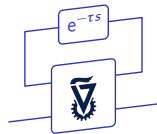


Time Delays in Control and Estimation (038806)



lecture no. 2

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Outline

Delay element

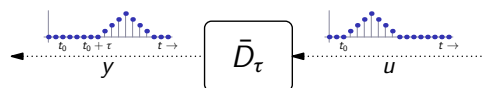
Delays as socializers

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Finite-dimensional approximations

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Discrete-time delay element

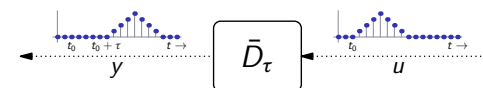


Minimal realization

$$\bar{D}_\tau(z) = z^{-\tau} I_m = \left[\begin{array}{ccc|c|c} 0 & I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_m & 0 \\ 0 & 0 & \cdots & 0 & I_m \\ \hline I_m & 0 & 0 & 0 & 0 \end{array} \right]$$

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Discrete-time delay element: state



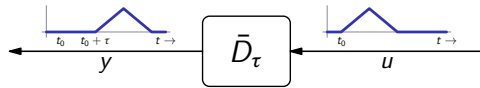
State propagation:

$$\bar{D}_\tau : \left\{ \begin{array}{l} \overbrace{\begin{bmatrix} x[t+1] \\ u[t-\tau+1] \\ \vdots \\ u[t-1] \\ u[t] \end{bmatrix}}^{x[t+1]} = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} \underbrace{\begin{bmatrix} u[t-\tau] \\ u[t-\tau+1] \\ \vdots \\ u[t-1] \end{bmatrix}}_{x[t]} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u[t] \\ y[t] = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} x[t] \end{array} \right.$$

i.e. the state $x[t]$ is the **history** of its inputs in $\mathbb{Z}_{t-\tau..t-1}$.

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Continuous-time delay element



Transfer function:

$$\bar{D}_\tau(s) = e^{-\tau s} I_m.$$

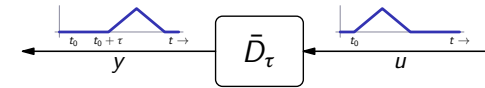
State is the history of u in the whole $[t - \tau, t]$, i.e.

$$\check{u}_t : [-\tau, 0] \rightarrow \mathbb{R}^m \quad \text{such that } \check{u}_t(s) = u(t + s), \quad \forall s \in [-\tau, 0].$$

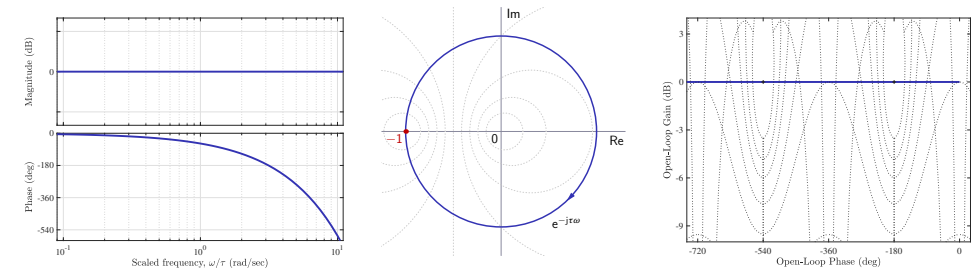
(set of all $\check{u}_t$ is **infinite-dimensional** space)

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Continuous-time delay element: frequency response



Standard plots:



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I/O delays

If P is LTI, then

$$P_\tau = \bar{D}_{\tau_o} P \bar{D}_{\tau_i} = P \bar{D}_{\tau_o + \tau_i} = P \bar{D}_\tau \quad \Rightarrow \quad \begin{cases} P_\tau(s) = P(s) e^{-\tau s} \\ P_\tau(z) = P(z) z^{-\tau} \end{cases}$$

aka **dead-time systems**.

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State of dead-time systems: discrete insight

Transfer function

$$P(z)z^{-\tau} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} = \left[\begin{array}{c|c|c|c|c|c} A & B & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & \dots & 0 & I \\ \hline C & D & 0 & 0 & 0 & 0 \end{array} \right]$$

and its state

$$x[t] = \begin{bmatrix} x_P[t] \\ u[t - \tau] \\ u[t - \tau + 1] \\ \vdots \\ u[t - 1] \end{bmatrix} \in \mathbb{R}^{n+\tau m}.$$

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State of dead-time systems

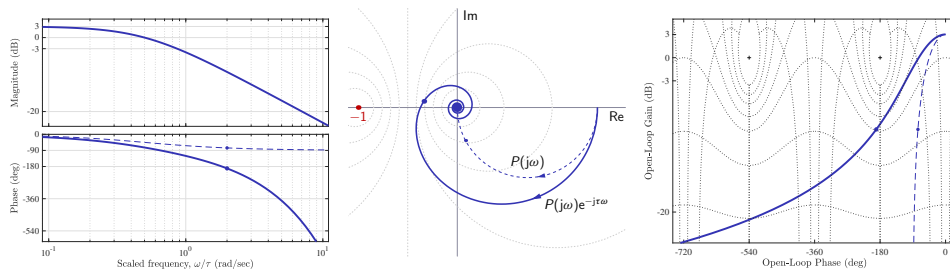
State of $P_\tau = P\bar{D}_\tau$ at a time instance t is

$$(x_P(t), \check{u}_t) \in \mathbb{R}^n \times \{[-\tau, 0] \rightarrow \mathbb{R}^m\}.$$

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Dead-time systems: frequency response

Standard plots:



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Internal delay: 1-dimensional wave equation

Reads

$$\frac{\partial^2 w(x, t)}{\partial t^2} = c^2 \frac{\partial^2 w(x, t)}{\partial x^2} \quad \text{for } 0 < x < L \text{ and } t \geq 0$$

($c > 0$ is wave propagation speed), where

— w is a variable of interest

$p_x = -\rho_0 c^2 \partial w / \partial x$ is the pressure of the air in a duct (ρ_0 is the air density)

Boundary control:

$$\frac{\partial w(0, t)}{\partial t} = u(t)$$

Environment:

$$p_L(t) = -c Z_m \frac{\partial w(x, t)}{\partial x} \Big|_{x=L} = Z_L \frac{\partial w(L, t)}{\partial t},$$

where

- $Z_m = \rho_0 c$ propagation impedance
- Z_L end impedance operator

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Wave equation: model

Laplace wrt t :

$$c^2 \partial_x^2 W(x, s) = s^2 W(x, s)$$

with

$$sW(0, s) = U(s) \quad \text{and} \quad sZ_L(s)W(L, s) + cZ_m \partial_x W(L, s) = 0$$

where $\partial_x f := \partial f / \partial x$ and $\partial_x^2 f := \partial^2 f / \partial x^2$. Second-order **ODE** solution:

$$\begin{aligned} \begin{bmatrix} W(x, s) \\ \partial_x W(x, s) \end{bmatrix} &= \exp \left(\begin{bmatrix} 0 & 1 \\ s^2/c^2 & 0 \end{bmatrix} x \right) \begin{bmatrix} W(0, s) \\ \partial_x W(0, s) \end{bmatrix} \\ &= \begin{bmatrix} \cosh(sx/c)/s & c \sinh(sx/c)/s \\ \sinh(sx/c)/c & \cosh(sx/c) \end{bmatrix} \begin{bmatrix} U(s) \\ \partial_x W(0, s) \end{bmatrix} \end{aligned}$$

where

$$\cosh(sx/c) = \frac{e^{sx/c} + e^{-sx/c}}{2} \quad \text{and} \quad \sinh(sx/c) = \frac{e^{sx/c} - e^{-sx/c}}{2}$$

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Wave equation: model (contd)

Boundary condition $sZ_L(s)W(L, s) + cZ_m \partial_x W(L, s) = 0$ reads then

$$\begin{bmatrix} Z_L(s) & Z_m \end{bmatrix} \begin{bmatrix} \cosh(sL/c) & \sinh(sL/c) \\ \sinh(sL/c) & \cosh(sL/c) \end{bmatrix} \begin{bmatrix} U(s) \\ c \partial_x W(0, s) \end{bmatrix} = 0.$$

Hence,

$$c \partial_x W(0, s) = - \overbrace{\frac{1 + R(s)e^{-2(L/c)s}}{1 - R(s)e^{-2(L/c)s}}}^{:= V_0(s)} U(s), \quad \text{where } R(s) := \frac{Z_L(s) - Z_m}{Z_L(s) + Z_m}$$

(reflectance). Thus,

$$\begin{bmatrix} sW(x, s) \\ c \partial_x W(x, s) \end{bmatrix} = \begin{bmatrix} \cosh(sx/c) & \sinh(sx/c) \\ \sinh(sx/c) & \cosh(sx/c) \end{bmatrix} \begin{bmatrix} 1 \\ V_0(s) \end{bmatrix} U(s).$$

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Wave equation: transfer function of $u \mapsto p_0$

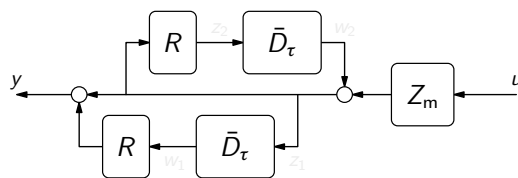
$G_0 : u \mapsto p_0$ has

$$G_0(s) = -V_0(s)Z_m = \frac{1 + R(s)e^{-\tau s}}{1 - R(s)e^{-\tau s}} Z_m,$$

where

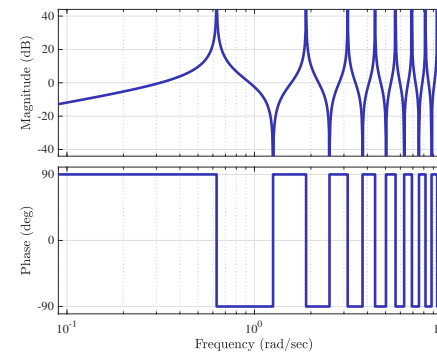
$$\tau := \frac{2L}{c} > 0$$

(travel time).

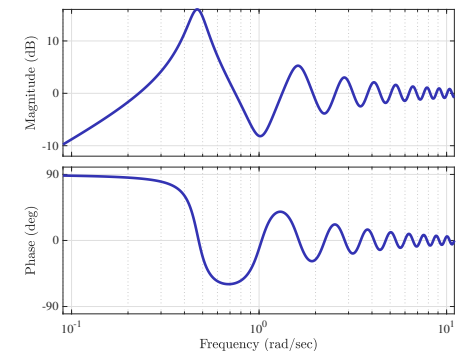


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Wave equation: complexity



$$Z_L(s) = 0, R(s) = -1$$



$$Z_L(s) = s/(s+1), R(s) = -1/(2s+1)$$

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Wave equation: transfer function of $u \mapsto p_x$

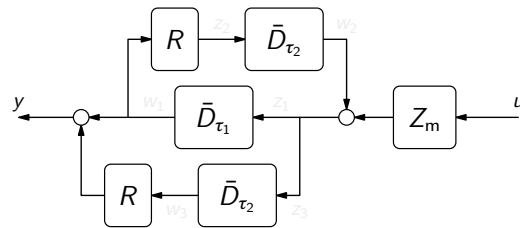
$G_x : u \mapsto p_x$ has

$$G_x(s) = \frac{e^{-\tau_1 s} + R(s)e^{-\tau_2 s}}{1 - R(s)e^{-(\tau_1 + \tau_2)s}} Z_m,$$

where

$$\tau_1 := \frac{x}{c} \quad \text{and} \quad \tau_2 := \frac{2L - x}{c}$$

(travel times of forward and reflected waves).



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Time-delay systems so far

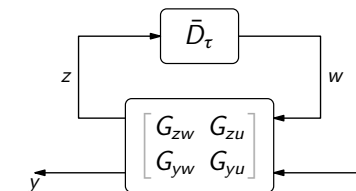
- pure delay: $\bar{D}_\tau$
- input delay: $P_\tau = P\bar{D}_\tau$ for a finite-dimensional P
- internal delays, like

$$G_0 = \frac{1 + R\bar{D}_\tau}{1 - R\bar{D}_\tau} Z_m \quad \text{or even} \quad G_x = \frac{\bar{D}_{\tau_1} + R\bar{D}_{\tau_2}}{1 - R\bar{D}_{\tau_1 + \tau_2}} Z_m$$

- ...

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Isolating delays



for delay-free G . The “plant” $u \mapsto y$ is

$$P_\tau = \mathcal{F}_u(G, \bar{D}_\tau) = G_{yu} + G_{yw}\bar{D}_\tau(I - G_{zw}\bar{D}_\tau)^{-1}G_{zu}.$$

Fits all we studied so far...

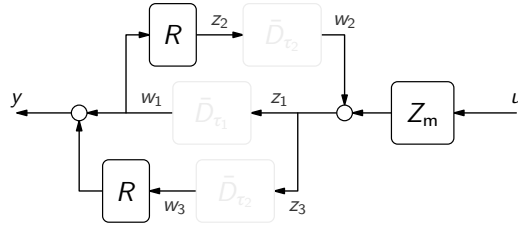
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For multiple delays, with $\text{diag}\{\bar{D}_{\tau_i}\}$

If

$$G_x = \frac{\bar{D}_{\tau_1} + R\bar{D}_{\tau_2}}{1 - R\bar{D}_{\tau_1 + \tau_2}} Z_m,$$

then

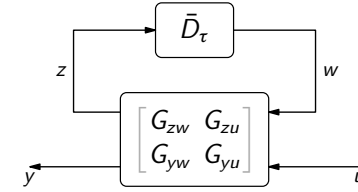


Hence:

$$G = G_{\text{wave},x} := \left[\begin{array}{ccc|c} 0 & 1 & 0 & Z_m \\ R & 0 & 0 & 0 \\ 0 & 1 & 0 & Z_m \\ \hline 1 & 0 & R & 0 \end{array} \right]$$

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State-space representation



Let

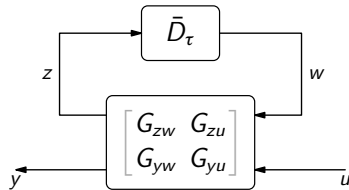
$$G(s) = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array} \right].$$

Involved relations:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t) \\ z(t) = C_z x(t) + D_{zw} w(t) + D_{zu} u(t) \\ y(t) = C_y x(t) + D_{yw} w(t) + D_{yu} u(t) \end{cases} \quad \text{and} \quad w(t) = z(t - \tau).$$

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State-space representation (contd)



For $P_\tau : u \mapsto y$,

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t - \tau) \end{bmatrix} + \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_y & D_{yw} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t - \tau) \end{bmatrix} + D_{yu} u(t) \end{cases}$$

called **delay-differential equation** (DDE, subclass of FDE). Its state at each t

$$(x(t), \check{z}_t) \in \mathbb{R}^n \times \{[-\tau, 0] \rightarrow \mathbb{R}^{m_\tau}\}.$$

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State-space representation: special cases

Let

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t - \tau) \end{bmatrix} + \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix} u(t)$$

If

– $D_{zw} = 0$, then

$$\dot{x}(t) = Ax(t) + A_\tau x(t - \tau) + B_u u(t) + B_\tau u(t - \tau)$$

($A_\tau := B_w C_z$ and $B_\tau := B_w D_{zu}$) known as **retarded** DDE;

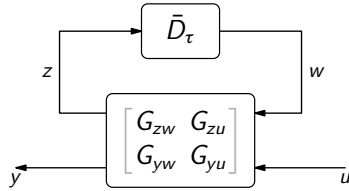
– $\exists E_\tau \in \mathbb{R}^{n \times n}$ such that $B_w D_{zw} = -E_\tau B_w$, then

$$\dot{x}(t) + E_\tau \dot{x}(t - \tau) = Ax(t) + A_\tau x(t - \tau) + B_u u(t) + B_\tau u(t - \tau)$$

($A_\tau := B_w C_z + E_\tau A$ and $B_\tau := B_w D_{zu} + E_\tau B_u$) known as **neutral** DDE.

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Transfer function



If $(x(t), \dot{z}_t) = 0$,

$$P_\tau(s) = D_{yu} + \begin{bmatrix} C_y & D_{yw}e^{-\tau s} \end{bmatrix} \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw}e^{-\tau s} \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix}$$

each element of which is a

- quotient of quasi-polynomials (powers of both s and $e^{-\tau s}$).

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Approximating pure delay

$\exists$ increasing sequence $\{\omega_i\}$ such that

$$\arg R_\tau(j\omega_i) - \arg e^{-j\tau\omega_i} = \pi + 2\pi k \quad \text{or} \quad \begin{array}{c} \text{Diagram showing a point } R_\tau(j\omega_i) \text{ on a circle in the complex plane, with a dashed line connecting it to } e^{-j\tau\omega_i} \text{ on the unit circle.} \end{array}$$

Hence,

$$|\bar{D}_\tau(j\omega_i) - R_\tau(j\omega_i)| = |\bar{D}_\tau(j\omega_i)| + |R_\tau(j\omega_i)| = 1 + |R_\tau(j\omega_i)|$$

and

$$\|\bar{D}_\tau - R_\tau\|_\infty =: \epsilon_R \geq 1 + \sup_{i \in \mathbb{N}} |R_\tau(j\omega_i)| \geq 1 = \|\bar{D}_\tau - 0\|_\infty$$

i.e. never better than $R_\tau = 0$ (makes no sense).

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Padé equality

Given $\phi(s)$, analytic around $s = 0$, construct

$$R_{m,n}(s) = \frac{P_m(s)}{Q_n(s)} = \frac{p_m s^m + \dots + p_1 s + p_0}{q_n s^n + \dots + q_1 s + 1}$$

(i.e. choose p_i and q_i) to

- match first $n + m + 1$ terms of Maclaurin series of $\phi(s)$

This reads

$$\begin{aligned} \phi(s) - R_{m,n}(s) &= \underbrace{\gamma_0}_{=0} + \underbrace{\gamma_1}_{=0} s + \dots + \underbrace{\gamma_{n+m}}_{=0} s^{n+m} + \gamma_{n+m-1} s^{n+m-1} + \dots \\ &= O(s^{m+n+1}) \quad \text{as } s \rightarrow 0 \end{aligned}$$

or, equivalently,

$$\phi(s)Q_n(s) - P_m(s) = O(s^{m+n+1}) \quad \text{as } s \rightarrow 0.$$

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Preliminary: multiplication of polynomials

Consider $A(s) = a_n s^n + \dots + a_1 s + a_0$ and $B(s) = b_m s^m + \dots + b_1 s + b_0$. If

$$C(s) = A(s)B(s) = c_{n+m} s^{n+m} + \dots + c_1 s + c_0$$

then

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \\ c_{m+1} \\ \vdots \\ c_n \\ c_{n+1} \\ \vdots \\ c_{n+m} \end{bmatrix} = \begin{bmatrix} a_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & a_{m-1} & \dots & a_0 \\ a_{m+1} & a_m & \dots & a_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & \dots & a_{n-m} \\ 0 & a_n & \dots & a_{n-m+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

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Padé equality in matrix form (general)

Relation $\phi(s)Q_n(s) - P_m(s) = O(s^{m+n+1})$ reads

$$\begin{bmatrix} \phi_0 & 0 & \dots & 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ \phi_1 & \phi_0 & \dots & 0 & 0 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_m & \phi_{m-1} & \dots & \phi_0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 \\ \phi_{m+1} & \phi_m & \dots & \phi_1 & \phi_0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_n & \phi_{n-1} & \dots & \phi_{n-m} & \phi_{n-m-1} & \dots & \phi_0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{n+m} & \phi_{n+m-1} & \dots & \phi_n & \phi_{n-1} & \dots & \phi_m & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ \vdots \\ q_m \\ q_{m+1} \\ \vdots \\ q_n \\ p_0 \\ p_1 \\ \vdots \\ p_m \end{bmatrix} = 0,$$

where

$$\phi_i = \frac{\phi^{(i)}(0)}{i!}$$

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Padé equality in matrix form ($m = n$)

$$\begin{bmatrix} \phi_0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ \phi_1 & \phi_0 & \dots & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_n & \phi_{n-1} & \dots & \phi_0 & 0 & 0 & \dots & -1 \\ \phi_{n+1} & \phi_n & \dots & \phi_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{2n} & \phi_{2n-1} & \dots & \phi_n & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ \vdots \\ q_n \\ p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = 0.$$

$$\phi(s) = e^s \implies \phi_i = \frac{1}{i!}.$$

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Key step

Define

$$\psi_i(t) := \frac{t^i}{i!}$$

($\psi_i(0) = 0$ and $\psi_i(1) = \phi_i$ for all $i \in \mathbb{N}$) and then

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_n(t) \end{bmatrix} := \begin{bmatrix} \psi_{n+1}(t) & \psi_n(t) & \dots & \psi_1(t) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{2n}(t) & \psi_{2n-1}(t) & \dots & \psi_n(t) \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ \vdots \\ q_n \end{bmatrix}.$$

Last n rows of Padé equality read $\xi(1) = 0$. Also,

$$\psi_i(t) = \dot{\psi}_{i+1}(t) \implies \xi_i(t) = \dot{\xi}_{i+1}(t) \implies \xi_i(t) = \xi_n^{(n-i)}(t)$$

We thus look for

— $2n$ -order polynomial $\xi_n(t)$ such that $\xi_n^{(i)}(t) = 0$ at $t = 0$ and $t = 1$ for all $i \in \mathbb{Z}_{0..n-1}$. It is $\alpha t^n (t-1)^n$ for some $\alpha \neq 0$.

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Key step (contd)

Thus, coefficients satisfy

$$\xi_n(t) = \frac{t^{2n}}{(2n)!} + \sum_{i=1}^n \frac{t^{2n-i}}{(2n-i)!} q_i = \alpha t^n (t-1)^n, \quad \forall t \in [0, 1]$$

By the binomial theorem,

$$t^n (t-1)^n = \sum_{i=0}^n \binom{n}{i} (-1)^i t^{2n-i}.$$

Hence,

$$\alpha = 1/(2n)! \quad \text{and} \quad q_i = (-1)^i \binom{n}{i} \frac{(2n-i)!}{(2n)!} = (-1)^i \frac{(2n-i)! n!}{(2n)! (n-i)! i!}$$

With a little more algebra,

$$P_n(s) = Q_n(-s).$$

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Padé approximant

Using the substitution $s \rightarrow -\tau s$,

$$e^{-\tau s} \approx R_{n,n}(\tau s) = \frac{Q_n(-\tau s)}{Q_n(\tau s)}, \quad \text{where } Q_n(s) = \sum_{i=0}^n \binom{n}{i} \frac{(2n-i)!}{(2n)!} s^i$$

First 6 of them:

$$Q_1(s) = 1 + \frac{1}{2} s,$$

$$Q_2(s) = 1 + \frac{1}{2} s + \frac{1}{12} s^2,$$

$$Q_3(s) = 1 + \frac{1}{2} s + \frac{1}{10} s^2 + \frac{1}{120} s^3,$$

$$Q_4(s) = 1 + \frac{1}{2} s + \frac{3}{28} s^2 + \frac{1}{84} s^3 + \frac{1}{1680} s^4,$$

$$Q_5(s) = 1 + \frac{1}{2} s + \frac{1}{9} s^2 + \frac{1}{72} s^3 + \frac{1}{1008} s^4 + \frac{1}{30240} s^5,$$

$$Q_6(s) = 1 + \frac{1}{2} s + \frac{5}{44} s^2 + \frac{1}{66} s^3 + \frac{1}{792} s^4 + \frac{1}{15840} s^5 + \frac{1}{665280} s^6.$$

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Properties

Lemma

$Q_n(s)$ is Hurwitz for all $n \in \mathbb{N}$.

Lemma

$|e^{-j\omega} - R_{n,n}(j\omega)| \leq 2.02 |H_{2n+1}(j\omega)|$ for all $\omega \in \mathbb{R}$, where

$$H_{2n+1}(s) = \frac{(s/\omega_n)^{2n+1}}{B_{2n+1}(s/\omega_n)}$$

where

$$\omega_n := \left(2 \frac{(2n)!(2n+1)!}{(n!)^2} \right)^{1/(2n+1)}$$

and $B_k(s)$ is the k -order Butterworth polynomial, $|B_k(j\omega)|^2 = 1 + \omega^{2k}$.

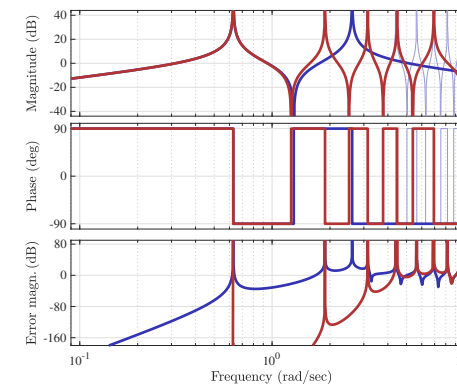
Remark: Note that

$$\{\omega_n\} = \{2.8845, 4.2823, 5.7251, 7.1814, 8.6435, \dots\} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\omega_n}{n} = \frac{4}{e} \approx 1.4715.$$

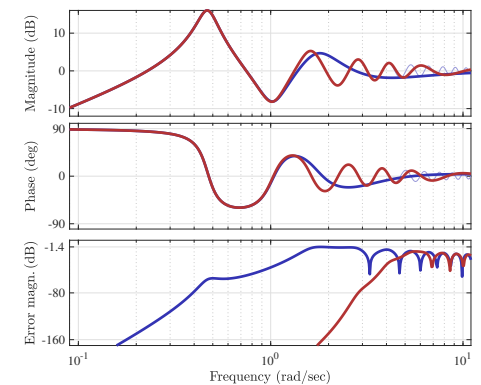
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Example

$[4, 4]$ - and $[12, 12]$ -Padé approximants of $G_0(s) = \frac{1 + R(s)e^{-\tau s}}{1 - R(s)e^{-\tau s}}$ are



$$Z_L(s) = 0, \quad R(s) = -1$$



$$Z_L(s) = s/(s+1), \quad R(s) = -1/(2s+1)$$

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