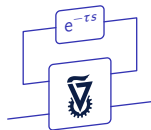


Time Delays in Control and Estimation (038806)



lecture no. 4

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Previously on TDCE...

Stability of

$$P_\tau(s) = D_{yu} + \begin{bmatrix} C_y & D_{yw}e^{-\tau s} \end{bmatrix} \begin{bmatrix} sI_n - A & -B_w e^{-\tau s} \\ -C_z & I_{m_\tau} - D_{zw}e^{-\tau s} \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix}$$

can be linked with roots of the characteristic function

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw}e^{-\tau s} \end{bmatrix} \quad \text{and}$$

in $\bar{\mathbb{C}}_0$ whenever $\rho(D_{zw}) \neq 1$.

Outline

Direct method

Rekašius transformation

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Rekašius transformation

Continuity

If

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix} \quad \text{and} \quad \rho(D_{zw}) < 1$$

then

$$\sup_{\chi_\tau(s)=0} \operatorname{Re} s$$

is a continuous function of τ for all $\tau \geq 0$.

Hence, as $\tau > 0$ changes

– roots migrate LHP $\rightarrow$ RHP or RHP $\rightarrow$ LHP only through $j\mathbb{R}$

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Idea

Two steps:

1. analyze roots of $\chi_0(s)$ in $\mathbb{C}_0$
2. increase τ and count imaginary crossings by roots of $\chi_\tau(s)$
 - if roots cross from left to right $\longrightarrow$ switch
 - if roots cross from right to left $\longrightarrow$ reversal

Idea and terminology

Two steps:

1. analyze roots of $\chi_0(s)$ in $\mathbb{C}_0$
2. increase τ and count imaginary crossings by roots of $\chi_\tau(s)$
 - if roots cross from left to right $\longrightarrow$ **switch**
 - if roots cross from right to left $\longrightarrow$ **reversal**

Assumptions

Consider

$$\chi_\tau(s) = Q_0(s) + Q_1(s)e^{-\tau s}$$

such that

$$\mathcal{A}_1: \deg Q_0(s) \geq \deg Q_1(s),$$

$$\mathcal{A}_2: \left| \lim_{s \rightarrow \infty} \frac{Q_1(s)}{Q_0(s)} \right| < 1,$$

$$\mathcal{A}_3: Q_0(s) \text{ and } Q_1(s) \text{ have no common roots in } \bar{\mathbb{C}}_0,$$

$$\mathcal{A}_4: Q_0(0) + Q_1(0) \neq 0.$$

If

$$Q_0(s) = q_{0n}s^n + \dots + q_{01}s + q_{00} \text{ and } Q_1(s) = q_{1n}s^n + \dots + q_{11}s + q_{10}$$

then $\mathcal{A}_2$ says that $|q_{0n}| > |q_{1n}|$ and $\mathcal{A}_4$ says that $q_{00} \neq -q_{10}$.

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$$\chi_\tau(s) = Q_0(s) + Q_1(s)e^{-\tau s}$$

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$$\mathcal{A}_5: \deg Q_0(s) \geq \deg Q_1(s),$$

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Pure imaginary roots

Satisfy

$$Q_0(j\omega) + Q_1(j\omega)e^{-j\tau\omega} = 0 \quad \Longleftrightarrow \quad -e^{j\tau\omega} = \frac{Q_1(j\omega)}{Q_0(j\omega)}.$$

Magnitude relation:

$$\left| \frac{Q_1(j\omega)}{Q_0(j\omega)} \right| = 1 \quad \Longleftrightarrow \quad \phi(\omega) := Q_0(j\omega)Q_0(-j\omega) - Q_1(j\omega)Q_1(-j\omega) = 0$$

independent of τ , finite number of roots, say $\omega_i > 0$.

Phase relation:

$$\tau\omega = \arg Q_1(j\omega) - \arg Q_0(j\omega) + (2k - 1)\pi \quad \text{for some } k \in \mathbb{Z}$$

always solvable in τ .

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Crossing directions

Lemma

Let $\omega_i > 0$ be a solution of $\phi(\omega_i) = 0$. Each pure imaginary point $j\omega_i$ is

1. a switch iff sign $\phi(\omega_i)$ changes from minus to plus as ω increases,
2. a reversal iff sign $\phi(\omega)$ changes from plus to minus as ω increases,
3. a tangential point (no roots cross the imaginary axis) otherwise.

In particular,

- $\left. \frac{d\phi(\omega)}{d\omega} \right|_{\omega=\omega_i} > 0 \implies \text{switch}$
- $\left. \frac{d\phi(\omega)}{d\omega} \right|_{\omega=\omega_i} < 0 \implies \text{reversal}$

Crossing directions: proof

Directions from $\sigma(\omega) = \text{sign Re } \frac{ds}{d\tau} \Big|_{s=j\omega}$. Along any root of $\chi_\tau(s)$,

$$0 = \frac{d\chi_\tau(s)}{d\tau} = \frac{dQ_0(s)}{ds} \frac{ds}{d\tau} + \frac{dQ_1(s)}{ds} \frac{ds}{d\tau} e^{-\tau s} - Q_1(s) e^{-\tau s} \left(\tau \frac{ds}{d\tau} + s \right).$$

Hence, using $Q_1(s) e^{-\tau s} = -Q_0(s)$

$$\frac{ds}{d\tau} = \frac{sQ_1(s)e^{-\tau s}}{Q'_0(s) + Q'_1(s)e^{-\tau s} - \tau Q_1(s)e^{-\tau s}} = -s \left(\frac{Q'_0(s)}{Q_0(s)} - \frac{Q'_1(s)}{Q_1(s)} + \tau \right)^{-1}$$

Since $\text{sign Re } z^{-1} = \text{sign Re } z$,

$$\begin{aligned} \sigma(\omega) &= \text{sign Re} \left[-\frac{1}{j\omega} \left(\frac{Q'_0(j\omega)}{Q_0(j\omega)} - \frac{Q'_1(j\omega)}{Q_1(j\omega)} + \tau \right) \right] \\ &= \text{sign Re} \left[\frac{j}{\omega} \left(\frac{Q'_0(j\omega)}{Q_0(j\omega)} - \frac{Q'_1(j\omega)}{Q_1(j\omega)} \right) \right] \\ &= \text{sign Re} \left[j \left(\frac{Q'_0(j\omega)}{Q_0(j\omega)} - \frac{Q'_1(j\omega)}{Q_1(j\omega)} \right) \right] \end{aligned}$$

because $\text{Re} \frac{1}{z} = \frac{\text{Re } z}{|z|^2}$

because $\text{Re } jz = \text{Im } z$

which does not depend on τ .

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because $\text{Re } \frac{\tau}{j\omega} = 0$

because $\omega > 0$

which does **not depend on τ** .

Crossing directions: proof (contd)

Multiplying what's under "sign" by $Q_0(j\omega)Q_0(-j\omega) = Q_1(j\omega)Q_1(-j\omega) > 0$,

$$\begin{aligned}\sigma(\omega) &= \text{sign Re} \left[j \left(\frac{dQ_0(j\omega)}{d(j\omega)} Q_0(-j\omega) - \frac{dQ_1(j\omega)}{d(j\omega)} Q_1(-j\omega) \right) \right] \\ &= \text{sign Re} \left[\frac{dQ_0(j\omega)}{d\omega} Q_0(-j\omega) - \frac{dQ_1(j\omega)}{d\omega} Q_1(-j\omega) \right].\end{aligned}$$

Because $\text{sign Re } z = \text{sign}(z + \bar{z})$,

$$\begin{aligned}\sigma(\omega) &= \text{sign} \left(\frac{dQ_0(j\omega)}{d\omega} Q_0(-j\omega) - \frac{dQ_1(j\omega)}{d\omega} Q_1(-j\omega) \right. \\ &\quad \left. + \frac{dQ_0(-j\omega)}{d\omega} Q_0(j\omega) - \frac{dQ_1(-j\omega)}{d\omega} Q_1(j\omega) \right) \\ &= \text{sign} \frac{d\phi(\omega)}{d\omega}.\end{aligned}$$



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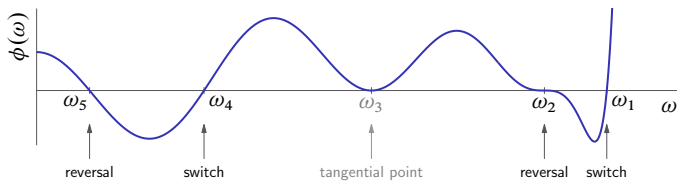
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Qualitative properties of $\phi(\omega)$

$$\phi(\omega) = \overbrace{(q_{0n}^2 - q_{1n}^2)}^{>0, \text{ by } \mathcal{A}_2} \omega^{2n} + \dots$$



- the **largest** crossing frequency is **always a switch**
- switches and reversals alternate

Crossing delays

At each ω_i ,

$$\tau_{ik} = \tau_{i0} + \frac{\pi}{\omega_i} 2k, \quad k \in \mathbb{N}$$

The number v_j of $\mathbb{C}_0$ roots of $\chi_\tau(s)$ as τ passes each τ_j :

$$v_j = \begin{cases} v_{j-1} + 2 & \text{if switch} \\ v_{j-1} - 2 & \text{if reversal} \\ v_{j-1} & \text{if tangential point} \end{cases}$$

and

- $\chi_\tau(s)$ is Hurwitz in each $\tau \in (\tau_j, \tau_{j+1})$ with $v_j = 0$.

Moreover,

- $\omega_1 > \omega_i \implies \frac{2\pi}{\omega_1} = \tau_{1,k+1} - \tau_{1k} < \tau_{i,k+1} - \tau_{ik} = \frac{2\pi}{\omega_i}$ for all $i > 1$

meaning

- $\exists j_* \in \mathbb{Z}_+$ such that $v_j > 0$ for all $j \geq j_*$ and we stop.

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- $\exists j_\star \in \mathbb{Z}_+$ such that $v_j > 0$ for all $j \geq j_\star$ and we stop.

Example 1

Let

$$\chi_\tau(s) = s^2 + 0.1s + 1 + q_{10}e^{-\tau s}, \quad q_{10} > 0.$$

Set $\nu_0 = 0$ because $\chi_0(s) = s^2 + 0.1s + 1$ is Hurwitz.

Now

$$\begin{aligned} \phi(\omega) &= (-\omega^2 + j0.1\omega + 1)(-\omega^2 - j0.1\omega + 1) - q_{10}^2 \\ &= (\omega^2 - 0.995)^2 + 1 - 0.995^2 - q_{10}^2 \end{aligned}$$

and $\dot{\phi}(\omega) = 4\omega(\omega^2 - 0.995) = 0 \iff \omega^2 = 0.995$.

Phase relation:

$$\begin{aligned} \arg Q_1(j\omega) - \arg Q_0(j\omega) &= \arg q_{10} - \arg(1 - \omega^2 + j0.1\omega) \\ &= \begin{cases} \arctan \frac{0.1\omega}{\omega^2 - 1} & \text{if } 0 \leq \omega \leq 1 \\ \arctan \frac{0.1\omega}{\omega^2 - 1} - \pi & \text{if } \omega \geq 1 \end{cases} \end{aligned}$$

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Example 1: if $0 < q_{10} < \sqrt{1 - 0.995^2} \approx 0.0999$

No crossing frequencies and

- $\chi_\tau(s)$ is **Hurwitz for all $\tau \geq 0$** (delay-independent stable)

because $\nu_0 = 0$.

Example 1: if $q_{10} = \sqrt{1 - 0.995^2} \approx 0.0999$

Double positive root $\omega_{1,2}^2 = 0.995$ and

$$\phi(\omega) = (\omega^2 - 0.995)^2 \geq 0$$

does not change sign at $\omega = \omega_{1,2} \implies$ tangential point. Hence,

$$v_j = 0, \quad \text{for all } j \in \mathbb{Z}_+$$

Phase relation (as $\omega_{1,2} < 1$):

$$\tau_{1k} = \frac{1}{\omega_{1,2}} \left(\arctan \frac{0.1\omega_{1,2}}{\omega_{1,2}^2 - 1} + (2k - 1)\pi \right) \approx 1.62495 + 6.31476k$$

Hence

- $\chi_\tau(s)$ is Hurwitz iff $\tau \neq 1.62495 + 6.31476k$ for $k \in \mathbb{Z}_+$.

Example 1: if $\sqrt{1 - 0.995^2} \leq q_{10} < 1$

Two positive real roots of $\phi(\omega)$,

$$\omega_1^2 = 0.995 + \sqrt{0.995^2 - 1 + q_{10}^2} \quad \& \quad \omega_2^2 = 0.995 - \sqrt{0.995^2 - 1 + q_{10}^2}$$

where

- $\omega_1 > 0.995$ is switch
- $0 < \omega_2 < 0.995$ is reversal.

By phase relation,

$$\tau_{1k} = \frac{1}{\omega_1} \left(\arctan \frac{0.1\omega_1}{\omega_1^2 - 1} + \begin{cases} 2k\pi & \text{if } 0.1 \leq q_{10} < 1 \\ (2k - 1)\pi & \text{if } \sqrt{1 - 0.995^2} \leq q_{10} \leq 0.1 \end{cases} \right)$$

and

$$\tau_{2k} = \frac{1}{\omega_2} \left(\arctan \frac{0.1\omega_2}{\omega_2^2 - 1} + (2k - 1)\pi \right).$$

Example 1: if $q_{10} = 0.4$ (satisfies $\sqrt{1 - 0.995^2} \leq q_{10} < 1$)

Now $\omega_1 \approx 1.1757$ and $\omega_2 \approx 0.7795$ and the phase relation yields

$$\tau_{1k} \approx 0.2537 + 5.3441k = \{0.25, 5.6, 10.94, 16.29, 21.63, 26.97, \dots\}$$

for switches and

$$\tau_{2k} \approx 3.7785 + 8.0602k = \{3.78, 11.84, 19.9, 27.96, 36.02, \dots\}$$

for reversals. All crossing delays:

$$\{\tau_j\} = \{0, 0.25, 3.78, 5.6, 10.94, 11.84, 16.29, 19.9, 21.63, 26.97, 27.96, \dots\}$$

so that

$$\{\nu_j\} = \{0, 2, 0, 2, 4, 2, 4, 2, 4, 6, 4, \dots\}$$

and

$$\chi_\tau(s) \text{ is Hurwitz iff } \tau \in [0, 0.2537) \cup (3.7785, 5.5978).$$

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$$\chi_r(s) \text{ is Hurwitz iff } r \in [0, 0.2537) \cup (3.7785, 5.5978).$$

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- $\chi_\tau(s)$ is Hurwitz iff $\tau \in [0, 0.2537) \cup (3.7785, 5.5978)$.

Example 1: if $q_{10} \geq 1$

One positive real root of $\phi(\omega)$,

$$\omega_1^2 = 0.995 + \sqrt{0.995^2 - 1 + q_{10}^2} > 0.995$$

and it is switch. The phase relation yields

$$\tau_{1k} = \frac{1}{\omega_1} \left(\arctan \frac{0.1\omega_1}{\omega_1^2 - 1} + 2(k-1)\pi \right).$$

Hence,

→ $\chi_\tau(s)$ is Hurwitz iff $\tau \in [0, \frac{1}{\omega_1} \arctan \frac{0.1\omega_1}{\omega_1^2 - 1})$.

Example 1: if $q_{10} \geq 1$

One positive real root of $\phi(\omega)$,

$$\omega_1^2 = 0.995 + \sqrt{0.995^2 - 1 + q_{10}^2} > 0.995$$

and it is switch. The phase relation yields

$$\tau_{1k} = \frac{1}{\omega_1} \left(\arctan \frac{0.1\omega_1}{\omega_1^2 - 1} + 2(k-1)\pi \right).$$

Hence,

- $\chi_\tau(s)$ is Hurwitz iff $\tau \in [0, \frac{1}{\omega_1} \arctan \frac{0.1\omega_1}{\omega_1^2 - 1})$.

Necessary condition

Roots can cross the imaginary axis only in pairs. Therefore,

- if $\chi_0(s)$ has an odd number of RHP poles, $\chi_\tau(s)$ is unstable for all τ .

Lemma

If

$$\chi_0(s) = s^n + \chi_{0,n-1}s^{n-1} + \cdots + \chi_{01}s + \chi_{00},$$

then $\chi_\tau(s)$ is not Hurwitz for all $\tau \geq 0$ whenever $\chi_{00} \leq 0$.

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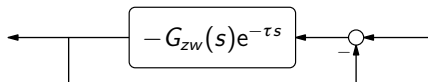
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Nyquist connections



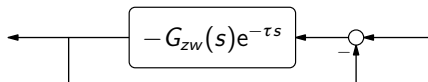
The characteristic quasi-polynomial is $\chi_\tau(s) = -M_{zw}(s) - N_{zw}(s)e^{-\tau s}$.

If $q_{10} = 0.4$, then $\chi_\tau(s)$ is Hurwitz iff $\tau \in [0, 0.2537] \cup (3.7765, 5.5948)$.

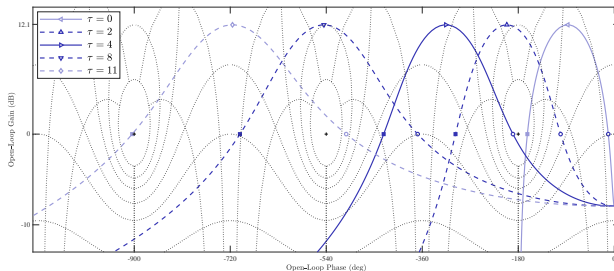
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- Nyquist is more intuitive
- delay sweeping is easier to algorithmize

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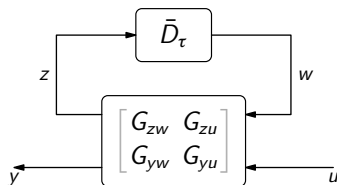


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Example 2

Remember (Lect. 3)



$$\text{with } G(s) = \frac{1}{s} \begin{bmatrix} \alpha e^\alpha & s - \alpha \\ \alpha e^\alpha & -\alpha \end{bmatrix}$$

so that

$$P_\tau(s) = \frac{\alpha(e^{\alpha-s} - 1)}{s - \alpha e^{\alpha-s}} \quad \text{and} \quad \chi_\tau(s) = s - \alpha e^{\alpha-s}.$$

The root of $\chi_\tau(s)$ at $s = \alpha$ is a removable singularity of $P_\tau(s)$,

$$\lim_{s \rightarrow \alpha} P_\tau(s) = \lim_{s \rightarrow \alpha} \frac{\alpha(e^{\alpha-s} - 1)}{s - \alpha e^{\alpha-s}} = -\frac{\alpha}{1 + \alpha}$$

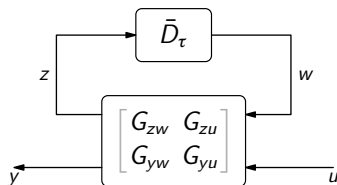
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Example 2 (contd)

Rewrite

$$P_\tau(s) = \frac{\alpha(e^{\alpha-s} - 1)}{s - \alpha e^{\alpha-s}} = -\frac{\alpha(1 - e^{\alpha-s})}{s - \alpha + \alpha(1 - e^{\alpha-s})} = -\frac{\phi(s)}{1 + \phi(s)}$$

where

$$\phi(s) := \alpha \frac{1 - e^{\alpha-s}}{s - \alpha} = \int_0^1 \alpha e^{(\alpha-s)t} dt$$

is holomorphic and

$$\begin{aligned} \sup_{\operatorname{Re} s > 0} |\phi(s)| &= \sup_{\operatorname{Re} s > 0} \left| \alpha \int_0^1 e^{(\alpha-s)t} dt \right| \leq \sup_{\operatorname{Re} s > 0} \int_0^1 |\alpha e^{(\alpha-s)t}| dt \\ &= \sup_{\operatorname{Re} s > 0} \int_0^1 \alpha e^{\alpha t} e^{-\operatorname{Re} s t} dt = \int_0^1 \alpha e^{\alpha t} dt = e^\alpha - 1 = \phi(0). \end{aligned}$$

Hence, $\phi \in H_\infty$ and $\|\phi\|_\infty = e^\alpha - 1$. By the Small Gain Theorem

P_τ is stable if $e^\alpha - 1 < 1 \iff \alpha < \ln 2 \approx 0.693$.

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For delay sweeping,

$$\chi_\tau(s) = s - \alpha e^\alpha e^{-\tau s},$$

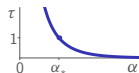
$\mathcal{A}_{1-4}$ hold, $\phi(\omega) = \omega^2 - \alpha^2 e^{2\alpha} \implies \omega_1 = \alpha e^\alpha$ (switch), and

$$\tau_k \omega_1 = \arg j\omega_1 - \arg \alpha e^\alpha + (2k-1)\pi = \frac{\pi}{2} - 0 + (2k-1)\pi$$

$\Downarrow$

$$\tau_k = \frac{(4k-1)\pi}{2\alpha e^\alpha}.$$

The smallest positive solution:

$$\tau = \tau_1 = \tau_* := \frac{3\pi}{2\alpha e^\alpha} =$$


Stable iff

$$\alpha_* > 1 \iff \alpha < \alpha_* \approx 1.29313 \approx 1.87 \ln 2 \quad (\text{solves } \alpha_* e^{\alpha_*} = 1.5\pi)$$

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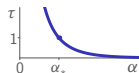
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Multiple delays

Pure imaginary roots of

$$\chi_\tau(s) = Q_0(s) + Q_1(s)e^{-\tau s} + Q_2(s)e^{-2\tau s}$$

are also roots of

$$e^{-2\tau s}\chi_\tau(-s) = Q_2(-s) + Q_1(-s)e^{-\tau s} + Q_0(-s)e^{-2\tau s}$$

and hence roots of

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Multiple delays: artifacts

The magnitude relation for $\tilde{\chi}_\tau(j\omega)$ reads

$$|Q_0(-j\omega)\chi_\tau(j\omega)| = |Q_2(j\omega)e^{-j2\tau\omega}\chi_\tau(-j\omega)|$$

or, equivalently,

$$(|Q_0(j\omega)| - |Q_2(j\omega)|)|\chi_\tau(j\omega)| = 0$$

i.e. there are *potentially* additional pure imaginary roots (artifacts) in $\tilde{\chi}_\tau(s)$, which are

- frequencies where $|Q_0(j\omega)| = |Q_2(j\omega)|$.

At each pure imaginary root $j\omega_i$ of $\tilde{\chi}_\tau(s)$ such that $|Q_0(j\omega_i)| \neq |Q_2(j\omega_i)|$,

- crossing directions of $\chi_\tau(s)$ and $\tilde{\chi}_\tau(s)$ coincide if $|Q_0(j\omega_i)| > |Q_2(j\omega_i)|$
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Multiple delays: example

Let

$$\chi_\tau(s) = s + e^{-\tau s} + e^{-2\tau s},$$

for Hurwitz $\chi_0(s) = s + 2$. Potential artifact frequencies solve

$$|Q_0(j\omega)| = |Q_2(j\omega)| \iff |\omega| = 1.$$

so we should only be suspicious about $\omega = 1$. Now,

$$\tilde{\phi}(\omega) = (1 - \omega^2)^2 - (1 + \omega^2) = \omega^2(\omega^2 - 3)$$

has only one positive root, $\omega_1 := \sqrt{3}$ and it is a switch of $\tilde{\chi}_\tau(s)$. Because

$$|Q_0(j\omega_1)| = \sqrt{3} > 1 = |Q_2(j\omega_1)|$$

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Outline

Direct method

Rekašius transformation

Key observation

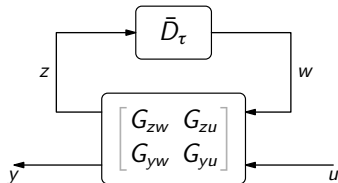
Let

$$\bar{R}_\alpha(s) := \frac{-s + \alpha}{s + \alpha}.$$

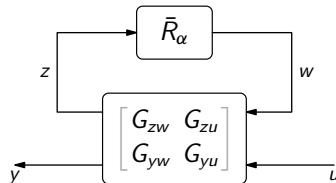
The locus of $e^{-j\tau\omega}$ in $\mathbb{C}$ as τ sweeps $\mathbb{R}_+$ coincides with that of $\bar{R}_\alpha(j\omega)$ as α sweeps $\mathbb{R}$. Moreover, at every $\omega_i \in \mathbb{R}_+ \setminus \{0\}$,

$$\bar{R}_\alpha(j\omega_i) = e^{-j\tau\omega_i} \iff \alpha_i = \omega_i \cot \frac{\tau_i \omega_i}{2}.$$

Hence, pure imaginary roots of characteristic functions $\chi_\tau(s)$ and $\chi_\alpha(s)$ of



and



respectively, coincide. But $\chi_\alpha(s)$ is a plain polynomial!

Crossing directions

Lemma

Let s_τ and s_α be roots of $\chi_\tau(s)$ and $\chi_\alpha(s)$, respectively. If $j\omega_i \neq 0$ is a root of both these functions, then

$$\text{sign Re } \frac{ds_\tau}{d\tau} \Big|_{s=j\omega_i} = - \text{sign Re } \frac{ds_\alpha}{d\alpha} \Big|_{s=j\omega_i}.$$

Example

Let

$$\chi_{\tau}(s) = s^2 + 0.1s + 1 + 0.4e^{-\tau s}$$

$\Downarrow$

$$s^2 + 0.1s + 1 + 0.4 \frac{-s + \alpha}{s + \alpha} = \frac{s^3 + (\alpha + 0.1)s^2 + 0.1(\alpha + 6)s + 1.4\alpha}{s + \alpha}$$

$\Downarrow$

$$\chi_{\alpha}(s) = s^3 + (\alpha + 0.1)s^2 + 0.1(\alpha + 6)s + 1.4\alpha$$

Nonzero imaginary roots iff

$$1.4\alpha = 0.1(\alpha + 6)(\alpha + 0.1) \quad \Rightarrow \quad \alpha_{1,2} = \frac{7.9 \pm \sqrt{60.01}}{2}.$$

Imaginary roots

$$s_{1,2} = j\sqrt{0.1(\alpha + 6)}|_{\alpha=\alpha_{1,2}} \approx j\{0.7795, 1.1757\}.$$

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