolerance imposes more severe constraints on ω_c .

Outline

Lyapunov's second method

PI and PD stabilization

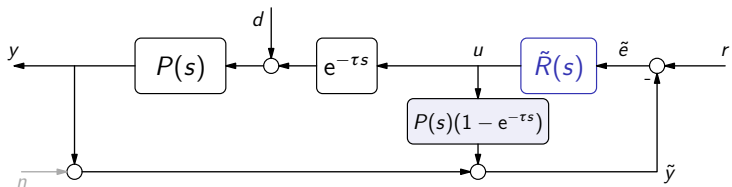
Smith predictor and modifications (Otto J. M. Smith, Watanabe–Itō)

Finite spectrum assignment (Manitius–Olbrot)

Reduction (Kwon–Pearson, Artstein)

Controller structures

Smith controller



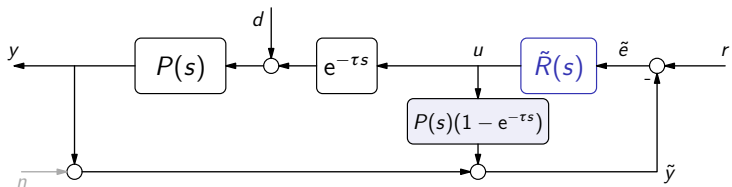
Comprises

- primary controller $\tilde{R}$
- Smith predictor $P(1 - \bar{D}_\tau)$

The overall controller $R : y \mapsto u$ has

$$R(s) = \frac{\tilde{R}(s)}{1 + \tilde{R}(s)P(s)(1 - e^{-\tau s})}.$$

Smith controller: rationale

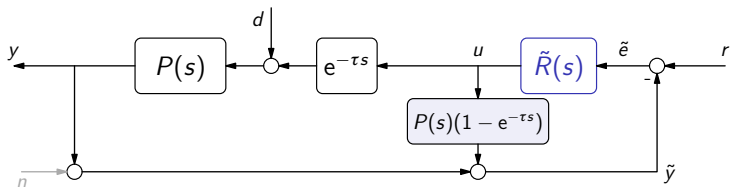


Signal

$$\tilde{y} = y + P(1 - \bar{D}_\tau)u = P(\bar{D}_\tau u + d) + P(1 - \bar{D}_\tau)u = P(u + d)$$

may be seen as a **predicted** version of y .

Smith controller: the magic



Closed-loop transfer functions:

- control sensitivity

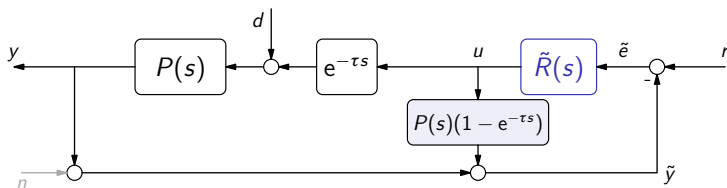
$$T_c(s) = \frac{\tilde{R}(s)}{1 + P(s)\tilde{R}(s)}$$

- complementary sensitivity

$$T(s) = \frac{P(s)\tilde{R}(s)}{1 + P(s)\tilde{R}(s)} e^{-\tau s}$$

have **polynomial** denominators.

Smith controller: the catch



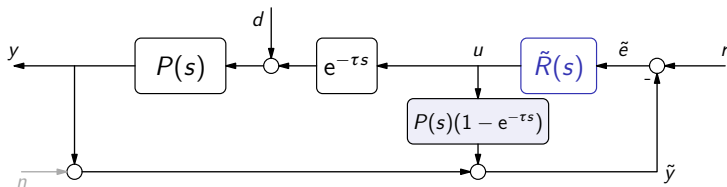
The disturbance sensitivity transfer function:

$$T_d(s) = \frac{P(s)}{1 + P(s)\tilde{R}(s)} + \frac{P(s)\tilde{R}(s)}{1 + P(s)\tilde{R}(s)} P(s)(1 - e^{-\tau s})$$

is unstable, unless $P(s)(1 - e^{-\tau s})$ is stable. The reason: unstable pole-zero cancellations, cf.

$$R(s) = \frac{\tilde{R}(s)}{1 + \tilde{R}(s)P(s)(1 - e^{-\tau s})} = \frac{M_P(s)N_{\tilde{R}}(s)}{M_P(s)M_{\tilde{R}}(s) + N_P(s)N_{\tilde{R}}(s)(1 - e^{-\tau s})}$$

Smith controller: the catch



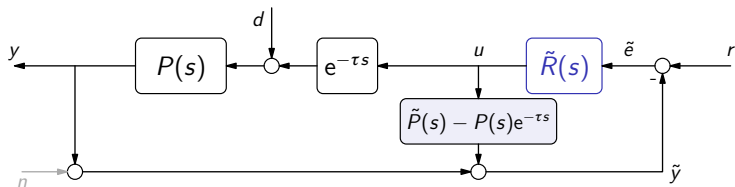
The disturbance sensitivity transfer function:

$$T_d(s) = \frac{P(s)}{1 + P(s)\tilde{R}(s)} + \frac{P(s)\tilde{R}(s)}{1 + P(s)\tilde{R}(s)} P(s)(1 - e^{-\tau s})$$

is unstable, unless $P(s)(1 - e^{-\tau s})$ is stable. The reason: unstable pole-zero cancellations, cf.

$$R(s) = \frac{\tilde{R}(s)}{1 + \tilde{R}(s)P(s)(1 - e^{-\tau s})} = \frac{M_P(s)N_{\tilde{R}}(s)}{M_P(s)M_{\tilde{R}}(s) + N_P(s)N_{\tilde{R}}(s)(1 - e^{-\tau s})}.$$

Modified Smith predictor of Watanabe & Ito



for a finite-dimensional $\tilde{P}$. Now,

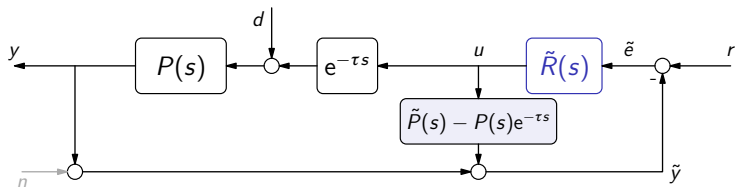
$$\tilde{y} = \tilde{P}u + Pd$$

no longer predicts y . Still, it is delay free, so the term

is called **dead-time compensator (DTC)**

for this class of controllers.

Modified Smith predictor of Watanabe & Ito



for a finite-dimensional $\tilde{P}$. Now,

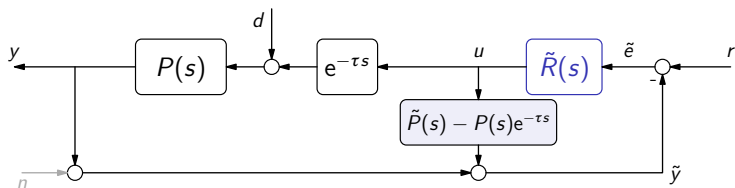
$$\tilde{y} = \tilde{P}u + Pd$$

no longer predicts y . Still, it is delay free, so the term

— **dead-time compensator** (DTC)

for this class of controllers.

Modified Smith predictor: closed-loop transfer functions



If $\Pi(s) := \tilde{P}(s) - P(s)e^{-\tau s}$, then

$$\begin{bmatrix} S(s) & T_c(s) \\ T_d(s) & T(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\Pi(s) & 1 \end{bmatrix} \begin{bmatrix} \tilde{S}(s) & \tilde{T}_c(s) \\ \tilde{T}_d(s) & \tilde{T}(s) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Pi(s) & 1 \end{bmatrix},$$

where

$$\begin{bmatrix} \tilde{S}(s) & \tilde{T}_c(s) \\ \tilde{T}_d(s) & \tilde{T}(s) \end{bmatrix} := \frac{1}{1 + \tilde{P}(s)\tilde{R}(s)} \begin{bmatrix} 1 \\ \tilde{P}(s) \end{bmatrix} \begin{bmatrix} 1 & \tilde{R}(s) \end{bmatrix}.$$

Modified Smith predictor: choice of $\tilde{P}$

Assuming $P(s) = C(sI - A)^{-1}B$, Watanabe & Ito suggested

$$\tilde{P}(s) = Ce^{-A\tau}(sI - A)^{-1}B$$

for which

$$\begin{aligned}\Pi(s) &= \tilde{P}(s) - P(s)e^{-\tau s} = C(e^{-A\tau} - e^{-\tau s}I)(sI - A)^{-1}B \\ &= Ce^{-A\tau} \int_0^\tau e^{-(sI - A)t} dt B\end{aligned}$$

is an entire function, bounded in $\mathbb{C}_0 \implies$ stable.

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe–Itō)

Finite spectrum assignment (Manitius–Olbrot)

Reduction (Kwon–Pearson, Artstein)

Controller structures

Idea

Let

$$\dot{x}(t) = Ax(t) + Bu(t - \tau),$$

with measurable x . Ideally,

$$u(t) = Kx(t + \tau) + K_v v(t)$$

and K such that $A + BK$ is Hurwitz. Because we do not measure $x(t + \tau)$, try to use its predicted value

$$x(t + \tau) = e^{A\tau}x(t) + \int_{t-\tau}^t e^{A(t-r)}Bu(r)dr,$$

which is causally implementable.

Idea

Let

$$\dot{x}(t) = Ax(t) + Bu(t - \tau),$$

with measurable x . Ideally,

$$u(t) = Kx(t + \tau) + K_v v(t)$$

and K such that $A + BK$ is Hurwitz. Because we do not measure $x(t + \tau)$, try to use its predicted value

$$x(t + \tau) = e^{A\tau}x(t) + \int_{t-\tau}^t e^{A(t-r)}Bu(r)dr,$$

which is **causally implementable**.

Predictor: impulse response and transfer function

The relation

$$u(t) \mapsto \int_{t-\tau}^t e^{A(t-r)} B u(r) dr = \int_{-\infty}^{\infty} e^{A(t-r)} B \mathbb{1}_{[0,\tau]}(t-r) u(r) dr$$

is the convolution of $u(t)$ and $e^{At} B \mathbb{1}_{[0,\tau]}(t)$. Hence, it's LTI and its transfer function is

$$\int_0^{\infty} e^{At} B \mathbb{1}_{[0,\tau]}(t) e^{-st} dt = \int_0^{\tau} e^{-(sI-A)t} dt B = (sI - A)^{-1} (I - e^{-(sI-A)\tau}) B$$

which is an entire function, bounded on $\mathbb{C}_0$, hence, belongs to H_{∞} .

Characteristic function

With $v = 0$

$$U(s) = Ke^{A\tau}X(s) + K \int_0^\tau e^{-(sI-A)t} dt BU(s)$$

and the closed-loop relation in the s -domain:

$$\begin{bmatrix} sI - A & -Be^{-\tau s} \\ -Ke^{A\tau} & I - K \int_0^\tau e^{-(sI-A)t} dt B \end{bmatrix} \begin{bmatrix} X(s) \\ U(s) \end{bmatrix} = 0.$$

Its characteristic function, corresponding to the nontrivial free response, is

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -Be^{-\tau s} \\ -Ke^{A\tau} & I - K \int_0^\tau e^{-(sI-A)t} dt B \end{bmatrix}.$$

Hmm, looks complicated...

Characteristic function (contd)

Because

$$(sI - A) \int_0^\tau e^{-(sI - A)t} dt B = (I - e^{-(sI - A)\tau}) B = e^{A\tau} (e^{-A\tau} B - e^{-\tau s} B),$$

we have

$$\begin{aligned} \begin{bmatrix} sI - A & -Be^{-\tau s} \\ -Ke^{A\tau} & I - K \int_0^\tau e^{-(sI - A)t} dt B \end{bmatrix} \\ = \begin{bmatrix} sI - A & -e^{-A\tau} B \\ -Ke^{A\tau} & I \end{bmatrix} \begin{bmatrix} I & e^{-A\tau} \int_0^\tau e^{-(sI - A)t} dt B \\ 0 & I \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \chi_\tau(s) &= \det \begin{bmatrix} sI - A & -e^{-A\tau} B \\ -Ke^{A\tau} & I \end{bmatrix} = \det(sI - A - e^{-A\tau} B K e^{A\tau}) \\ &= \det(sI - (A + BK)) \end{aligned}$$

is a plain n -degree polynomial.

Characteristic function (contd)

Because

$$(sI - A) \int_0^\tau e^{-(sI-A)t} dt B = (I - e^{-(sI-A)\tau}) B = e^{A\tau} (e^{-A\tau} B - e^{-\tau s} B),$$

we have

$$\begin{aligned} & \begin{bmatrix} sI - A & -Be^{-\tau s} \\ -Ke^{A\tau} & I - K \int_0^\tau e^{-(sI-A)t} dt B \end{bmatrix} \\ &= \begin{bmatrix} sI - A & -e^{-A\tau} B \\ -Ke^{A\tau} & I \end{bmatrix} \begin{bmatrix} I & e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B \\ 0 & I \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \chi_\tau(s) &= \det \begin{bmatrix} sI - A & -e^{-A\tau} B \\ -Ke^{A\tau} & I \end{bmatrix} = \det(sI - A - e^{-A\tau} B K e^{A\tau}) \\ &= \det(sI - (A + BK)) \end{aligned}$$

is a plain n -degree polynomial.

Characteristic function (contd)

Because

$$(sI - A) \int_0^\tau e^{-(sI-A)t} dt B = (I - e^{-(sI-A)\tau}) B = e^{A\tau} (e^{-A\tau} B - e^{-\tau s} B),$$

we have

$$\begin{aligned} \begin{bmatrix} sI - A & -Be^{-\tau s} \\ -Ke^{A\tau} & I - K \int_0^\tau e^{-(sI-A)t} dt B \end{bmatrix} \\ = \begin{bmatrix} sI - A & -e^{-A\tau} B \\ -Ke^{A\tau} & I \end{bmatrix} \begin{bmatrix} I & e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B \\ 0 & I \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \chi_\tau(s) &= \det \begin{bmatrix} sI - A & -e^{-A\tau} B \\ -Ke^{A\tau} & I \end{bmatrix} = \det(sI - A - e^{-A\tau} B K e^{A\tau}) \\ &= \det(sI - (A + BK)) \end{aligned}$$

is a plain n -degree polynomial.

Prediction logic is not always obvious

Let

$$\dot{x}(t) = Ax(t) + \sum_i B_i u(t - \tau_i).$$

The control law

$$u(t) = K \left(x(t) + \sum_i \int_{t-\tau_i}^t e^{A(t-\tau_i-r)} B_i u(r) dr \right) + K_v v(t)$$

renders

$$\chi_\tau(s) = \det \left(sI - A - \sum_i e^{-A\tau_i} B_i K \right)$$

finite polynomial again. But this is not an obvious predictor...

Observer-predictor

If $y(t) = Cx(t)$ is measurable, then

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t - \tau) - L(y(t) - C\hat{x}(t)) \\ u(t) &= K \left(e^{A\tau} \hat{x}(t) + \int_{t-\tau}^t e^{A(t-r)} Bu(r) dr \right) + K_v v(t),\end{aligned}$$

for some L such that $A + LC$ is Hurwitz. In this case

$$\begin{bmatrix} sI - A - LC & 0 & 0 \\ 0 & sI - A & -Be^{-s\tau} \\ Ke^{A\tau} & -Ke^{A\tau} & I - K \int_0^\tau e^{-(sI-A)t} dt B \end{bmatrix} \begin{bmatrix} E(s) \\ X(s) \\ U(s) \end{bmatrix} = 0$$

and its characteristic function is again a polynomial,

$$\chi_\tau(s) = \det(sI - A - LC) \det(sI - A - BK),$$

just now having the degree $2n$.

Observer-predictor

If $y(t) = Cx(t)$ is measurable, then

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t - \tau) - L(y(t) - C\hat{x}(t)) \\ u(t) &= K \left(e^{A\tau} \hat{x}(t) + \int_{t-\tau}^t e^{A(t-r)} Bu(r) dr \right) + K_v v(t),\end{aligned}$$

for some L such that $A + LC$ is Hurwitz. In this case

$$\begin{bmatrix} sI - A - LC & 0 & 0 \\ 0 & sI - A & -Be^{-\tau s} \\ Ke^{A\tau} & -Ke^{A\tau} & I - K \int_0^\tau e^{-(sI-A)t} dt B \end{bmatrix} \begin{bmatrix} E(s) \\ X(s) \\ U(s) \end{bmatrix} = 0$$

and its characteristic function is again a polynomial,

$$\chi_\tau(s) = \det(sI - A - LC) \det(sI - A - BK),$$

just now having the degree $2n$.

Outline

Lyapunov's second method

PI and PD stabilization

Smith predictor and modifications (Otto J. M. Smith, Watanabe–Itō)

Finite spectrum assignment (Manitius–Olbrot)

Reduction (Kwon–Pearson, Artstein)

Controller structures

Idea

Let again

$$\dot{x}(t) = Ax(t) + Bu(t - \tau),$$

with measurable x . Define

$$\tilde{x}(t) := x(t) + \int_{t-\tau}^t e^{A(t-\tau-r)} Bu(s) dr$$

(prediction of $e^{-A\tau}x(t + \tau)$). With a little help of the Leibniz integral rule,

$$\begin{aligned}\dot{\tilde{x}}(t) &= \dot{x}(t) + A \int_{t-\tau}^t e^{A(t-\tau-r)} Bu(r) dr + e^{-A\tau} Bu(t) - Bu(t - \tau) \\ &= Ax(t) + A \int_{t-\tau}^t e^{A(t-\tau-r)} Bu(r) dr + e^{-A\tau} Bu(t) \\ &= A\tilde{x}(t) + e^{-A\tau} Bu(t),\end{aligned}$$

which is finite dimensional and can be stabilized by $u(t) = \tilde{K}\tilde{x}(t)$.

Idea

Let again

$$\dot{x}(t) = Ax(t) + Bu(t - \tau),$$

with measurable x . Define

$$\tilde{x}(t) := x(t) + \int_{t-\tau}^t e^{A(t-\tau-r)} Bu(s) dr$$

(prediction of $e^{-A\tau}x(t + \tau)$). With a little help of the Leibniz integral rule,

$$\begin{aligned} \dot{\tilde{x}}(t) &= \dot{x}(t) + A \int_{t-\tau}^t e^{A(t-\tau-r)} Bu(r) dr + e^{-A\tau} Bu(t) - Bu(t - \tau) \\ &= Ax(t) + A \int_{t-\tau}^t e^{A(t-\tau-r)} Bu(r) dr + e^{-A\tau} Bu(t) \\ &= A\tilde{x}(t) + e^{-A\tau} Bu(t), \end{aligned}$$

which is finite dimensional and can be stabilized by $u(t) = \tilde{K}\tilde{x}(t)$.

Time-varying delays

Time-domain analysis $\implies$ not limited to time-invariant systems. Let

$$\dot{x}(t) = Ax(t) + Bu(\theta(t))$$

for absolutely continuous $\theta(t) \leq t$ with $\dot{\theta}(t) > 0$ (i.e. delay $\tau(t) = t - \theta(t)$). Let $\theta^{-1}(t)$ denote the inverse of $\theta(t)$, i.e. r for which $\theta(r) = t$. Then

$$\tilde{x}(t) = x(t) + \int_{\theta(t)}^t e^{A(t-\theta^{-1}(r))} B \theta^{-1}(r) u(r) dr$$

satisfies the delay free

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + e^{-A(\theta^{-1}(t)-t)} B \theta^{-1}(t) u(t).$$

Time-varying delays

Time-domain analysis $\implies$ not limited to time-invariant systems. Let

$$\dot{x}(t) = Ax(t) + Bu(\theta(t))$$

for absolutely continuous $\theta(t) \leq t$ with $\dot{\theta}(t) > 0$ (i.e. delay $\tau(t) = t - \theta(t)$). Let $\theta^{-1}(t)$ denote the inverse of $\theta(t)$, i.e. r for which $\theta(r) = t$. Then

$$\tilde{x}(t) = x(t) + \int_{\theta(t)}^t e^{A(t-\theta^{-1}(r))} B \dot{\theta}^{-1}(r) u(r) dr$$

satisfies the delay free

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + e^{-A(\theta^{-1}(t)-t)} B \dot{\theta}^{-1}(t) u(t).$$

Outline

Lyapunov's second method

PI and PD stabilization

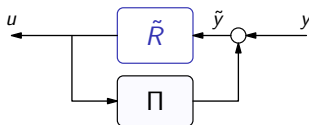
Smith predictor and modifications (Otto J. M. Smith, Watanabe–Itō)

Finite spectrum assignment (Manitius–Olbrot)

Reduction (Kwon–Pearson, Artstein)

Controller structures

Dead-time compensators



Smith controller has

$$\Pi(s) = -P(s)(1 - e^{-\tau s})$$

and the primary controller as $\tilde{R}$.

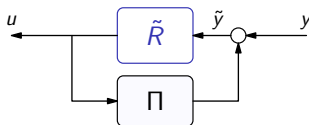
Modified Smith predictor has

$$\Pi(s) = -\tilde{P}(s) + P(s)e^{-\tau s} = -Ce^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B$$

and the primary controller as $\tilde{R}$. In this case

$$\tilde{y}(t) = y(t) - Ce^{-A\tau} \int_{t-\tau}^t e^{A(t-r)} Bu(r) dr.$$

Dead-time compensators



Smith controller has

$$\Pi(s) = -P(s)(1 - e^{-\tau s})$$

and the primary controller as $\tilde{R}$.

Modified Smith predictor has

$$\Pi(s) = -\tilde{P}(s) + P(s)e^{-\tau s} = -Ce^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B$$

and the primary controller as $\tilde{R}$. In this case

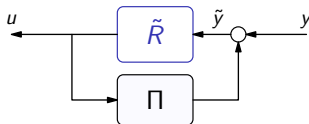
$$\tilde{y}(t) = y(t) - Ce^{-A\tau} \int_{t-\tau}^t e^{A(t-r)} Bu(r) dr.$$

FSA

Control law

$$u(t) = \tilde{K} \left(e^{A\tau} x(t) + \int_{t-\tau}^t e^{A(t-r)} B u(r) dr \right)$$

is also of the form



with

$$\tilde{R} = K e^{A\tau} \quad \text{and} \quad \Pi(s) = e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B$$

for which

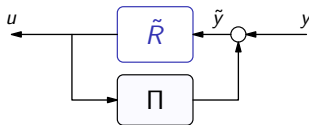
$$\tilde{y}(t) = x(t) + e^{-A\tau} \int_{t-\tau}^t e^{A(t-r)} B u(r) dr.$$

Reduction

Control law

$$u(t) = \tilde{K} \left(x(t) + \int_{t-\tau}^t e^{A(t-\tau-r)} B u(r) dr \right)$$

is also of the form



with

$$\tilde{R} = \tilde{K} \quad \text{and} \quad \Pi(s) = e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B$$

for which

$$\tilde{y}(t) = x(t) + e^{-A\tau} \int_{t-\tau}^t e^{A(t-r)} B u(r) dr,$$

again.

Observer-predictor

Let

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t - \tau) - L(y(t) - C\hat{x}(t)) \\ u(t) &= K \left(e^{A\tau} \hat{x}(t) + \int_{t-\tau}^t e^{A(t-r)} Bu(r) dr \right).\end{aligned}$$

Denote

$$\bar{x}(t) := e^{A\tau} \hat{x}(t) + \int_{t-\tau}^t e^{A(t-r)} Bu(r) dr,$$

with which $u(t) = K\bar{x}(t)$. By Leibnitz integral rule

$$\begin{aligned}\dot{\bar{x}}(t) &= e^{A\tau} (A\hat{x}(t) + Bu(t - \tau) - L(y(t) - C\hat{x}(t))) \\ &\quad + A \int_{t-\tau}^t e^{A(t-r)} Bu(r) dr + Bu(t) - e^{A\tau} Bu(t - \tau) \\ &= A\bar{x}(t) + Bu(t) - e^{A\tau} L(y(t) - C\hat{x}(t)).\end{aligned}$$

Observer-predictor (contd)

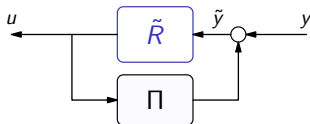
Thus, observer-predictor reads

$$\begin{aligned}\dot{\bar{x}}(t) &= (A + e^{A\tau} L C e^{-A\tau}) \bar{x}(t) + B u(t) - e^{A\tau} L \tilde{y}(t) \\ u(t) &= K \bar{x}(t).\end{aligned}$$

where

$$\tilde{y}(t) = y(t) + C e^{-A\tau} \int_{t-\tau}^t e^{A(t-r)} B u(r) dr$$

This is



with

$$\tilde{R}(s) = \left[\begin{array}{c|c} A + BK + e^{A\tau} L C e^{-A\tau} & -e^{A\tau} L \\ \hline K & 0 \end{array} \right], \quad \Pi(s) = e^{-A\tau} \int_0^\tau e^{-(sI-A)t} dt B.$$

Observer-predictor vs. MSP

Controller

$$\tilde{R}(s) = \left[\frac{A + BK + e^{A\tau} L C e^{-A\tau}}{K} \mid \frac{-e^{A\tau} L}{0} \right]$$

is the observer based-controller for

$$\tilde{P}(s) = \left[\frac{A}{C e^{-A\tau}} \mid \frac{B}{0} \right]$$

with the gain $\tilde{L} = e^{A\tau} L$, for which

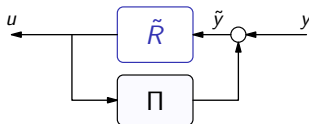
$$A + \tilde{L} C e^{-A\tau} = A + e^{A\tau} L C e^{-A\tau} = e^{A\tau} (A + L C) e^{-A\tau}.$$

Thus,

- MSP with observer-based $\tilde{R} \iff$ observer-predictor.

Summarizing

Common structure:



where

- $\tilde{R}$ is finite-dimensional, designed for a finite-dimensional plant
- Π is infinite-dimensional, fixed structure

Common theme seems to be

— prediction as a rationale for Π ,

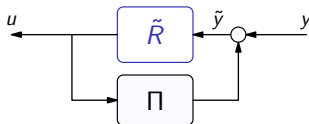
but prediction is a suspicious operation

— prediction is difficult, especially about the future (cf. [20])

Can it be justified in control-theoretic terms?

Summarizing

Common structure:



where

- $\tilde{R}$ is finite-dimensional, designed for a finite-dimensional plant
- Π is infinite-dimensional, fixed structure

Common theme seems to be

- prediction as a rationale for Π ,

but prediction is a suspicious operation

- “prediction is difficult, especially about the future” (N. Bohr)...

Can it be justified in control-theoretic terms?