

Sampled-Data Control Systems (036709)

lecture no. 13

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT



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Outline

Quantization effects

Oversampling in A/D conversion

State feedback H_∞

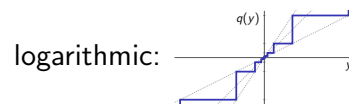
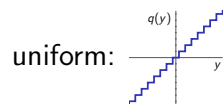
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Quantization

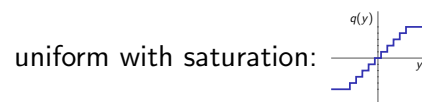
Digital signals were studied a bit simplistically in this course. In real-world applications signals processed by digital computers / transmitted by digital networks are not only live on discrete time, but also have

- finite precision (i.e. finite number of bits).

A (static scalar) quantizer is a function $q : \mathbb{R} \rightarrow \mathbb{Q} \subset \mathbb{R}$, for some countable (or even finite) set $\mathbb{Q} = \{q_i\}$. Examples of quantizers:



In practical situations $\mathbb{Q}$ is finite, having $\mu \in \mathbb{N} \setminus \{1\}$ distinct values, like



In this case quantized signal can be represented by $\lceil \log_2 \mu \rceil$ bits.

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Uniform quantization

This type is probably the most widespread. Its idealized version is

$$q(y) = \left\lfloor \frac{y}{\delta_q} + \frac{1}{2} \right\rfloor \delta_q : \text{graph of } q(y) \text{ showing a staircase function with step size } \delta_q$$

where δ_q is the step size. This is a **nonlinear** system, which is seen from

$$q(\alpha \delta_q) = \delta_q \neq \alpha \delta_q = \alpha q(\delta_q), \quad \forall \alpha \in (1, 1.5).$$

Moreover, if $\mathbb{Q}$ is **finite**, like in

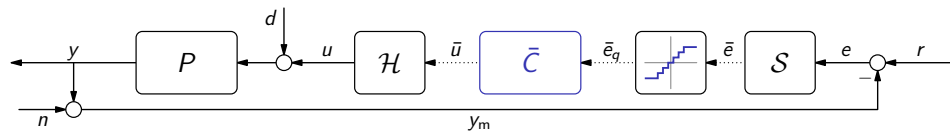
$$q(y) = \text{sat} \left(\left\lfloor \frac{y}{\delta_q} + \frac{1}{2} \right\rfloor \delta_q \right) : \text{graph of } q(y) \text{ showing a staircase function with saturation}$$

then a nonlinear **saturation** element is effectively a part of the quantizer.

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Uniform quantization in control

Because of that, the presence of a quantization element in control loop, like



might have far-reaching consequences, e.g.

- loss of stability,
- limit cycles,
- nonzero steady-state errors,

et cetera.

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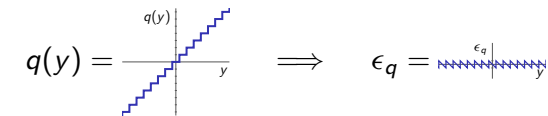
Quantization error

The mismatch between the input and output of a quantizer,

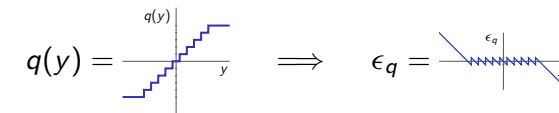
$$\epsilon_q = q(y) - y$$

is known as the quantization error.

- If $q(y)$ does not saturate, then ϵ_q is bounded, like $|\epsilon_q| \leq \delta_q/2$ in



- If $q(y)$ saturates, then ϵ_q is unbounded, like in



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Additive noise model

If the signal y does not saturate in the quantizer, then



i.e. we can replace a uniform quantizer with a bounded additive noise. It is known that if y is “sufficiently rich,” then ϵ_q

- is a zero-mean, i.e. $E[\epsilon_q] = 0$, white process independent of y ,
- has variance (power) $E[|\epsilon_q|^2] = \delta_q^2/12$.

In other words, under that assumption

$$|E_q(e^{j\theta})| = \frac{\delta_q}{2\sqrt{3}} \approx 0.2887 \delta_q$$

at every frequency $\theta \in [-\pi, \pi]$. This is a convenient model, as it

- facilitates the incorporation of quantizers into standard linear analysis.

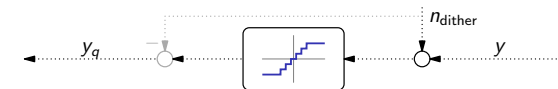
Yet it is valid only under negligible saturation and sufficiently rich signals.

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Dither

If y is not expected to be sufficiently rich, we may

- add artificial noise n_{dither} , uniformly distributed in $[-\delta_q/2, \delta_q/2]$, to y like



to “enrich” the input to the quantizer. If technically possible, the very same signal can then be subtracted from the quantized version. This procedure is known as **dithering**. Although its use might

- ☹ corrupt measurements,

it can help to

- ☺ whiten the quantization error ϵ_q and decorrelate it from y .

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What if the additive noise model is inadequate?

In simple terms, it could be a pain, with the need to dig deep into nonlinear dynamics. Some exemplars:

- not every system can even be stabilized if $\mathbb{Q}$ is finite
(e.g. $x[t+1] = ax[t] + bu[t]$ can be stabilized by μ -level quantizer only if $\mu > |a|$)
- logarithmic or dynamic quantizers may be advantageous
- nonlinear control laws may be advantageous

Still, tangible solutions tend to address stripped-down problems, where plant dynamics are of low order and control goals are simplistic.

It may be that the most practical solution is to bet on hardware progress to make related problems obsolete . . .

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Quantization effects

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State feedback H_∞

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What quantization resolution we need?

In theory, as high as possible. However, high-resolution quantizers might be

- expensive to construct
(hinge on the precision of analog components)
- fragile to operate
(might be vulnerable to noise and interference, thermal conditions, et cetera)

Oversampling enables to overcome these difficulties. Namely

- ☺ high-resolution converters can be built from low-fidelity components with the use of more sophisticated *digital* signal processing. However, it
- ☹ introduces large time lags

to the conversion, hindering their use in feedback control applications. We thus study them not because of their applicability, but rather because their operating principles are good examples of the

- use of sampling and feedback ideas in solving an engineering problem.

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A/D conversion with quantization

Let y be an analog signal to be sampled with some sampling period $h > 0$. Assume that

1. y is ω_N -bandlimited, i.e. $Y(j\omega) = 0$ for all $|\omega| \geq \omega_N := \pi/h$, and
2. y is rich enough to have white and decorrelated ϵ_q (or dither is added).

As no frequencies alias, the ideal sampling results in $\bar{y} = \mathcal{S}_{\text{id}} y$ such that

$$\bar{Y}(e^{j\theta}) = \frac{1}{h} Y\left(j\frac{\theta}{h}\right).$$

If the signal is also quantized, we may think of the discrete spectrum as

$$\bar{Y}_q(e^{j\theta}) = \frac{1}{h} Y\left(j\frac{\theta}{h}\right) + E_q(e^{j\theta}),$$

where $|E_q(e^{j\theta})| = \delta_q/(2\sqrt{3}) = \text{const}$ for all θ

Remark: Note that the signal ϵ_q is intrinsically discrete, it is not a sampled version of an analog signal.

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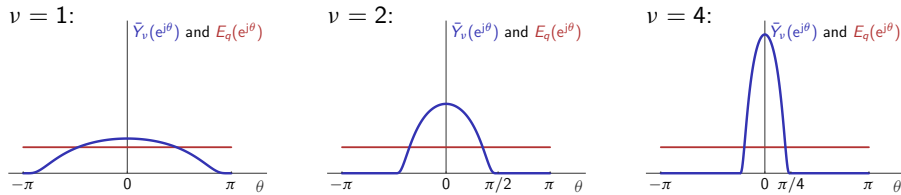
Effect of oversampling

Assume now that we sample y with the period h/ν for some $\nu \in \mathbb{N}$. In this case

$$\bar{Y}_q(e^{j\theta}) = \frac{\nu}{h} Y\left(j\nu \frac{\theta}{h}\right) + E_q(e^{j\theta}) =: \bar{Y}_\nu(e^{j\theta}) + E_q(e^{j\theta})$$

and ϵ_q is still white and with the same intensity, $\delta_q/(2\sqrt{3})$, which depends on the quantization resolution, but not on ν . As ν increases,

- $\bar{Y}_\nu(e^{j\theta})$ grows and squeezes into narrower range $\theta \in [-\pi/\nu, \pi/\nu]$,
- $\bar{E}_q(e^{j\theta})$ remains the same.



Consequently, as ν grows, the signal of interest, $\bar{Y}_\nu(e^{j\theta})$, becomes more and more dominant over $\bar{E}_q(e^{j\theta})$ in the frequencies $\theta \in [-\pi/\nu, \pi/\nu]$.

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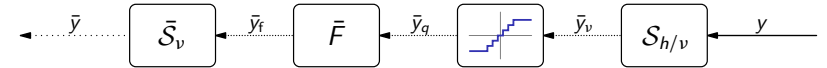
Effect of oversampling: use

So all we need now is to

- filter out components of $\bar{Y}_q(e^{j\theta})$ outside $\theta \in [-\pi/\nu, \pi/\nu]$ to get $\bar{y}_f$ (this is *digital* filtering, so can be implemented reliably)
- take every ν th instance of $\bar{y}_f$ to end up with $\bar{y}$ (this $\bar{y}$ is synchronized with ih , as required)

downsampling

The whole scheme is



and decreases the intensity of quantization noise by up to a factor of $1/\sqrt{\nu}$ (achievable if the ideal low-pass filter $\bar{F}$ is used). Equivalently, the effective quantization step becomes $\delta_q/\sqrt{\nu}$.

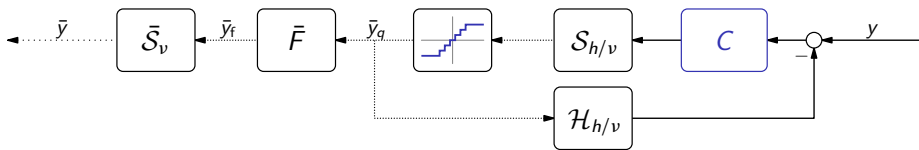
But we can do

- even better by more sophisticated processing of $\bar{y}_q$.

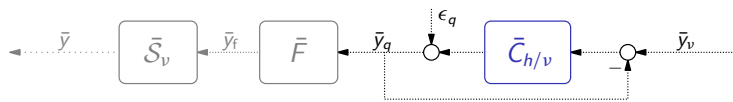
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$\Delta\Sigma$ conversion

The idea is to put the quantizer into the following feedback loop:



and design its “controller” C , which acts on the mismatch between y and $\mathcal{H}_{h/\nu}\bar{y}_q$. This system can be analyzed in the form



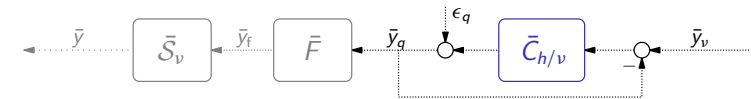
where

$$\bar{C}_{h/\nu} = S_{h/\nu} C \mathcal{H}_{h/\nu}$$

and ϵ_q represents the quantization error.

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$\Delta\Sigma$ conversion: linear analysis



It is readily seen that

$$\bar{y}_q = \frac{1}{1 + \bar{C}_{h/\nu}} \epsilon_q + \frac{\bar{C}_{h/\nu}}{1 + \bar{C}_{h/\nu}} \bar{y}_\nu =: \bar{S} \epsilon_q + \bar{T} \bar{y}_\nu,$$

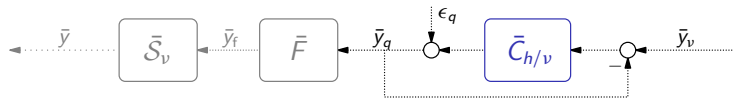
and our goals here are to

- keep $\bar{y}_\nu$ untouched in low frequencies (in already familiar $[-\pi/\nu, \pi/\nu]$)
- reduce the effect of ϵ_q in low frequencies

This looks like the sensitivity / complementary sensitivity game in classical feedback control, just the “plant” is elementary, $\bar{P} = 1$. Thus, the problem of designing an appropriate $\bar{C}_{h/\nu}$ appears simple.

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$\Delta\Sigma$ conversion: linear analysis (contd)



It makes sense to use an integral action in $\bar{C}_{h/v}$. For example, consider

$$\bar{C}_{h/v}(z) = \frac{k}{z-1} \implies \bar{S}(z) = \frac{z-1}{z-(1-k)}, \quad \bar{T}(z) = \frac{k}{z-(1-k)},$$

which are stable iff $k \in (0, 2)$. If $k = 1$, then $\bar{T}(z) = 1/z$, i.e. it is merely a delay, and

$$|\bar{S}(e^{j\theta})| = |1 - e^{-j\theta}| = \sqrt{2(1 - \cos \theta)} =$$

Hence, we can further reduce the noise intensity in $[-\pi/\nu, \pi/\nu]$,

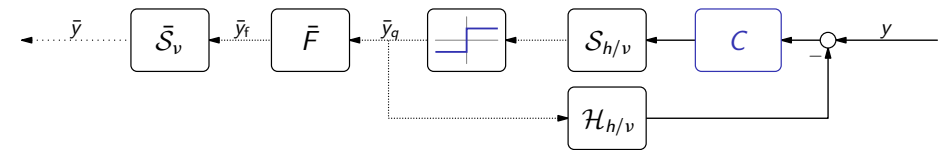
- effectively rendering $\bar{y}_q[t] = \bar{y}_v[t-1]$ regardless of δ_q .

So we can take $\mathbb{Q} = \{-\delta_q/2, \delta_q/2\}$, only **one bit** (super easy to implement).

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$\Delta\Sigma$ conversion: nonlinear analysis

But it is not that easy to analyze. The actual loop is nevertheless



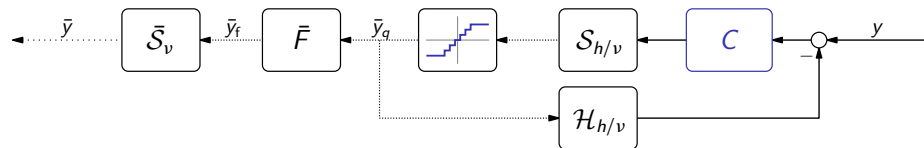
which is a nonlinear sampled-data system, whose

- stability is not to be taken for granted.

It seems that the closed-loop system is stable with $\bar{C}_{h/v}(z) = 1/(z-1)$, but this might no longer be true for more complex $\bar{C}_{h/v}$.

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$\Delta\Sigma$ conversion: tuning parameters



We can

- increase oversampling rate ν
(transparent effect, simple analysis, expensive implementation)
- choose higher-order dynamics of $\bar{C}_{h/v} = S_{h/v} C H_{h/v}$
(transparent effect, nontrivial stability analysis, relatively simpler implementation)
- add quantization levels to $\mathbb{Q}$ (multibit)
(transparent effect, nontrivial stability analysis)

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H_∞ norm

We know (Lect. 2, Slide 33) that a discrete system $G : u \mapsto y$ is ℓ_2 -stable if $u \in \ell_2 \implies y \in \ell_2$ and $\|y\|_2 \leq \gamma \|u\|_2$. The smallest γ for which the above inequality holds for all $u \in \ell_2$ is dubbed the ℓ_2 -induced norm of G , denoted $\|G\|_{\ell_2 \rightarrow \ell_2}$. If G is LSI and causal, then it is ℓ_2 -stable iff its transfer function $G(z)$ is holomorphic and bounded in $z \in \mathbb{C} \setminus \bar{\mathbb{D}}_1$, i.e. belongs to H_∞ , and

$$\|G\|_{\ell_2 \rightarrow \ell_2} = \|G\|_\infty := \sup_{z \in \mathbb{C} \setminus \bar{\mathbb{D}}_1} \|G(z)\| = \max_{\theta \in [-\pi, \pi]} \|G(e^{j\theta})\|.$$

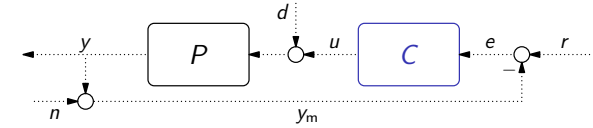
The quantity $\|G\|_\infty$ is known as the H_∞ norm of G , with

$$\|G\|_\infty < \gamma \iff \|G(e^{j\theta})\| < \gamma \quad \forall \theta \in [-\pi, \pi].$$

Control problems, in which a controller is designed to guarantee $\|T_{cl}\|_\infty < \gamma$ for a closed-loop system T_{cl} of interest and a given $\gamma > 0$, are known as H_∞ optimization problems.

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Some common H_∞ problems



- weighted sensitivity: $\|W_\sigma S\|_\infty < 1$ for a stable weight W_σ
(if successful, guarantees $|S(e^{j\theta})| < 1/|W_\sigma(e^{j\theta})|$ for all frequencies $\theta \in [-\pi, \pi]$)
- mixed sensitivity: $\left\| \begin{bmatrix} W_\sigma S \\ W_\pi T_c \end{bmatrix} \right\|_\infty < 1$ for stable weights W_σ and W_π
(if successful, guarantees $|S(e^{j\theta})| < 1/|W_\sigma(e^{j\theta})|$ and $|T_c(e^{j\theta})| < 1/|W_\pi(e^{j\theta})|$ for all frequencies $\theta \in [-\pi, \pi]$)
- balanced sensitivity:

$$\left\| \begin{bmatrix} S & T_d W \\ T_c/W & T \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} 1 \\ C/W \end{bmatrix} (1 + PC)^{-1} \begin{bmatrix} 1 & PW \end{bmatrix} \right\|_\infty < \gamma$$

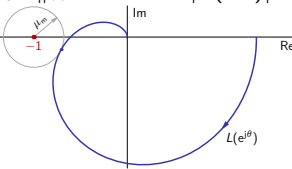
for a weight W (optimization in the H_∞ loop shaping procedure)

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Advantages of the H_∞ paradigm

Basically, two:

- guaranteed robustness
e.g. if $W_\sigma = \mu_m$, then $\|W_\sigma S\|_\infty < 1 \iff |S(e^{j\theta})| < 1/\mu_m$, for all θ , guaranteeing the modulus margin μ_m :



- guaranteed shaping of closed-loop frequency responses
e.g. if $W_\sigma(e^{j\theta}) = 1/\epsilon$ for $\theta \in [0, \theta_b]$ and $W_\sigma(e^{j\theta}) = \mu_m$ for $\theta \in (\theta_b, \pi]$, then

$$\|W_\sigma S\|_\infty < 1 \iff |S(e^{j\theta})| < \begin{cases} \epsilon & \theta \in [0, \theta_b] \\ 1/\mu_m & \theta \in (\theta_b, \pi] \end{cases}$$

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Preliminary: H_∞ DARE

Let

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix} \in \mathbb{R}^{(n+p) \times (n+m_1+m_2)}, \quad E_1 = \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0_{m_2} \end{bmatrix},$$

and $\gamma \geq 0$ be given. The matrix equation

$$X = A'XA + C'C - (C'D + A'XB)(D'D - \gamma^2 E_1 + B'XB)^{-1}(D'C + B'XA)$$

is dubbed the **H_∞ discrete-time algebraic Riccati equation** (H_∞ DARE). Its solution $X \in \mathbb{R}^{n \times n}$ is said to be stabilizing if $\det(D'D + B'XB - \gamma^2 E_1) \neq 0$ and

$$A_F := A - B(D'D - \gamma^2 E_1 + B'XB)^{-1}(D'C + B'XA)$$

is Schur. The stabilizing X , if exists, is unique¹ and such that $X = X'$.

¹MATLAB: `idare(A,B,C'*C,D'*D-gamma^2*E1,C'*D)`, since R2019a.

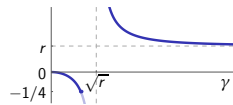
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H_∞ DARE: example

Consider

$$\begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & \sqrt{r} \end{bmatrix}$$

for some $r \geq 0$. The corresponding H_∞ DARE is

$$\frac{X^2 - X - r_\gamma}{X + r_\gamma} = 0, \quad \text{where } r_\gamma := \frac{r\gamma^2}{\gamma^2 - r}$$


and the corresponding

$$A_F = \frac{r_\gamma}{X + r_\gamma}$$

If $r_\gamma = 0$, then this is a linear equation, $X = 1$. Its solution is stabilizing, as $A_F = 0$ for it. If $r_\gamma \neq 0$, then this is a quadratic equation $X^2 - X - r_\gamma = 0$, so

$$X = \frac{1}{2} \pm \sqrt{r_\gamma + \frac{1}{4}} \implies A_F = \frac{r_\gamma}{r_\gamma + 1/2 \pm \sqrt{r_\gamma + 1/4}}.$$

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H_∞ DARE: example (contd)

Three cases may be considered for $X = 1/2 \pm \sqrt{r_\gamma + 1/4}$:

1. if $-1/4 \leq r_\gamma < 0$, then both solutions are real, positive, and only

$$A_F = \frac{r_\gamma}{r_\gamma + 1/2 + \sqrt{r_\gamma + 1/4}}$$

is Schur iff $r_\gamma \neq -1/4$, so that $X = 1/2 + \sqrt{r_\gamma + 1/4}$ is stabilizing;

2. if $r_\gamma < -1/4$, then no real solutions exist
3. if $r_\gamma \geq r$, then both solutions are real, only one is positive, and only

$$A_F = \frac{r_\gamma}{r_\gamma + 1/2 + \sqrt{r_\gamma + 1/4}}$$

is Schur iff $r_\gamma \neq \infty$, so that $X = 1/2 + \sqrt{r_\gamma + 1/4}$ is again stabilizing.

Thus, the stabilizing solution exists iff ($r_\gamma = -1/4 \iff \gamma = \sqrt{r/(1+4r)}$)

$$\gamma \in \left[0, \sqrt{\frac{r}{1+4r}}\right) \cup (\sqrt{r}, \infty).$$

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H_∞ DARE: another property

Let

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -(D'D - \gamma^2 E_1 + B'XB)^{-1}(D'C + B'XA),$$

where the partitioning is compatible with that of $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$. We have:

$$(D_2'D_2 + B_2'XB_2 > 0) \wedge (D_1'D_1 + B_1'XB_1 < \gamma^2 I) \implies A + B_2F_2 \text{ is Schur}$$

and not only $A_F = A + B_1F_1 + B_2F_2$.

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H_∞ DARE: example (contd)

In the example above,

$$D_2'D_2 + B_2'XB_2 = \frac{1}{2} + r + \frac{\sqrt{1+4r_\gamma}}{2} > 0$$

and

$$D_1'D_1 + B_1'XB_1 = \frac{1 + \sqrt{1+4r_\gamma}}{2} < \gamma^2 \iff \gamma > \sqrt{1+r} > \sqrt{r}.$$

Now, there are two intervals of γ for which a stabilizing X exists:

- if $\gamma \in [0, \sqrt{r/(1+4r)})$, then

$$A + B_2F_2 = 1 + \frac{1 - \sqrt{1 - 4\gamma^2 r / (r - \gamma^2)}}{2r} > 1 \implies \text{not Hurwitz};$$

- if $\gamma > \sqrt{r}$, then

$$A + B_2F_2 = 1 - \frac{\sqrt{1 + 4\gamma^2 r / (\gamma^2 - r)} - 1}{2r} < 1$$

and $A + B_2F_2 > -1$ iff $\gamma > \sqrt{2r(1+2r)/(1+4r)} \in [\sqrt{r}, \sqrt{1+r})$.

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H_∞ DARE as Schur complement

Let

$$\begin{bmatrix} \Phi_{11}(X) & \Phi_{12}(X) \\ \Phi_{21}(X) & \Phi_{22}(X) \end{bmatrix} := \begin{bmatrix} C'C + A'XA - X & C'D + A'XB \\ D'C + B'XA & D'D - \gamma^2 E_1 + B'XB \end{bmatrix} \\ = \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} X \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & \gamma^2 E_1 \end{bmatrix}.$$

The H_∞ DARE reads then, much like the LQR DARE,

$$\Phi_{11}(X) - \Phi_{12}(X)\Phi_{22}^{-1}(X)\Phi_{21}(X) = 0,$$

i.e. DARE solutions zero the Schur complement of $\Phi_{22}(X)$. If $X = X'$ solves the DARE, then for all compatibly dimensioned x and u

$$\begin{bmatrix} x' & u' \end{bmatrix} \begin{bmatrix} \Phi_{11}(X) & \Phi_{12}(X) \\ \Phi_{21}(X) & \Phi_{22}(X) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \|u - Fx\|_{\Phi_{22}(X)}^2,$$

where $F = -\Phi_{22}^{-1}(X)\Phi_{21}(X)$ and for $M = M'$ we denote $\|v\|_M^2 := v'Mv$.

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State-feedback H_∞ problem statement

Consider

$$\begin{cases} x[t+1] = Ax[t] + B_w w[t] + B_u u[t], & x[0] = 0 \\ z[t] = C_z x[t] + D_{zw} w[t] + D_{zu} u[t] \end{cases}$$

where

z is the regulated signal, it reflects control performance requirements

w is the exogenous input

u is the control input, which we can use to re

The discrete-time state-feedback H_∞ problem:

- supposing that the state vector $x[t]$ is measurable,
- design $u[t]$, stabilizing the system and guaranteeing that

$$\|T_{zw}\|_\infty < \gamma \iff \|z\|_2 < \gamma \|w\|_2, \quad \forall w \in \ell_2, w \neq 0$$

for a given $\gamma > 0$.

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Discrete-time H_∞ problem: assumptions

Standard assumptions:

$\mathcal{A}_1$: (A, B) is stabilizable

$\mathcal{A}_2$: $\begin{bmatrix} A - e^{j\theta}I & B \\ C_z & D_z \end{bmatrix}$ has full column rank $\forall \theta \in [-\pi, \pi]$

Assumption

- $\mathcal{A}_1$ is necessary for the existence of stabilizing control law.

Assumption

- $\mathcal{A}_2$ guarantees the existence of the DARE solution for $\gamma \rightarrow \infty$ and is technical.

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Necessary solvability condition

The discrete-time H_∞ problem is solvable **only if** the H_∞ DARE

$$X = A'XA + C_z'C_z - \begin{bmatrix} C_z'D_w + A'XB_w & C_z'D_u + A'XB_u \end{bmatrix} \\ \times \begin{bmatrix} D_w'D_w + B_w'XB_w - \gamma^2 I & D_w'D_u + B_w'XB_u \\ D_u'D_w + B_u'XB_w & D_u'D_u + B_u'XB_u \end{bmatrix}^{-1} \begin{bmatrix} D_w'C_z + B_w'XA \\ D_u'C_z + B_u'XA \end{bmatrix}$$

has a stabilizing solution X such that

1. $X = X' \geq 0$,
2. $D_u'D_u + B_u'XB_u > 0$,
3. $D_w'D_w + B_w'XB_w < \gamma^2 I$.

If $\mathcal{A}_{1,2}$ hold, then these conditions hold whenever γ is sufficiently large.

Remark: As a matter of fact, if the last condition is replaced with the milder

- 3'. $D_w'D_w + B_w'XB_w - (D_w'D_u + B_w'XB_u)(D_u'D_u + B_u'XB_u)^{-1}(D_u'D_w + B_u'XB_w) < \gamma^2 I$,

then the corresponding *full-information* problem is solvable, in which we can measure not only $x[t]$, but also $w[t]$.

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Completing squares

Assuming now that the DARE has a stabilizing solution, define

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12,w} & R_{12,u} \\ R_{21,w} & R_{22,ww} & R_{22,wu} \\ R_{21,u} & R_{22,uw} & R_{22,uu} \end{bmatrix}$$

$$:= \begin{bmatrix} C_z' \\ D_{zw}' \\ D_{zu}' \end{bmatrix} \begin{bmatrix} C_z & D_{zw} & D_{zu} \end{bmatrix} + \begin{bmatrix} A' \\ B_w' \\ B_u' \end{bmatrix} X \begin{bmatrix} A & B_w & B_u \end{bmatrix} - \begin{bmatrix} X & 0 & 0 \\ 0 & \gamma^2 I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By this definition, and taking into account that

$$x[t+1] = \begin{bmatrix} A & B_w & B_u \end{bmatrix} \begin{bmatrix} x[t] \\ w[t] \\ u[t] \end{bmatrix} \quad \text{and} \quad z[t] = \begin{bmatrix} C_z & D_{zw} & D_{zu} \end{bmatrix} \begin{bmatrix} x[t] \\ w[t] \\ u[t] \end{bmatrix},$$

we get

$$\alpha[t] := \left\| \begin{bmatrix} x[t] \\ w[t] \\ u[t] \end{bmatrix} \right\|_R^2 = \|z[t]\|^2 - \gamma^2 \|w[t]\|^2 + \|x[t+1]\|_X^2 - \|x[t]\|_X^2.$$

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Completing squares (contd)

But by the property of every DARE to zero the Schur complement of R_{22} ,

$$\alpha[t] = \left\| \begin{bmatrix} w[t] \\ u[t] \end{bmatrix} - \begin{bmatrix} F_w \\ F_u \end{bmatrix} x[t] \right\|_{R_{22}}^2,$$

where

$$\begin{bmatrix} F_w \\ F_u \end{bmatrix} := - \overbrace{\begin{bmatrix} D_w' D_w + B_w' X B_w - \gamma^2 I & D_w' D_u + B_w' X B_u \\ D_u' D_w + B_u' X B_w & D_u' D_u + B_u' X B_u \end{bmatrix}}^{R_{22}}^{-1} \overbrace{\begin{bmatrix} D_w' C_z + B_w' X A \\ D_u' C_z + B_u' X A \end{bmatrix}}^{R_{21}}$$

Equating these two expressions for $\alpha[t]$ we have that the equality

$$\|z[t]\|^2 - \gamma^2 \|w[t]\|^2 + \|x[t+1]\|_X^2 - \|x[t]\|_X^2 = \left\| \begin{bmatrix} w[t] - F_w x[t] \\ u[t] - F_u x[t] \end{bmatrix} \right\|_{R_{22}}^2$$

holds for all u and w . Hence, whenever u renders $\lim_{t \rightarrow \infty} x[t] = 0$, we have

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = \sum_{t \in \mathbb{Z}_+} \begin{bmatrix} w[t] - F_w x[t] \\ u[t] - F_u x[t] \end{bmatrix}' R_{22} \begin{bmatrix} w[t] - F_w x[t] \\ u[t] - F_u x[t] \end{bmatrix}.$$

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Strategy for u

By the necessary solvability conditions, $R_{22,ww} < 0$ (nonsingular), so that

$$R_{22} = \begin{bmatrix} I & 0 \\ R_{22,uw} R_{22,ww}^{-1} & I \end{bmatrix} \begin{bmatrix} R_{22,ww} & 0 \\ 0 & \Delta_w \end{bmatrix} \begin{bmatrix} I & R_{22,ww}^{-1} R_{22,wu} \\ 0 & I \end{bmatrix},$$

where $\Delta_w := R_{22,uu} - R_{22,uw} R_{22,ww}^{-1} R_{22,wu}$ (the Schur complement). By the same conditions, $R_{22,uu} > 0$, so that $\Delta_w > 0$. Thus,

$$\begin{bmatrix} w[t] - F_w x[t] \\ u[t] - F_u x[t] \end{bmatrix}' R_{22} \begin{bmatrix} w[t] - F_w x[t] \\ u[t] - F_u x[t] \end{bmatrix} = \|\Delta_w^{1/2} (u[t] - F_u x[t])\|^2$$

$$- \|(-R_{22,ww})^{1/2} (w[t] - F_w x[t] + R_{22,ww}^{-1} R_{22,wu} (u[t] - F_u x[t]))\|^2$$

and then

$$\|z\|_2^2 - \gamma^2 \|w\|_2^2 = \|\Delta_w^{1/2} (u - F_u x)\|_2^2$$

$$- \|(-R_{22,ww})^{1/2} (w - F_w x + R_{22,ww}^{-1} R_{22,wu} (u - F_u x))\|_2^2.$$

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Strategy for u (contd)

Remember, our goal is to guarantee

$$\|z\|_2^2 < \gamma^2 \|w\|_2^2, \quad \forall w \in \ell_2, w \neq 0.$$

This is equivalent to keeping

$$\|\Delta_w^{1/2} (u - F_u x)\|_2^2 < \|(-R_{22,ww})^{1/2} (w - F_w x + R_{22,ww}^{-1} R_{22,wu} (u - F_u x))\|_2^2$$

for all such w . A safe bet is to minimize the **right-hand side** by choosing

$$u[t] = F_u x[t],$$

which is stabilizing (as $A + B_u F_u$ is Schur) and for which

$$\|z\|_2^2 = \gamma^2 \|w\|_2^2 - \|(-R_{22,ww})^{1/2} (w - F_w x)\|_2^2 \leq \gamma^2 \|w\|_2^2$$

and the *equality* above holds only if the disturbance $w[t] = F_w x[t]$.

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Strategy for w

The choice

$$w[t] = w_*[t] := F_w x[t]$$

is the only hope for w to spoil the success of u . But if this is the case, the closed-loop dynamics are

$$x[t+1] = Ax[t] + B_w F_w x[t] + B_u F_u x[t] = A_F x[t],$$

which yields $x \equiv 0$ because $x[0] = 0$ by assumption. Hence, we end up with $w \equiv 0$, so that

$$\|z\|_2^2 < \gamma^2 \|w\|_2^2, \quad \forall w \in \ell_2, w \neq 0$$

as required. Still, the signal w_* may be seen as the **worst-case disturbance**, because any deviation from it reduces $\|z\|$,

$$\|z\|_2^2 = \gamma^2 \|w\|_2^2 - \|(-R_{22,ww})^{1/2}(w - w_*)\|_2^2.$$

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Solution

Theorem

If $\mathcal{A}_{1,2}$ hold, then the state-feedback H_∞ problem for a given γ is solvable iff the H_∞ DARE has a stabilizing solution X such that

$$X = X' \geq 0, \quad D_u' D_u + B_u' X B_u > 0, \quad \text{and} \quad D_w' D_w + B_w' X B_w < \gamma^2 I.$$

In this case the state feedback control law

$$u[t] = F_u x[t]$$

solves the problem and renders

$$\|z\|_2^2 = \gamma^2 \|w\|_2^2 - \|(\gamma^2 I - D_w' D_w - B_w' X B_w)^{1/2}(w - F_w x)\|_2^2,$$

where

$$\begin{bmatrix} F_w \\ F_u \end{bmatrix} = - \begin{bmatrix} D_w' D_w + B_w' X B_w - \gamma^2 I & D_w' D_u + B_w' X B_u \\ D_u' D_w + B_u' X B_w & D_u' D_u + B_u' X B_u \end{bmatrix}^{-1} \begin{bmatrix} D_w' C_z + B_w' X A \\ D_u' C_z + B_u' X A \end{bmatrix}.$$

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