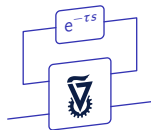


Time Delays in Control and Estimation (038806)



lecture no. 2

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Outline

Delay element

Delays as socializers

Uniform interconnection

Finite-dimensional approximations

Outline

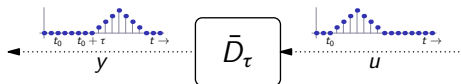
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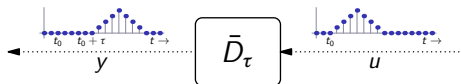
Discrete-time delay element



Minimal realization

$$\bar{D}_\tau(z) = z^{-\tau} I_m = \left[\begin{array}{cccc|c} 0 & I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_m & 0 \\ 0 & 0 & \cdots & 0 & I_m \\ \hline I_m & 0 & 0 & 0 & 0 \end{array} \right]$$

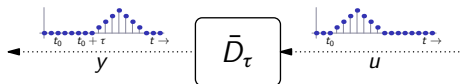
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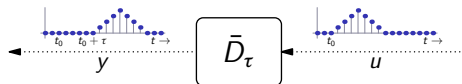
Discrete-time delay element: state



State propagation:

$$\bar{D}_\tau : \left\{ \begin{array}{l} \overbrace{\begin{bmatrix} \end{bmatrix}}^{x[t+1]} \\ y[t] = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} x[t] \end{array} \right. = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} \overbrace{\begin{bmatrix} \end{bmatrix}}^{x[t]} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u[t]$$

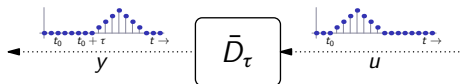
Discrete-time delay element: state



State propagation:

$$\bar{D}_\tau : \left\{ \begin{array}{l} \overbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}^{x[t+1]} \\ u[t] \\ \underbrace{y[t]}_{= \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} x[t]} \end{array} \right\} = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} \overbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}^{x[t]} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u[t]$$

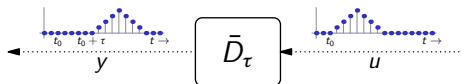
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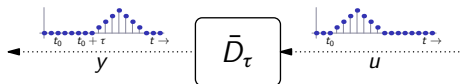
Discrete-time delay element: state



State propagation:

$$\bar{D}_\tau : \left\{ \begin{array}{l} \overbrace{\begin{bmatrix} x[t+1] \\ u[t-1] \\ u[t] \end{bmatrix}}^{x[t+1]} \\ y[t] = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} x[t] \end{array} \right. = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} \overbrace{\begin{bmatrix} x[t] \\ u[t-1] \end{bmatrix}}^{x[t]} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u[t]$$

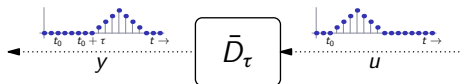
Discrete-time delay element: state



State propagation:

$$\bar{D}_\tau : \left\{ \begin{array}{l} \overbrace{\begin{bmatrix} u[t - \tau + 1] \\ \vdots \\ u[t - 1] \\ u[t] \end{bmatrix}}^{x[t+1]} \\ y[t] = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} x[t] \end{array} \right. = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} \overbrace{\begin{bmatrix} u[t - \tau] \\ u[t - \tau + 1] \\ \vdots \\ u[t - 1] \end{bmatrix}}^{x[t]} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u[t]$$

Discrete-time delay element: state

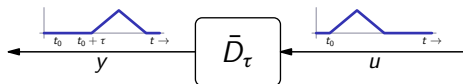


State propagation:

$$\bar{D}_\tau : \begin{cases} \overbrace{\begin{bmatrix} u[t - \tau + 1] \\ \vdots \\ u[t - 1] \\ u[t] \end{bmatrix}}^{x[t+1]} = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} \overbrace{\begin{bmatrix} u[t - \tau] \\ u[t - \tau + 1] \\ \vdots \\ u[t - 1] \end{bmatrix}}^{x[t]} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u[t] \\ y[t] = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} x[t] \end{cases}$$

i.e. the state $x[t]$ is the **history** of its inputs in $\mathbb{Z}_{t-\tau..t-1}$.

Continuous-time delay element



Transfer function:

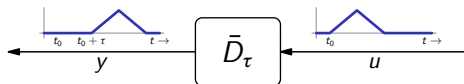
$$\bar{D}_\tau(s) = e^{-\tau s} I_m.$$

State is the history of u in the whole $[t - \tau, t]$, i.e.

$$\check{u}_t : [-\tau, 0] \rightarrow \mathbb{R}^m \quad \text{such that} \quad \check{u}_t(s) = u(t+s), \quad \forall s \in [-\tau, 0].$$

(set of all $\check{u}_t$ is infinite-dimensional space)

Continuous-time delay element



Transfer function:

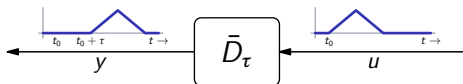
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Continuous-time delay element



Transfer function:

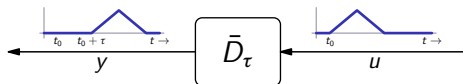
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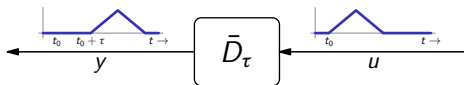
(set of all $\check{u}_t$ is **infinite-dimensional** space)

Continuous-time delay element: frequency response

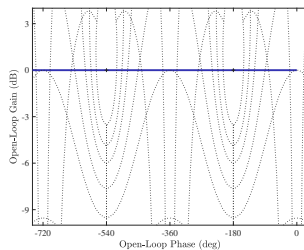
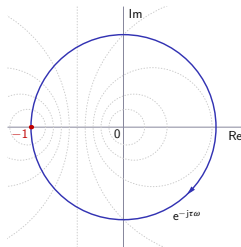
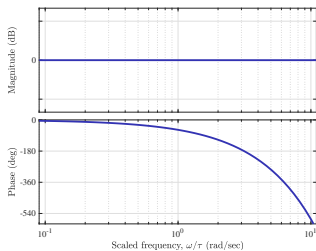


Standard plots:

Continuous-time delay element: frequency response



Standard plots:



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I/O delays

If P is LTI, then

$$P_\tau = \bar{D}_{\tau_o} P \bar{D}_{\tau_i} = P \bar{D}_{\tau_o + \tau_i} = P \bar{D}_\tau \quad \implies \quad \begin{cases} P_\tau(s) = P(s)e^{-\tau s} \\ P_\tau(z) = P(z)z^{-\tau} \end{cases}$$

aka **dead-time systems**.

State of dead-time systems: discrete insight

Transfer function

$$P(z)z^{-\tau} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \left[\begin{array}{cccc|c} 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \\ 0 & 0 & \dots & 0 & I \\ \hline I & 0 & 0 & 0 & 0 \end{array} \right] = \left[\begin{array}{ccccc|c} A & B & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & \dots & 0 & I \\ \hline C & D & 0 & 0 & 0 & 0 \end{array} \right]$$

and its state

$$x[t] = \begin{bmatrix} x_P[t] \\ \vdots \\ u[t - \tau] \\ u[t - \tau + 1] \\ \vdots \\ u[t - 1] \end{bmatrix} \in \mathbb{R}^{n+\tau m}.$$

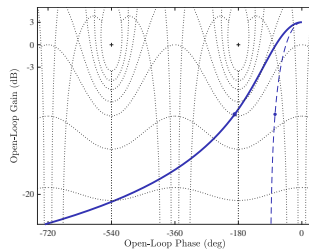
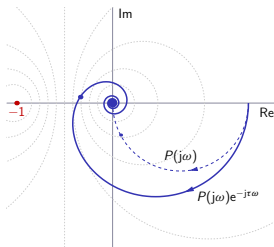
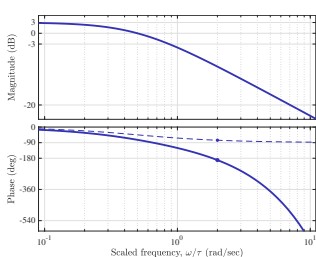
State of dead-time systems

State of $P_\tau = P\bar{D}_\tau$ at a time instance t is

$$(x_P(t), \check{u}_t) \in \mathbb{R}^n \times \{[-\tau, 0] \rightarrow \mathbb{R}^m\}.$$

Dead-time systems: frequency response

Standard plots:



Internal delay: 1-dimensional wave equation

Reads

$$\frac{\partial^2 w(x, t)}{\partial t^2} = c^2 \frac{\partial^2 w(x, t)}{\partial x^2} \quad \text{for } 0 < x < L \text{ and } t \geq 0$$

($c > 0$ is wave propagation speed), where

- w is a variable of interest

$p_x = -\rho_0 c^2 \partial w / \partial x$ is the pressure of the air in a duct (ρ_0 is the air density)

Boundary control:

$$\frac{\partial w(0, t)}{\partial t} = u(t)$$

Environment:

$$p_L(t) = -cZ_m \left. \frac{\partial w(x, t)}{\partial x} \right|_{x=L} = Z_L \frac{\partial w(L, t)}{\partial t},$$

where

- $Z_m = \rho_0 c$ propagation impedance
- Z_L end impedance operator

Wave equation: model

Laplace wrt t :

$$c^2 \partial_x^2 W(x, s) = s^2 W(x, s)$$

with

$$sW(0, s) = U(s) \quad \text{and} \quad sZ_L(s)W(L, s) + cZ_m \partial_x W(L, s) = 0$$

where $\partial_x f := \partial f / \partial x$ and $\partial_x^2 f := \partial^2 f / \partial x^2$. Second-order ODE solution:

$$\begin{aligned} \begin{bmatrix} W(x, s) \\ \partial_x W(x, s) \end{bmatrix} &= \exp \left(\begin{bmatrix} 0 & 1 \\ s^2/c^2 & 0 \end{bmatrix} x \right) \begin{bmatrix} W(0, s) \\ \partial_x W(0, s) \end{bmatrix} \\ &= \begin{bmatrix} \cosh(sx/c)/s & c \sinh(sx/c)/s \\ \sinh(sx/c)/c & \cosh(sx/c) \end{bmatrix} \begin{bmatrix} U(s) \\ \partial_x W(0, s) \end{bmatrix} \end{aligned}$$

where

$$\cosh(sx/c) = \frac{e^{sx/c} + e^{-sx/c}}{2} \quad \text{and} \quad \sinh(sx/c) = \frac{e^{sx/c} - e^{-sx/c}}{2}$$

Wave equation: model

Laplace wrt t :

$$c^2 \partial_x^2 W(x, s) = s^2 W(x, s)$$

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Wave equation: model (contd)

Boundary condition $sZ_L(s)W(L, s) + cZ_m\partial_x W(L, s) = 0$ reads then

$$\begin{bmatrix} Z_L(s) & Z_m \end{bmatrix} \begin{bmatrix} \cosh(sL/c) & \sinh(sL/c) \\ \sinh(sL/c) & \cosh(sL/c) \end{bmatrix} \begin{bmatrix} U(s) \\ c\partial_x W(0, s) \end{bmatrix} = 0.$$

Hence,

$$c\partial_x W(0, s) = - \overbrace{\frac{1 + R(s)e^{-2(L/c)s}}{1 - R(s)e^{-2(L/c)s}}}^{:= V_0(s)} U(s), \quad \text{where } R(s) := \frac{Z_L(s) - Z_m}{Z_L(s) + Z_m}$$

(reflectance). Thus,

$$\begin{bmatrix} sW(x, s) \\ c\partial_x W(x, s) \end{bmatrix} = \begin{bmatrix} \cosh(sx/c) & \sinh(sx/c) \\ \sinh(sx/c) & \cosh(sx/c) \end{bmatrix} \begin{bmatrix} 1 \\ V_0(s) \end{bmatrix} U(s).$$

Wave equation: model (contd)

Boundary condition $sZ_L(s)W(L, s) + cZ_m\partial_x W(L, s) = 0$ reads then

$$\begin{bmatrix} Z_L(s) & Z_m \end{bmatrix} \begin{bmatrix} \cosh(sL/c) & \sinh(sL/c) \\ \sinh(sL/c) & \cosh(sL/c) \end{bmatrix} \begin{bmatrix} U(s) \\ c\partial_x W(0, s) \end{bmatrix} = 0.$$

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Wave equation: transfer function of $u \mapsto p_0$

$G_0 : u \mapsto p_0$ has

$$G_0(s) = -V_0(s)Z_m = \frac{1 + R(s)e^{-\tau s}}{1 - R(s)e^{-\tau s}} Z_m,$$

where

$$\tau := \frac{2L}{c} > 0$$

(travel time).

Wave equation: transfer function of $u \mapsto p_0$

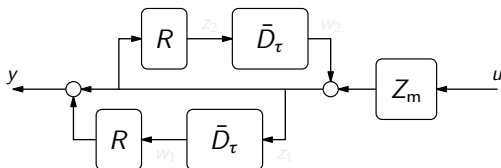
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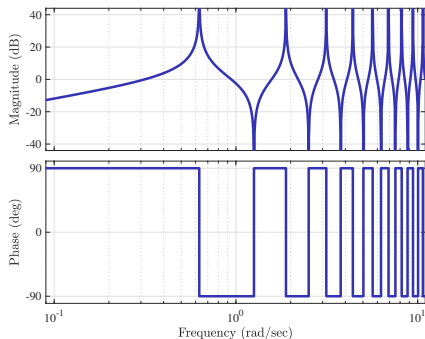
where

$$\tau := \frac{2L}{c} > 0$$

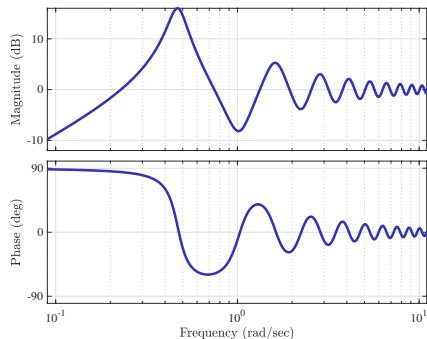
(travel time).



Wave equation: complexity



$$Z_L(s) = 0, R(s) = -1$$



$$Z_L(s) = s/(s+1), R(s) = -1/(2s+1)$$

Wave equation: transfer function of $u \mapsto p_x$

$G_x : u \mapsto p_x$ has

$$G_x(s) = \frac{e^{-\tau_1 s} + R(s)e^{-\tau_2 s}}{1 - R(s)e^{-(\tau_1 + \tau_2)s}} Z_m,$$

where

$$\tau_1 := \frac{x}{c} \quad \text{and} \quad \tau_2 := \frac{2L - x}{c}$$

(travel times of forward and reflected waves).

Wave equation: transfer function of $u \mapsto p_x$

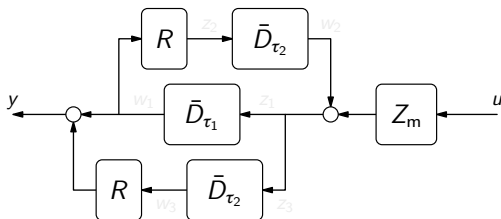
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Finite-dimensional approximations

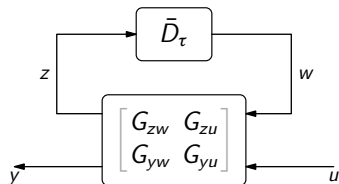
Time-delay systems so far

- pure delay: $\bar{D}_\tau$
- input delay: $P_\tau = P\bar{D}_\tau$ for a finite-dimensional P
- internal delays, like

$$G_0 = \frac{1 + R\bar{D}_\tau}{1 - R\bar{D}_\tau} Z_m \quad \text{or even} \quad G_x = \frac{\bar{D}_{\tau_1} + R\bar{D}_{\tau_2}}{1 - R\bar{D}_{\tau_1 + \tau_2}} Z_m$$

- ...

Isolating delays

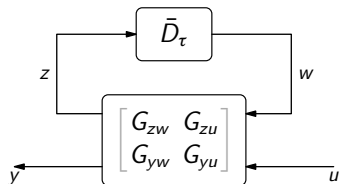


for delay-free G . The “plant” $u \mapsto y$ is

$$P_\tau = \mathcal{F}_u(G, \bar{D}_\tau) = G_{yu} + G_{yw} \bar{D}_\tau (I - G_{zw} \bar{D}_\tau)^{-1} G_{zu}.$$

Fits all we studied so far...

Isolating delays



for delay-free G . The “plant” $u \mapsto y$ is

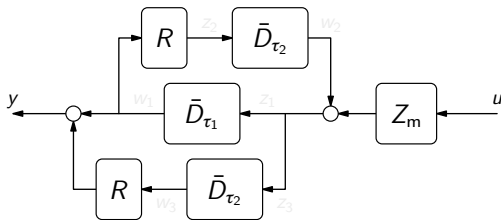
$$P_\tau = \mathcal{F}_u(G, \bar{D}_\tau) = G_{yu} + G_{yw} \bar{D}_\tau (I - G_{zw} \bar{D}_\tau)^{-1} G_{zu}.$$

Fits all we studied so far...

If

$$G_x = \frac{\bar{D}_{\tau_1} + R\bar{D}_{\tau_2}}{1 - R\bar{D}_{\tau_1 + \tau_2}} Z_m,$$

then

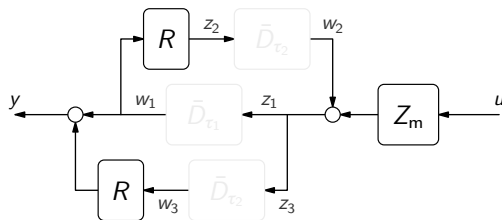


For multiple delays, with $\text{diag}\{\bar{D}_{\tau_i}\}$

If

$$G_x = \frac{\bar{D}_{\tau_1} + R\bar{D}_{\tau_2}}{1 - R\bar{D}_{\tau_1 + \tau_2}} Z_m,$$

then



Hence:

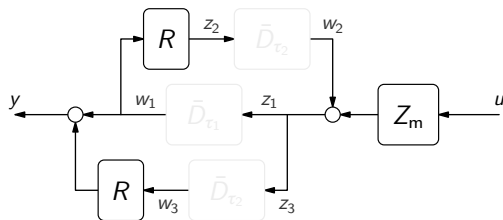
$$G = G_{\text{wave},x} := \left[\begin{array}{ccc|c} 0 & 1 & 0 & Z_m \\ R & 0 & 0 & 0 \\ 0 & 1 & 0 & Z_m \\ 1 & 0 & R & 0 \end{array} \right]$$

For multiple delays, with $\text{diag}\{\bar{D}_{\tau_i}\}$

If

$$G_x = \frac{\bar{D}_{\tau_1} + R\bar{D}_{\tau_2}}{1 - R\bar{D}_{\tau_1+\tau_2}} Z_m,$$

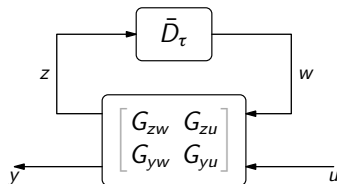
then



Hence:

$$G = G_{\text{wave},x} := \left[\begin{array}{ccc|c} 0 & 1 & 0 & Z_m \\ R & 0 & 0 & 0 \\ 0 & 1 & 0 & Z_m \\ \hline 1 & 0 & R & 0 \end{array} \right]$$

State-space representation



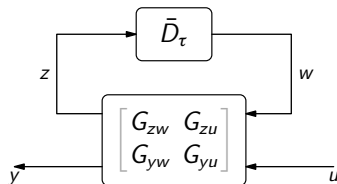
Let

$$G(s) = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array} \right].$$

Involved relations:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t) \\ z(t) = C_z x(t) + D_{yw} w(t) + D_{zu} u(t) \\ y(t) = C_y x(t) + D_{yw} w(t) + D_{yu} u(t) \end{cases} \quad \text{and} \quad w(t) = z(t - \tau).$$

State-space representation (contd)



For $P_\tau : u \mapsto y$,

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t-\tau) \end{bmatrix} + \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_y & D_{yw} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t-\tau) \end{bmatrix} + D_{yu} u(t) \end{cases}$$

called **delay-differential equation** (DDE, subclass of FDE). Its state at each t

$$(x(t), \check{z}_t) \in \mathbb{R}^n \times \{[-\tau, 0] \rightarrow \mathbb{R}^{m_\tau}\}.$$

State-space representation: special cases

Let

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t - \tau) \end{bmatrix} + \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix} u(t)$$

If

- $D_{zw} = 0$, then

$$\dot{x}(t) = Ax(t) + A_\tau x(t - \tau) + B_u u(t) + B_\tau u(t - \tau)$$

($A_\tau := B_w C_z$ and $B_\tau := B_w D_{zu}$) known as **retarded** DDE;

- $\exists E_\tau \in \mathbb{R}^{n \times n}$ such that $B_w D_{zw} = -E_\tau B_w$, then

$$\dot{x}(t) + E_\tau \dot{x}(t - \tau) = Ax(t) + A_\tau x(t - \tau) + B_u u(t) + B_\tau u(t - \tau)$$

($A_\tau := B_w C_z + E_\tau A$ and $B_\tau := B_w D_{zu} + E_\tau B_u$) known as **neutral** DDE.

State-space representation: special cases

Let

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t - \tau) \end{bmatrix} + \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix} u(t)$$

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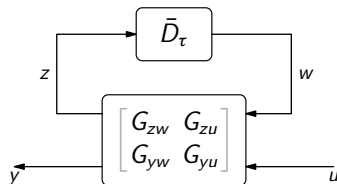
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($A_\tau := B_w C_z + E_\tau A$ and $B_\tau := B_w D_{zu} + E_\tau B_u$) known as **neutral** DDE.

Transfer function



If $(x(t), \check{z}_t) = 0$,

$$P_\tau(s) = D_{yu} + \begin{bmatrix} C_y & D_{yw}e^{-\tau s} \end{bmatrix} \begin{bmatrix} sI - A & -B_w e^{-\tau s} \\ -C_z & I - D_{zw}e^{-\tau s} \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ D_{zu} \end{bmatrix}$$

each element of which is a

— quotient of quasi-polynomials

(powers of both s and $e^{-\tau s}$).

Outline

Delay element

Delays as socializers

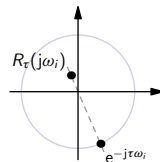
Uniform interconnection

Finite-dimensional approximations

Approximating pure delay

$\exists$ increasing sequence $\{\omega_i\}$ such that

$$\arg R_\tau(j\omega_i) - \arg e^{-j\tau\omega_i} = \pi + 2\pi k \quad \text{or}$$



Hence,

$$|\bar{D}_\tau(j\omega_i) - R_\tau(j\omega_i)| = |\bar{D}_\tau(j\omega_i)| + |R_\tau(j\omega_i)| = 1 + |R_\tau(j\omega_i)|$$

and

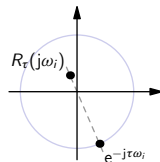
$$\|\bar{D}_\tau - R_\tau\|_\infty = \epsilon_R \geq 1 + \sup_{i \in \mathbb{N}} |R_\tau(j\omega_i)| \geq 1 = \|\bar{D}_\tau - 0\|_\infty$$

i.e. never better than $R_\tau = 0$ (makes no sense).

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Padé equality

Given $\phi(s)$, analytic around $s = 0$, construct

$$R_{m,n}(s) = \frac{P_m(s)}{Q_n(s)} = \frac{p_m s^m + \cdots + p_1 s + p_0}{q_n s^n + \cdots + q_1 s + 1}$$

(i.e. choose p_i and q_i) to

- match first $n + m + 1$ terms of Maclaurin series of $\phi(s)$

This reads

$$\begin{aligned} \phi(s) - R_{m,n}(s) &= \underbrace{\gamma_0}_{=0} + \underbrace{\gamma_1}_{=0} s + \cdots + \underbrace{\gamma_{n+m}}_{=0} s^{n+m} + \gamma_{n+m+1} s^{n+m+1} + \gamma_{n+m+2} s^{n+m+2} + \cdots \\ &= O(s^{n+m+1}) \quad \text{as } s \rightarrow 0 \end{aligned}$$

or, equivalently,

$$\phi(s) Q_n(s) - P_m(s) = O(s^{n+m+1}) \quad \text{as } s \rightarrow 0.$$

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Preliminary: multiplication of polynomials

Consider $A(s) = a_n s^n + \cdots + a_1 s + a_0$ and $B(s) = b_m s^m + \cdots + b_1 s + b_0$. If

$$C(s) = A(s)B(s) = c_{n+m} s^{n+m} + \cdots + c_1 s + c_0$$

then

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \\ c_{m+1} \\ \vdots \\ c_n \\ c_{n+1} \\ \vdots \\ c_{n+m} \end{bmatrix} = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & a_{m-1} & \cdots & a_0 \\ a_{m+1} & a_m & \cdots & a_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & \cdots & a_{n-m} \\ 0 & a_n & \cdots & a_{n-m+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Padé equality in matrix form (general)

Relation $\phi(s)Q_n(s) - P_m(s) = O(s^{m+n+1})$ reads

$$\begin{bmatrix} \phi_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \phi_1 & \phi_0 & \cdots & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_m & \phi_{m-1} & \cdots & \phi_0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1 \\ \phi_{m+1} & \phi_m & \cdots & \phi_1 & \phi_0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \phi_n & \phi_{n-1} & \cdots & \phi_{n-m} & \phi_{n-m-1} & \cdots & \phi_0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \phi_{n+m} & \phi_{n+m-1} & \cdots & \phi_n & \phi_{n-1} & \cdots & \phi_m & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ \vdots \\ q_m \\ q_{m+1} \\ \vdots \\ q_n \\ p_0 \\ p_1 \\ \vdots \\ p_m \end{bmatrix} = 0,$$

where

$$\phi_i = \frac{\phi^{(i)}(0)}{i!}$$

Padé equality in matrix form ($m = n$)

$$\left[\begin{array}{cccc|cccc} \phi_0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \phi_1 & \phi_0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_n & \phi_{n-1} & \cdots & \phi_0 & 0 & 0 & \cdots & -1 \\ \hline \phi_{n+1} & \phi_n & \cdots & \phi_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \phi_{2n} & \phi_{2n-1} & \cdots & \phi_n & 0 & 0 & \cdots & 0 \end{array} \right] \left[\begin{array}{c} 1 \\ q_1 \\ \vdots \\ q_n \\ \hline p_0 \\ p_1 \\ \vdots \\ p_n \end{array} \right] = 0.$$

$$\phi(s) = e^s \implies \phi_i = \frac{1}{i!}$$

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$$\phi(s) = e^s \quad \Longrightarrow \quad \phi_i = \frac{1}{i!}.$$

Key step

Define

$$\psi_i(t) := \frac{t^i}{i!}$$

($\psi_i(0) = 0$ and $\psi_i(1) = \phi_i$ for all $i \in \mathbb{N}$) and then

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_n(t) \end{bmatrix} := \begin{bmatrix} \psi_{n+1}(t) & \psi_n(t) & \cdots & \psi_1(t) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{2n}(t) & \psi_{2n-1}(t) & \cdots & \psi_n(t) \end{bmatrix} \begin{bmatrix} 1 \\ q_1 \\ \vdots \\ q_n \end{bmatrix}.$$

Last n rows of Padé equality read $\xi(1) = 0$. Also,

$$\psi_i(t) = \psi_{i+1}(t) \implies \xi_i(t) = \xi_{i+1}(t) \implies \xi_i(t) = \xi_n^{(n-i)}(t)$$

We thus look for

2n-order polynomial $\xi_n(t)$ such that $\xi_n^{(i)}(t) = 0$ at $t = 0$ and $t = 1$ for all $i \in \mathbb{Z}_{0,n-1}$. It is $\alpha t^\alpha (t-1)^\alpha$ for some $\alpha \neq 0$.

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Key step (contd)

Thus, coefficients satisfy

$$\xi_n(t) = \frac{t^{2n}}{(2n)!} + \sum_{i=1}^n \frac{t^{2n-i}}{(2n-i)!} q_i = \alpha t^n (t-1)^n, \quad \forall t \in [0, 1]$$

By the binomial theorem,

$$t^n (t-1)^n = \sum_{i=0}^n \binom{n}{i} (-1)^i t^{2n-i}.$$

Hence,

$$\alpha = 1/(2n)! \quad \text{and} \quad q_i = (-1)^i \binom{n}{i} \frac{(2n-i)!}{(2n)!} = (-1)^i \frac{(2n-i)! n!}{(2n)! (n-i)! i!}$$

With a little more algebra,

$$P_n(s) = Q_n(-s).$$

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Padé approximant

Using the substitution $s \rightarrow -\tau s$,

$$e^{-\tau s} \approx R_{n,n}(\tau s) = \frac{Q_n(-\tau s)}{Q_n(\tau s)}, \quad \text{where } Q_n(s) = \sum_{i=0}^n \binom{n}{i} \frac{(2n-i)!}{(2n)!} s^i$$

First 6 of them:

$$Q_1(s) = 1 + \frac{1}{2} s,$$

$$Q_2(s) = 1 + \frac{1}{2} s + \frac{1}{12} s^2,$$

$$Q_3(s) = 1 + \frac{1}{2} s + \frac{1}{10} s^2 + \frac{1}{120} s^3,$$

$$Q_4(s) = 1 + \frac{1}{2} s + \frac{3}{28} s^2 + \frac{1}{84} s^3 + \frac{1}{1680} s^4,$$

$$Q_5(s) = 1 + \frac{1}{2} s + \frac{1}{9} s^2 + \frac{1}{72} s^3 + \frac{1}{1008} s^4 + \frac{1}{30240} s^5,$$

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Properties

Lemma

$Q_n(s)$ is Hurwitz for all $n \in \mathbb{N}$.

Lemma

$|e^{-j\omega} - R_{n,n}(j\omega)| \leq 2.02|H_{2n+1}(j\omega)|$ for all $\omega \in \mathbb{R}$, where

$$H_{2n+1}(s) = \frac{(s/\omega_n)^{2n+1}}{B_{2n+1}(s/\omega_n)}$$

where

$$\omega_n := \left(2 \frac{(2n)!(2n+1)!}{(n!)^2} \right)^{1/(2n+1)}$$

and $B_k(s)$ is the k -order Butterworth polynomial, $|B_k(j\omega)|^2 = 1 + \omega^{2k}$.

Remark: Note that

$$\{\omega_n\} = \{2.5542, 4.0822, 5.7251, 7.3914, 8.9430, \dots\} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\omega_n}{n} = \sqrt[2]{2} \approx 1.4142$$

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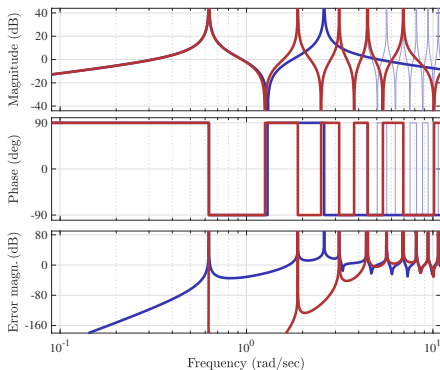
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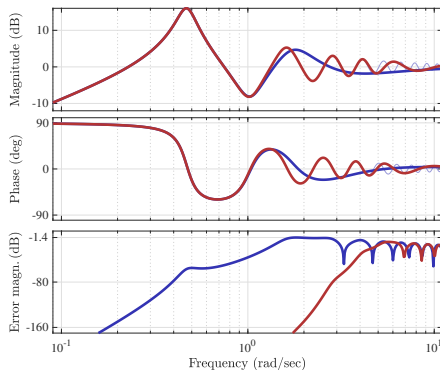
$$\{\omega_n\} = \{2.8845, 4.2823, 5.7251, 7.1814, 8.6435, \dots\} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\omega_n}{n} = \frac{4}{e} \approx 1.4715.$$

Example

[4, 4]- and [12, 12]-Padé approximants of $G_0(s) = \frac{1 + R(s)e^{-\tau s}}{1 - R(s)e^{-\tau s}}$ are



$$Z_L(s) = 0, R(s) = -1$$



$$Z_L(s) = s/(s + 1), R(s) = -1/(2s + 1)$$