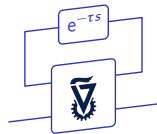


Time Delays in Control and Estimation (038806)



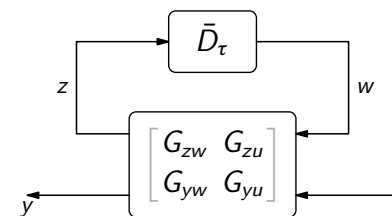
lecture no. 3

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1/24

Setup



where

$$G(s) = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix}$$

so that

$$P_\tau(s) = D_{yu} + \begin{bmatrix} C_y & D_{yw}e^{-\tau s} \end{bmatrix} \begin{bmatrix} sI_n - A & -B_w e^{-\tau s} \\ -C_z & I_{m_\tau} - D_{zw}e^{-\tau s} \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix}$$

2/24

Outline

I/O stability

3/24

L_2 stability

An LTI

- G is L_2 -stable iff its transfer function $G(s)$ is in H_∞ .

If $G(s) = D + C(sI - A)^{-1}B$, then

- $G \in H_\infty$ iff $\text{spec}(A) \in \mathbb{C} \setminus \bar{\mathbb{C}}_0$ (plus stabilizability and detectability)

Can say the same for

$$P_\tau(s) = D_{yu} + \begin{bmatrix} C_y & D_{yw}e^{-\tau s} \end{bmatrix} \begin{bmatrix} sI_n - A & -B_w e^{-\tau s} \\ -C_z & I_{m_\tau} - D_{zw}e^{-\tau s} \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix} ?$$

Unlike rational case,

- potential complications in $\{s \in \mathbb{C}_0 \mid |s| \rightarrow \infty\}$.

4/24

Characteristic function

Defined as

$$\chi_\tau(s) := \det \underbrace{\begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix}}_{\Phi_\tau(s)}$$

represents free motion of

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t - \tau) \end{bmatrix}$$

from initial conditions $(x(0), z_0) \in \mathbb{R}^n \times \{[-\tau, 0] \rightarrow \mathbb{R}^{m_\tau}\}$. Solutions of

$$\chi_\tau(s) = 0$$

known as **modes**.

5/24

Quasi-polynomials

Always (here $Q_0(s) = \det(sI - A)$)

$$\chi_\tau(s) = \sum_{i=0}^k Q_i(s) e^{-i\tau s}, \quad \deg Q_0(s) \geq \deg Q_i(s), \quad \forall i \in \mathbb{Z}_{1..k}$$

Classification:

- **retarded** if $\deg Q_0(s) > \deg Q_i(s), \forall i \in \mathbb{Z}_{1..k}$
- **neutral** if $\deg Q_0(s) = \deg Q_j(s)$ for some $j \in \mathbb{Z}_{1..k}$
- advanced if $\deg Q_0(s) < \deg Q_j(s)$ for some $j \in \mathbb{Z}_{1..k}$ (not here)
- **single delay** if $k = 1$
- (multiple) **commensurate delays** if $k > 1$
- incommensurate if $Q_0(s) + \sum_i Q_i(s) e^{-\tau_i s}$ with $\frac{\tau_{i1}}{\tau_{i2}} \notin \mathbb{Q}$ (not here)

6/24

Quasi-polynomials: examples

With

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix},$$

Input delay (assuming $P(s) = D + C(sI - A)^{-1}B$)

$$G_{\text{inpd}}(s) = \begin{bmatrix} 0 & I \\ P(s) & 0 \end{bmatrix} = \left[\begin{array}{c|cc} A & B & 0 \\ \hline 0 & 0 & I \\ C & D & 0 \end{array} \right]$$

$\Downarrow$

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B e^{-\tau s} \\ 0 & I \end{bmatrix} = \det(sI - A)$$

7/24

Quasi-polynomials: examples (contd)

With

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix},$$

Wave equation (assuming $R(s) = D + C(sI - A)^{-1}B$)

$$G_{\text{wave},0}(s) = \begin{bmatrix} R(s) & Z_m \\ 2R(s) & Z_m \end{bmatrix} = \left[\begin{array}{c|cc} A & B & 0 \\ \hline C & D & Z_m \\ 2C & 2D & Z_m \end{array} \right]$$

$\Downarrow$

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B e^{-\tau s} \\ -C & 1 - D e^{-\tau s} \end{bmatrix}$$

and

- $R(s) = -1 \implies \chi_\tau(s) = 1 + e^{-\tau s}$
- $R(s) = -1/(2s + 1) \implies \chi_\tau(s) = (s + 0.5) + 0.5e^{-\tau s}$

8/24

Quasi-polynomials: examples (contd)

With

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix},$$

General 2-order (for some $\alpha \in \mathbb{R}$)

$$G(s) = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 1 & 0 \\ -q_{00} & -q_{01} & -q_{10} & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

$\Downarrow$

$$\chi_\tau(s) = s^2 + q_{01}s + q_{00} + (-\alpha s + q_{10} + \alpha q_{00})e^{-\tau s} + \alpha q_{10}e^{-2\tau s}.$$

9/24

Quasi-polynomials: examples (contd)

With

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix},$$

Retarded with $D_{zw} \neq 0$

$$G(s) = \left[\begin{array}{c|ccc} 0 & -1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array} \right]$$

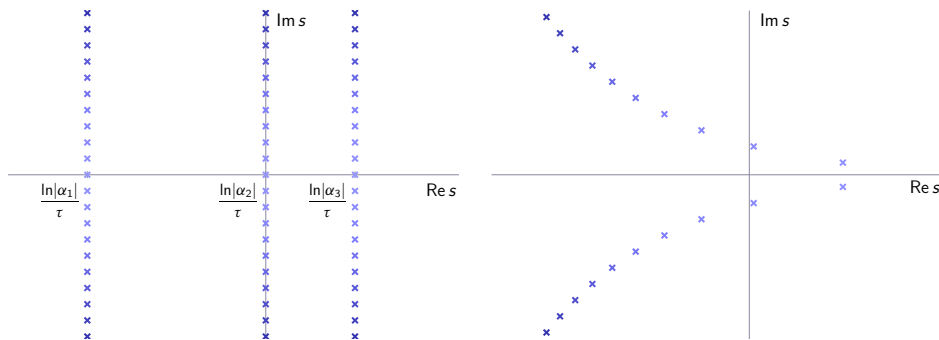
$\Downarrow$

$$\chi_\tau(s) = s + e^{-2\tau s}$$

10/24

Asymptotic root properties

As $|s| \rightarrow \infty$, modes approach



neutral, solutions of $e^{\tau s} = \alpha_i$

$$\frac{\ln|\alpha_i|}{\tau} \pm j \frac{2\pi k + \arg \alpha_i}{\tau}$$

retarded, solutions of $s e^{\tau s} = \alpha_i$

$$-\frac{\ln(2\pi k) - \ln|\tau \alpha_i|}{\tau} \pm j \frac{2\pi k + \arg \alpha_i - \frac{\pi}{2}}{\tau}$$

11/24

Neutral chains at large enough $|s|$

Two observations:

1. $|s| \rightarrow \infty$, but $|e^{-\tau s}|$ is bounded.
- 2.

$$\begin{aligned} \chi_\tau(s) &= \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix} \\ &= \underbrace{\det(sI - A)}_{\neq 0} \det(I - D_{zw} e^{-\tau s} - \underbrace{C_z(sI - A)^{-1} B_w e^{-\tau s}}_{\rightarrow 0}) \end{aligned}$$

Hence,

— neutral chains $\xrightarrow{|s| \rightarrow \infty} \text{Re } s = \frac{\ln|\lambda_i(D_{zw})|}{\tau}.$

12/24

Stability and roots of $\chi_\tau(s)$

If

$$P_\tau(s) = D_{yu} + \begin{bmatrix} C_y & D_{yw}e^{-\tau s} \end{bmatrix} \begin{bmatrix} sI_n - A & -B_w e^{-\tau s} \\ -C_z & I_{m_\tau} - D_{zw}e^{-\tau s} \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix},$$

then

- $\chi_\tau(s) = 0$ for some $s \in \bar{\mathbb{C}}_0 \implies P_\tau \notin H_\infty$ (normally)
- $\chi_\tau(s) \neq 0$ for all $s \in \bar{\mathbb{C}}_0 \xRightarrow{?} P_\tau \in H_\infty$

Associated with

$$\text{rank} \begin{bmatrix} A - sI_n & B_w & B_u \\ C_z & D_{zw} - e^{\tau s} I_{m_\tau} & D_{zu} \end{bmatrix} < n + m_\tau$$

and

$$\text{rank} \begin{bmatrix} A - sI_n & B_w \\ C_z & D_{zw} - e^{\tau s} I_{m_\tau} \\ C_y & D_{yw} \end{bmatrix} < n + m_\tau$$

13/24

If $\rho(D_{zw}) > 1$

Neutral chain of

$$\chi_\tau(s) = \det \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw} e^{-\tau s} \end{bmatrix}$$

in $\mathbb{C}_0$. Hence,

$$P_\tau(s) = D_{yu} + \begin{bmatrix} C_y & D_{yw}e^{-\tau s} \end{bmatrix} \begin{bmatrix} sI_n - A & -B_w e^{-\tau s} \\ -C_z & I_{m_\tau} - D_{zw}e^{-\tau s} \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix}$$

is not in H_∞ (normally).

14/24

If $\rho(D_{zw}) < 1$

With

$$\Phi_\tau(s) = \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw}e^{-\tau s} \end{bmatrix} \text{ and } G_{zw}(s) = D_{zw} + C_z(sI - A)^{-1}B_w$$

we have

$$\begin{aligned} \Phi_\tau(s) &= \begin{bmatrix} I & 0 \\ -C_z(sI - A)^{-1} & I \end{bmatrix} \\ &\quad \times \begin{bmatrix} sI - A & 0 \\ 0 & I - G_{zw}(s)e^{-\tau s} \end{bmatrix} \begin{bmatrix} I & -(sI - A)^{-1}B_w e^{-\tau s} \\ 0 & I \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \Phi_\tau^{-1}(s) &= \begin{bmatrix} I & (sI - A)^{-1}B_w e^{-\tau s} \\ 0 & I \end{bmatrix} \\ &\quad \times \begin{bmatrix} (sI - A)^{-1} & 0 \\ 0 & (I - G_{zw}(s)e^{-\tau s})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ C_z(sI - A)^{-1} & I \end{bmatrix}, \end{aligned}$$

15/24

If $\rho(D_{zw}) < 1$ (contd)

Because

$$I - G_{zw}(s)e^{-\tau s} = I - D_{zw}e^{-\tau s} - C_z(sI - A)^{-1}B_w e^{-\tau s},$$

we have that

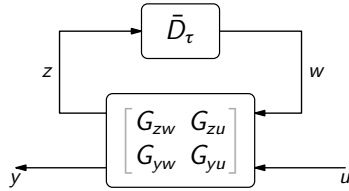
$$\begin{aligned} \limsup_{s \in \mathbb{C}_0, |s| \rightarrow \infty} \left\| \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw}e^{-\tau s} \end{bmatrix}^{-1} \right\| &= \limsup_{s \in \mathbb{C}_0, |s| \rightarrow \infty} \|(I - D_{zw}e^{-\tau s})^{-1}\| \\ &= \max_{\zeta \in \mathbb{D}} \|(I - D_{zw}\zeta)^{-1}\| \\ &< \infty \end{aligned}$$

and

- $\chi_\tau(s) \neq 0$ for all $s \in \bar{\mathbb{C}}_0 \implies P_\tau \in H_\infty$ like in rational case

16/24

If $\rho(D_{zw}) < 1$: example



Let

$$G(s) = \frac{1}{s} \begin{bmatrix} \alpha e^\alpha & s - \alpha \\ \alpha e^\alpha & -\alpha \end{bmatrix} = \left[\begin{array}{c|cc} 0 & \alpha e^\alpha & -\alpha \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right]$$

so that

$$P_\tau(s) = \frac{\alpha(e^{\alpha-s} - 1)}{s - \alpha e^{\alpha-s}}$$

and

$$\chi_\tau(s) = \det \begin{bmatrix} s & -\alpha e^{\alpha-s} \\ -1 & 1 \end{bmatrix} = s - \alpha e^{\alpha-s}$$

17/24

If $\rho(D_{zw}) < 1$: example (contd)

Because

$$\chi_\tau(\alpha) = \alpha - \alpha e^{\alpha-\alpha} = 0,$$

we may expect that $P_\tau(s)$ has a pole at $s = \alpha$. However,

$$\lim_{s \rightarrow \alpha} P_\tau(s) = \lim_{s \rightarrow \alpha} \frac{\alpha(e^{\alpha-s} - 1)}{s - \alpha e^{\alpha-s}} = -\frac{\alpha}{1 + \alpha}$$

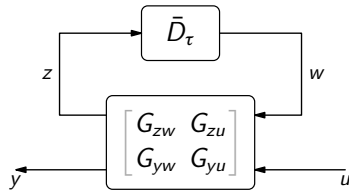
so $s = \alpha$ is not a pole.

PBH controllability:

$$\begin{aligned} \left[\begin{array}{ccc|c} A - sI & B_w & B_u \\ C_z & D_{zw} - e^{\tau s} I & D_{zu} \end{array} \right] \Big|_{s=\alpha} &= \left[\begin{array}{ccc|c} -s & \alpha e^\alpha & -\alpha \\ 1 & -e^s & 1 \end{array} \right] \Big|_{s=\alpha} \\ &= \left[\begin{array}{ccc|c} -\alpha & \alpha e^\alpha & -\alpha \\ 1 & -e^\alpha & 1 \end{array} \right] \end{aligned}$$

18/24

If $\rho(D_{zw}) = 1$: example



Let

$$G(s) = \begin{bmatrix} -s/(s+1) & -s/(s+1) \\ 1/(s+1) & 1/(s+1) \end{bmatrix} = \left[\begin{array}{c|cc} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & 0 \end{array} \right]$$

so that

$$P_\tau(s) = \frac{1}{s+1+se^{-s}}$$

and

$$\chi_\tau(s) = \det \begin{bmatrix} s+1 & -e^{-s} \\ -1 & 1+e^{-s} \end{bmatrix} = s+1+se^{-s}.$$

is neutral.

19/24

If $\rho(D_{zw}) = 1$: example (contd)

Have

$$\chi_\tau(s) = 0 \iff -e^{-s} = 1 + \frac{1}{s} \implies |e^{-s}| = \left| 1 + \frac{1}{s} \right|$$

Let $s = \sigma + j\omega$ with $\sigma \geq 0$.

– If $\sigma > 0$, then contradiction:

$$1 > e^{-\sigma} = \left| 1 + \frac{1}{\sigma + j\omega} \right| \geq \left| \operatorname{Re} \left(1 + \frac{1}{\sigma + j\omega} \right) \right| = \left| 1 + \frac{\sigma}{\sigma^2 + \omega^2} \right| > 1$$

– If $\sigma = 0$, then contradiction:

$$1 = e^0 = \left| 1 + \frac{1}{j\omega} \right| = 1 + \frac{1}{|\omega|} > 1$$

Hence,

– there are no roots in $\sigma \geq 0$

20/24

If $\rho(D_{zw}) = 1$: example (contd)

There is a neutral chain of modes $\{s_i\}$ such that

$$s_i + 1 + s_i e^{-s_i} = 0, \quad \text{with } \lim_{i \rightarrow \infty} |s_i| = \infty.$$

Now, $-s_i \in \mathbb{C}_0$ and

$$P_\tau(-s_i) = \frac{1}{-s_i + 1 - s_i e^{s_i}} = \frac{1}{-s_i + 1 + s_i^2/(s_i + 1)} = 1 + s_i$$

and

$$\lim_{i \rightarrow \infty} |P_\tau(-s_i)| = \infty \implies P_\tau \notin H_\infty.$$

As a matter of fact

$$P_\tau(s) = \frac{1}{(s+1)(s+1+se^{-s})} \text{ is an } H_\infty \text{ function.}$$

21/24

If $\rho(D_{zw}) = 1$: on the safe side

Anyhow,

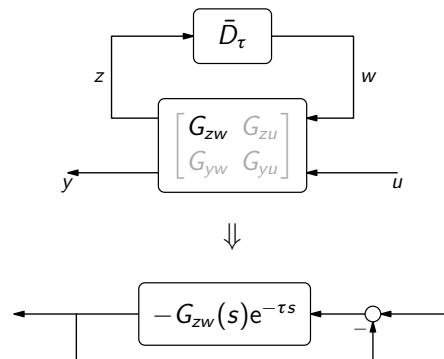
- $\rho(D_{zw}) = 1 \implies$ practical instability!

Thus,

- we're only interested in cases with $\rho(D_{zw}) < 1$ in control applications.

22/24

Nyquist criterion: setup



Important:

- $\rho(D_{zw}) < 1 \iff G_{zw}(\infty) < 1$

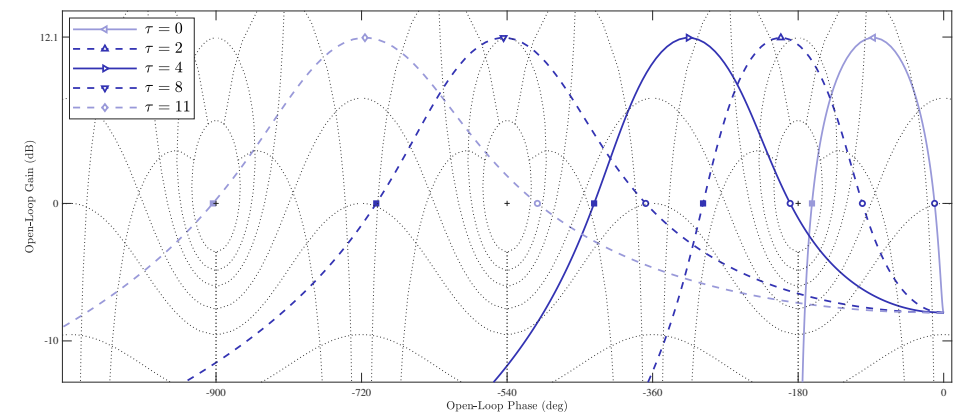
23/24

Nyquist criterion: example

Let

$$G(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.1 & -0.4 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{s^2 + 0.1s + 1} \begin{bmatrix} -0.4 & 1 \\ -0.4 & 1 \end{bmatrix}$$

Then



24/24