

Sampled-Data Control Systems (036709)

lecture no. 12

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Spatially distributed systems

The availability of powerful and cheap sensing / actuation / data processing technologies made

- spatial distribution of various components of feedback control systems technologically feasible. This is advantageous in many applications, e.g.
- telerobotics (e.g. remote surgeries, hazardous environments, etc)
- smart power grids
- traffic control
- transport systems, like platooning
- ...

From the control point of view, these realities present new opportunities (in facilitating better sensing and actuating), but pose new challenges in

- the need to handle large-scale systems,
- the need to account for communication networks and their limitations.

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Control over communication networks

Challenges:

- network transmission expenses,
(both direct and indirect, like effects of transmission on power consumption)
- intermittent sampling,
- variable network-induced delays,
- information losses via packet dropout,
- communication capacity limitations,
- security issues,

as well as decentralization (which is less related to sampled-data systems).

All this

- prompts new formulations and
- calls for new tools.

This lecture aims at providing a flavor of handling the first two issues.

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Outline

Robustness to variable sampling times

Emulation-based architecture

Event-triggered control

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A toy problem to start with

Consider

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and assume that $u(t) = Fx(t)$ is satisfactory, with Hurwitz $A_F := A + BF$. Apply it now digitally, as

$$u(t) = Fx(t_i), \quad \forall t \in [t_i, t_{i+1})$$

i.e. as $u = \mathcal{H}_{\text{ZOH}} F \mathcal{S}_{\text{idl}}$ under intermittent sampling sequence $\{t_i\}$. Our goal is to

- find under what conditions on $\{t_i\}$ the closed-loop system

$$\dot{x}(t) = Ax(t) + BFx(t_i), \quad \forall t \in [t_i, t_{i+1})$$

is stable.

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Analysis logic

Handling time-varying hybrid dynamics

$$\dot{x}(t) = Ax(t) + BFx(t_i)$$

might be problematic. As such, the “troublemaker” part, some function of $BFx(t_i)$, is embedded into a wider class of systems, whose

☺ analysis is simpler.

Typically, this wider class contains norm-bounded uncertainty, analyzed via **robust stability** tools. However, such an embedding

☹ introduces conservatism,

as the infeasibility of the wider problem does not imply that of the original one. For that reason, the approach makes sense only for preliminary checks.

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Toy problem: isolating the troublemaker

The trick is to rewrite the closed-loop dynamics as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BFx(t_i) = (A + BF)x(t) + BF(x(t_i) - x(t)) \\ &= A_F x(t) - B \int_{t_i}^t F \dot{x}(s) ds. \end{aligned}$$

Introduce two signals

$$e_1(t) := -F \dot{x}(t) \quad \text{and} \quad e_2(t) := - \int_{t_i}^t F \dot{x}(s) ds = \int_{t_i}^t e_1(s) ds$$

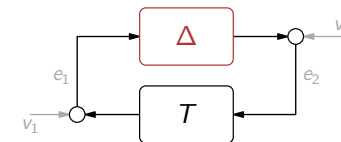
Because e_2 satisfies $\dot{e}_2(t) = e_1(t)$ with the reset $e_2(t_i) = 0$, the closed-loop dynamics read

$$\begin{cases} \dot{x}(t) = A_F x(t) + B e_2(t) \\ \dot{e}_2(t) = -F \dot{x}(t) = -F A_F x(t) - F B e_2(t), & e_2(t_i) = 0 \end{cases}$$

which is the feedback interconnection of $T : e_2 \mapsto e_1$ and $\Delta : e_1 \mapsto e_2$.

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Toy problem: isolating the troublemaker (contd)



The system

$$T : \begin{cases} \dot{x}(t) = A_F x(t) + B e_2(t) \\ e_1(t) = -F A_F x(t) - F B e_2(t) \end{cases}$$

is LTI, with the proper transfer function

$$T(s) = - \left[\begin{array}{c|c} A_F & B \\ \hline F A_F & F B \end{array} \right].$$

The system $\Delta : e_1 \mapsto e_2$ is the time-varying integrator resetting at each t_i . Its interconnection with T is hard to analyze accurately. But Δ happens to be stable, with a bounded gain, making it a good fit for **small-gain analyses**.

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Preliminary: L_2 -induced norm

A continuous-time signal $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is said to belong to L_2 if

$$\|f\|_2 := \left(\int_{\mathbb{R}_+} \|f(t)\|^2 dt \right)^{1/2} < \infty.$$

A system $G : u \mapsto y$ is said to be L_2 -stable if

$$u \in L_2 \implies y \in L_2 \quad \text{and} \quad \|y\|_2 \leq \gamma \|u\|_2$$

for all u and some $\gamma \geq 0$ independent of u . The smallest such γ , i.e.

$$\sup_{u \in L_2, u \neq 0} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{u \in L_2, \|u\|_2=1} \|Gu\|_2$$

is known as the L_2 -induced norm of G , denoted $\|G\|_{L_2 \rightarrow L_2}$. If G is LTI, then

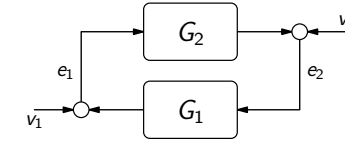
$$\|G\|_{L_2 \rightarrow L_2} = \|G\|_\infty := \sup_{s \in \mathbb{C}_0} \|G(s)\| = \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|,$$

provided $G \in H_\infty$, i.e. $G(s)$ is holomorphic and bounded in $\mathbb{C}_0$.

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Preliminary: stability of feedback interconnection

The feedback interconnection



is said to be internally stable if all systems $(v_1, v_2) \mapsto (e_1, e_2)$ are stable.

Theorem (the Small Gain Theorem)

If G_1 and G_2 are L_2 -stable, then their feedback interconnection is internally stable if $\|G_1\|_{L_2 \rightarrow L_2} \|G_2\|_{L_2 \rightarrow L_2} < 1$.

Theorem (the Robust Stability Theorem)

The interconnection is stable **for all** stable G_2 such that $\|G_2\|_{L_2 \rightarrow L_2} \leq \alpha$ **iff** G_1 is stable and such that $\|G_1\|_{L_2 \rightarrow L_2} < 1/\alpha$.

Proof: Sufficiency follows by the SGT. Necessity—by constructing feasible destabilizing G_2 for a given G_1 . □

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Example of the RST

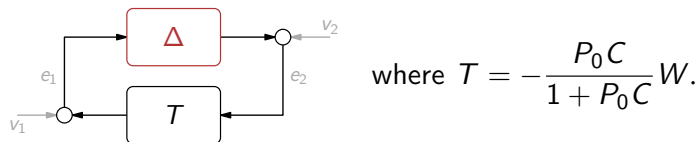
Consider a system P , whose frequency response is described as

$$P(j\omega) = P_0(j\omega)(1 + W(j\omega)\Delta(j\omega))$$

for given P_0 (nominal plant) and stable W (weight) and a stable Δ such that $|\Delta(j\omega)| \leq 1$. This description corresponds to the

- multiplicative uncertainty, where $|P(j\omega) - P_0(j\omega)| \leq |P_0(j\omega)| |W(j\omega)|$.

If a controller C is used to stabilize this plant, the closed-loop system is



The RST yields that the system is stable iff T is stable and $\|T\|_\infty < 1$, i.e.

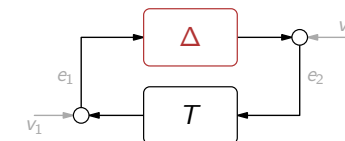
$$\left| \frac{P_0(j\omega)C(j\omega)}{1 + P_0(j\omega)C(j\omega)} \right| < \frac{1}{|W(j\omega)|}, \quad \forall \omega.$$

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Toy problem: small-gain analysis

It can be shown that

- Δ is stable and its L_2 -induced norm $\|\Delta\|_{L_2 \rightarrow L_2} = 2\bar{h}/\pi$, where $\bar{h} := \sup_i (t_{i+1} - t_i)$. Hence, the interconnection



is stable if it is stable for all stable Δ such that $\|\Delta\|_{L_2 \rightarrow L_2} \leq 2\bar{h}/\pi$. By the SGT, the later is equivalent to

$$\|T\|_\infty < \frac{\pi}{2 \sup_i (t_{i+1} - t_i)},$$

where $T(s) = -sF(sI - A_F)^{-1}B$. This condition for T is readily verifiable.

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Example

Consider the first-order plant $\dot{x}(t) = x(t) + u(t)$ and the analog control law $u(t) = -fx(t)$ with $f > 1$. The sampled-data controller

$$u(t) = -fx(t_i), \quad \forall t \in [t_i, t_{i+1})$$

stabilizes the closed-loop system if

$$\left\| \frac{sf}{s + (f - 1)} \right\|_{\infty} = f < \frac{\pi}{2 \sup_i (t_{i+1} - t_i)},$$

so we can guarantee the closed-loop stability for any sampling pattern such that

$$\sup_i (t_{i+1} - t_i) < \frac{\pi}{2f} < \frac{\pi}{2}.$$

This result

- ☹ is conservative
- ☹ encourages low-gain feedback, $f \downarrow 1$, so poor performance

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Extensions

There are more possibilities along the line outlined above. For example,

- Δ is also passive, so passivity analysis can be combined with the SGT, leading to less conservative results;
- even less conservative, asymptotically accurate, analysis results can be found in the literature, where
 - accuracy is gained by the increase of computational complexity
 - design of controllers becomes less tangible and not immediate

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Outline

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Extremis malis extrema remedia

Networked environments pose new challenges to sampled-data control. So

- why shall we confine ourselves to classical sampled-data architectures?

In networked control systems

- computations are assumed to be cheap, while
- communication is supposed to be expensive.

As such, there seems to be

- no justification to keep using $\mathcal{H}_{\text{ZOH}}$ in the whole $[t_i, t_{s+1}]$
- no justification to sample the output y by $\mathcal{S}_{\text{idl}}$
- no justification to limit processing only to one digital part

In other words,

- it's time to a paradigm shift.

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Inspiration: sampled-data LQR with optimal hold

Given

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

and measured $x(ih)$, the optimal LQR control law is of the form

$$u(t) = Fe^{A_F(t-ih)}x(ih), \quad \forall t \in [ih, (i+1)h)$$

where the gain F (depends on CARE) is such that $A_F := A + BF$ is Hurwitz. This control law can be presented as the system with **discrete jumps**

$$\begin{aligned} \dot{x}_a(t) &= A_F x_a(t), & x_a(ih) &= x(ih^-) \\ u(t) &= F x_a(t) \end{aligned}$$

This law may be interpreted as an endeavor to **emulate** the optimal analog closed-loop behavior. This logic appears to be universal, so

- could it be used under an intermittent sampling sequence $\{t_i\}$?

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Emulator-based intermittent state feedback

Consider

$$\dot{x}(t) = Ax(t) + Bu(t) + Bd(t)$$

and apply the control law $u(t) = Fe^{A_F(t-t_i)}x(t_i)$ or, equivalently,

$$\begin{aligned} \dot{x}_a(t) &= A_F x_a(t), & x_a(t_i) &= x(t_i) \\ u(t) &= F x_a(t) \end{aligned}$$

under some F such that $A_F = A + BF$ is Hurwitz. As $x(t)$ is continuous for all bounded u and d , the closed-loop dynamics are

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} A & BF \\ 0 & A_F \end{bmatrix} \begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} d(t), \quad \begin{bmatrix} x(t_i) \\ x_a(t_i) \end{bmatrix} = \begin{bmatrix} x(t_i^-) \\ x(t_i^-) \end{bmatrix}$$

or, denoting $\varepsilon := x - x_a$,

$$\begin{bmatrix} \dot{\varepsilon}(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_F \end{bmatrix} \begin{bmatrix} \varepsilon(t) \\ x_a(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} d(t), \quad \begin{bmatrix} \varepsilon(t_i) \\ x_a(t_i) \end{bmatrix} = \begin{bmatrix} 0 \\ x_a(t_i^-) + \varepsilon(t_i^-) \end{bmatrix}$$

which comprises two dynamically independent parts.

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Emulator-based intermittent state feedback (contd)

The first part is completely decoupled from the second part and its solution

$$\varepsilon(t_i + \tau) = \int_0^\tau e^{A(\tau-\sigma)} B d(t_i + \sigma) d\sigma$$

for all $\tau \in [t_i, t_{i+1})$ (as $\varepsilon(t_i) = 0$). By arguments used in Lect. 6, Slide 18,

$$\|\varepsilon(t_i + \tau)\| \leq \tau e^{\|A\|\tau} \|B\| \sup_{\sigma \in [0, \tau)} \|d(t_i + \sigma)\|,$$

so ε is **bounded** whenever so are d and $t_{i+1} - t_i$. The second part has

$$\begin{aligned} x_a(t_i^-) &= e^{A_F(t_i-t_{i-1})}x_a(t_{i-1}) = e^{A_F(t_i-t_{i-1})}x_a(t_{i-1}^-) + e^{A_F(t_i-t_{i-1})}\varepsilon(t_{i-1}^-) \\ &= e^{A_F t_i}x_0 + \sum_{j=0}^{i-1} e^{A_F(t_i-t_j)}\varepsilon(t_j^-) \end{aligned}$$

(as $e^{A_F t} e^{A_F s} = e^{A_F(t+s)}$). Since A_F is Hurwitz, **sampled $x_a(t_i^-)$ are bounded** if so are d and $t_{i+1} - t_i$. Hence, the analog $x_a(t)$ is bounded as well.

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Emulator-based intermittent state feedback (contd)

Sumarizing, if $A + BF$ is Hurwitz, then the control law

$$u(t) = Fe^{(A+BF)(t-t_i)}x(t_i)$$

stabilizes

$$\dot{x}(t) = Ax(t) + Bu(t) + Bd(t)$$

if the sequence of sampling intervals $\{t_{i+1} - t_i\}$ is uniformly bounded.

Remark: In fact, remembering that (Lect. 9, Slide 52)

$$\|z\|_2^2 = x_0' X x_0 + \int_0^\infty \|u(t) - Fx(t)\|_{D_2' D_2}^2 dt$$

for all u , the choice of u above renders $\|z\|_2^2 = x_0' X x_0$ if no disturbances are present. Thus, the control law above is *LQR optimal* for every $\{t_i\}$, provided F is chosen as the optimal gain for the corresponding continuous-time problem.

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Adding observer

If only

$$y(t) = Cx(t)$$

is measured, then some sort of observer is expected to be needed. But

- adding a discrete observer based on $y(t_i)$ might not be easy.

Rather, let's

- replace $u(t) = Fe^{A_F(t-t_i)}x(t_i)$ with $u(t) = Fe^{A_F(t-t_i)}\hat{x}(t_i)$,

where $\hat{x}$ is generated by the (analog) observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)),$$

for which $A_L := A + LC$ is Hurwitz. The dynamics of this controller are

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{x}}_a(t) \end{bmatrix} = \begin{bmatrix} A_L & BF \\ 0 & A_F \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ x_a(t) \end{bmatrix} - \begin{bmatrix} L \\ 0 \end{bmatrix} y(t), \quad \begin{bmatrix} \hat{x}(t_i) \\ x_a(t_i) \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix} \hat{x}(t_i^-)$$

and they contain state jumps at each sampling instance t_i .

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Adding observer: closed-loop dynamics

Combining the plant and the controller,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} A & 0 & BF \\ -LC & A_L & BF \\ 0 & 0 & A_F \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ x_a(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} d(t), \quad \begin{bmatrix} x(t_i) \\ \hat{x}(t_i) \\ x_a(t_i) \end{bmatrix} = \begin{bmatrix} x(t_i^-) \\ \hat{x}(t_i^-) \\ \hat{x}(t_i^-) \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} \dot{\epsilon}(t) \\ \dot{\epsilon}(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} A_L & 0 & 0 \\ -LC & A & 0 \\ 0 & 0 & A_F \end{bmatrix} \begin{bmatrix} \epsilon(t) \\ \epsilon(t) \\ x_a(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} d(t),$$

$$\begin{bmatrix} \epsilon(t_i) \\ \epsilon(t_i) \\ x_a(t_i) \end{bmatrix} = \begin{bmatrix} \epsilon(t_i^-) \\ 0 \\ x_a(t_i^-) + \epsilon(t_i^-) \end{bmatrix}$$

$$\text{in terms of } \begin{bmatrix} \epsilon \\ \epsilon \\ x_a \end{bmatrix} := \begin{bmatrix} I & -I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ x_a \end{bmatrix}.$$

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Adding observer: closed-loop dynamics (contd)

The observer error dynamics are stable and autonomous, with no jumps:

$$\dot{\epsilon}(t) = A_L\epsilon(t) + Bd(t),$$

and thus bounded whenever so is d . The signal ϵ resets to zero at each t_i ,

$$\dot{\epsilon}(t) = A\epsilon(t) - LC\epsilon(t), \quad \epsilon(t_i) = 0$$

and thus bounded whenever so are d and $\{t_{i+1} - t_i\}$, similarly to the state-feedback case. The boundedness of x_a , satisfying

$$\dot{x}_a(t) = A_F x_a(t), \quad x_a(t_i) = x_a(t_i^-) + \epsilon(t_i^-)$$

can be shown exactly as in the state-feedback case. Hence, the

- sampled-data observer-based controller stabilizes the system whenever the sequence of sampling intervals $\{t_{i+1} - t_i\}$ is uniformly bounded.

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Controller architecture

The observer-base controller can be presented in the form



where the sensor-side observer acts as (here $x_s := \hat{x}$)

$$\dot{x}_s(t) = (A + LC)x_s(t) + Bu(t) - Ly(t)$$

and the actuator-side emulator acts as

$$\dot{x}_a(t) = (A + BF)x_a(t), \quad x_a(t_i) = x_s(t_i)$$

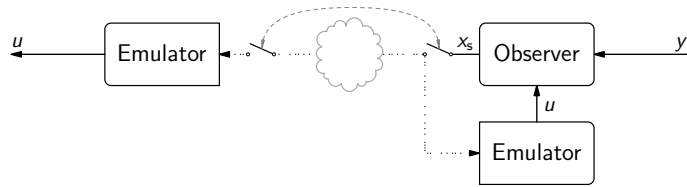
$$u(t) = Fx_a(t)$$

with **intermittent information flow** between them. One problem is that

- the **sensor-side** part needs to know the control signal $u(t)$, which is generated at the other end of the network.

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Architecture I: generating u at the sensor side



A way to resolve the problem is to

- run another copy of the emulator at the sensor side.

In this case the sensor-side part becomes

$$\begin{aligned}\dot{x}_s(t) &= (A + LC)x_s(t) + BF\tilde{x}_a(t) - Ly(t) \\ \dot{\tilde{x}}_a(t) &= (A + BF)\tilde{x}_a(t), \quad \tilde{x}_a(t_i) = x_s(t_i)\end{aligned}$$

and the actuator-side part remains unchanged. This works as long as

- two emulators remains synchronized, which might not be trivial over an unreliable network.

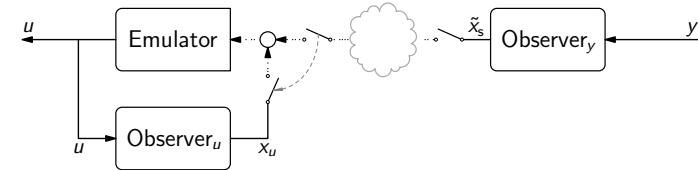
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Architecture II: splitting observer

By linearity, the observation $x_s = \tilde{x}_s + x_u$, where

$$\dot{\tilde{x}}_s(t) = (A + LC)\tilde{x}_s(t) - Ly(t) \quad \text{and} \quad \dot{x}_u(t) = (A + LC)x_u(t) + Bu(t).$$

We need y to generate $\tilde{x}_s$, so it has to be implemented at the sensor side. But x_u , driven by u , can be implemented at the actuator side, leading to



with the actuator-side part dynamics

$$\begin{bmatrix} \dot{x}_a(t) \\ \dot{x}_u(t) \end{bmatrix} = \begin{bmatrix} A + BF & 0 \\ BF & A + LC \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_u(t) \end{bmatrix}, \quad \begin{bmatrix} x_a(t_i) \\ x_u(t_i) \end{bmatrix} = \begin{bmatrix} \tilde{x}_s(t_i) + x_u(t_i) \\ x_u(t_i^-) \end{bmatrix}$$

and $u(t) = Fx_a(t)$.

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Outline

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ETC: idea

The reason for exchanging information between sensor(s) and actuator(s) in control systems is **uncertainty**, like disturbances, modeling mismatches, etc. If everything behaves (more-or-less) as expected, viz. in agreement with the model, control tasks can be accomplished in open loop. So,

- why not exchange information only when deemed necessary from the control point of view? This can be translated as
- why not sample only when deemed necessary?

Sampled-data control configurations, in which

- sampling instances are generated on the basis of system behavior are dubbed **event-triggered control** (ETC) systems. They
- ☺ can potentially outperform time-triggered control algorithms (performance vs. sampling rate tradeoff, e.g. the same cost under slower sampling)
- ☹ but might be way harder to analyze (intrinsically nonlinear)

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Problem: continuous-time case

Consider the plant

$$\dot{x}(t) = u(t) + w(t)$$

where $w(t)$ is a unit intensity white noise process. The goal is to minimize

$$\overline{\text{var}}(z) := \lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \int_0^T (|x(t)|^2 + r|u(t)|^2) dt \right]$$

for $r > 0$ (i.e. $z(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \sqrt{r} \end{bmatrix} u(t)$). It may not appear obvious, but this is an LQR problem, which is solved in continuous time by

$$u(t) = -(D_z' D_z)^{-1} (D_z' C_z + B' X) x(t) = -\frac{1}{\sqrt{r}} x(t),$$

where the CARE solution $X = \sqrt{r}$ (cf. Lect. 9, Slide 9), and the optimal

$$\overline{\text{var}}(z) = \text{tr}(X) = \sqrt{r}.$$

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Problem: sampled-data case

If only $x(ih)$ is measurable for some $h > 0$, then the optimal solution is

$$u(t) = -\frac{1}{\sqrt{r}} e^{-(t-ih)/\sqrt{r}} x(ih), \quad \forall t \in [ih, (i+1)h)$$

(cf. Lect. 9, Slide 53) and the optimal performance

$$\overline{\text{var}}(z) = \sqrt{r} + \frac{h}{2}$$

i.e. $h/2$ is the cost of sampling.

Remark: As $r \downarrow 0$, the regulated variable becomes $z(t) = x(t)$ and the optimal control law becomes $u = -\mathcal{H}_{\text{imp}} \mathcal{S}_{\text{idl}} x$, i.e.

$$u(t) = -\sum_{i \in \mathbb{Z}_+} \delta(t - ih) x(ih),$$

for which $x(ih) = 0$ for every i . In other words, the impulse hold is optimal in that case.

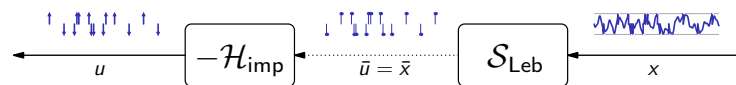
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Lebesgue sampling

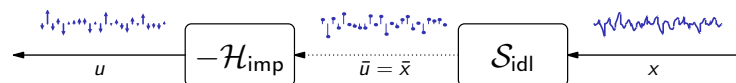
Under $r \downarrow 0$, Åström and Bernhardsson (1999) proposed to trigger sampling instances t_i according to

$$t_i = \min \{ t > t_{i-1} \mid |x(t)|^2 \geq h \}$$

for a given $h > 0$ and to use the same $u = -\mathcal{H}_{\text{imp}} \bar{x}$. This scheme was called **Lebesgue sampling**, in analogy with the Lebesgue integral. This



has the average sampling period $h_{\text{avg}} = h$ and improves the performance of



by a factor of 3, namely its $\overline{\text{var}}(x) = h/6$, vs. $\overline{\text{var}}(x) = h/2$ in the periodic sampling case. Moreover, no other sampled-data law can do better.

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More general Lebesgue sampling

For $r > 0$, consider triggering sampling instances t_i according to

$$t_i = \min \{ t > t_{i-1} \mid |x(t) + \sqrt{r} u(t)|^2 \geq h \},$$

still with

$$u(t) = -\frac{1}{\sqrt{r}} e^{-(t-t_i)/\sqrt{r}} x(t_i), \quad \forall t \in [t_i, t_{i+1}).$$

This can be interpreted as

- triggering on a “significant” deviation of $u(t)$ from the analog optimal $u_{\text{opt}}(t) = -x(t)/\sqrt{r}$.

This control law still has $h_{\text{avg}} = h$ and renders

$$\overline{\text{var}}(z) = \sqrt{r} + \frac{h}{6},$$

i.e. the cost of sampling reduces by a **factor of 3** vis-à-vis periodic sampling. Moreover, no other sampled-data law can do better.

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