



## TIME DELAYS IN CONTROL AND ESTIMATION (038806)

### HOMEWORK 1

(submission deadline: 18/6/2024, 23:59; do make an effort to be concise, clear, and accurate\*)

**Problem 1** (30pt). Form the finite-dimensional  $G$  such that the system in Fig. 1.7 with diagonal delays  $\text{diag}\{\bar{D}_{\tau_1}, \bar{D}_{\tau_0}\}$  (as explained in Remark 1.6 in the notes) generates the system

$$P_\tau = \begin{bmatrix} \bar{D}_{\tau_0} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \bar{D}_{\tau_1} \end{bmatrix}$$

for finite-dimensional  $P_{ij}$  and some  $\tau_1 > 0$  and  $\tau_0 > 0$ . Show that the resulted  $\mathcal{F}_u(G, \text{diag}\{\bar{D}_{\tau_1}, \bar{D}_{\tau_0}\}) = P_\tau$  indeed.

**Problem 2** (30pt). Derive  $R_{[2,3]}(s)$ , which is the  $[2, 3]$ -Padé approximation of

$$\phi(s) = \frac{e^{-s}}{s+1},$$

showing all steps leading to Eqn. (1.32) in the notes and solving the resulting linear equation numerically. Compute (numerically)  $\|\phi - R_{[2,3]}\|_\infty$ . Compare this quantity with  $\|\phi - \bar{R}_{[2,2]}F\|_\infty$ , where  $\bar{R}_{[2,2]}(s)$  is the  $[2, 2]$ -Padé approximation of  $e^{-s}$  and  $F(s) = 1/(s+1)$ . Try to explain differences.

**Problem 3** (40pt). Consider the function

$$\phi(s) = \frac{e^{-1} - e^{-s}}{s-1}.$$

1. Show that  $\phi \in H_\infty$
2. Approximate  $\phi(s)$  as

$$\phi_1(s) = \frac{e^{-1} - R_{2,2}(s)}{s-1},$$

where  $R_{2,2}(s)$  is the  $[2, 2]$ -Padé approximant of  $e^{-s}$ , as in §1.3.1. Does the resulting 3-order  $\phi_1(s)$  belong to  $H_\infty$ ?

3. Approximate  $\phi(s)$  as

$$\phi_2(s) = \frac{e^{-1} - R_{2,2,1}(s)}{s-1},$$

where  $R_{2,2,1}(s)$  is the  $[2, 2]$ -Padé approximant of  $e^{-s}$  with the additional (interpolation) constraint that  $R_{2,2,1}(1) = 1/e$ . How many Maclaurin series terms of  $\phi(s)$  can we match now with such  $\phi_2$ ? Does the resulting 3-order  $\phi_2(s)$  belong to  $H_\infty$ ?

4. Approximate  $\phi(s)$  with its  $[2, 3]$ -Padé approximant  $\phi_3(s)$ . Does the resulting 3-order  $\phi_3(s)$  belong to  $H_\infty$ ?

Plot the Bode magnitude plots of  $|\phi(j\omega) - \phi_i(j\omega)|$  for each  $i \in \mathbb{Z}_{1..3}$ .

**Problem 4** (10pt, bonus). Consider time-varying delay element  $\bar{D}_{\tau(t)}$  defined as

$$y(t) = \bar{D}_{\tau(t)}u(t) \iff y(t) = u(t - \tau(t))$$

for some function  $\tau(t) \in [0, 0.1]$ . Is  $\bar{D}_{\tau(t)}$  necessarily bounded as an operator  $L_2(\mathbb{R}_+) \rightarrow L_2(\mathbb{R}_+)$ ?

\*Computations may be done with whatever numerical or symbolic tools. But the logic should be explained and proofs given.