

# Sampled-Data Control Systems (036709)

## lecture no. 12

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## Spatially distributed systems

The availability of powerful and cheap sensing / actuation / data processing technologies made

- spatial distribution of various components of feedback control systems technologically feasible. This is advantageous in many applications, e.g.
- telerobotics (e.g. remote surgeries, hazardous environments, etc)
- smart power grids
- traffic control
- transport systems, like platooning
- ...

From the control point of view, these realities present new opportunities (in facilitating better sensing and actuating), but pose new challenges in

- the need to handle large-scale systems,
- the need to account for communication networks and their limitations.

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## Control over communication networks

### Challenges:

- network transmission expenses,  
(both direct and indirect, like effects of transmission on power consumption)
- intermittent sampling,
- variable network-induced delays,
- information losses via packet dropout,
- communication capacity limitations,
- security issues,

as well as decentralization (which is less related to sampled-data systems).

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- prompts new formulations and
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This lecture aims at providing a flavor of handling the first two issues.

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# Outline

Robustness to variable sampling times

Emulation-based architecture

Event-triggered control

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## A toy problem to start with

Consider

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and assume that  $u(t) = Fx(t)$  is satisfactory, with Hurwitz  $A_F := A + BF$ .  
Apply it now digitally, as

$$u(t) = Fx(t_i), \quad \forall t \in [t_i, t_{i+1})$$

i.e. as  $u = \mathcal{H}_{\text{ZOH}} F \mathcal{S}_{\text{idl}}$  under intermittent sampling sequence  $\{t_i\}$ . Our goal is to

- find under what conditions on  $\{t_i\}$  the closed-loop system

$$\dot{x}(t) = Ax(t) + BFx(t_i), \quad \forall t \in [t_i, t_{i+1})$$

is stable.

## Analysis logic

Handling time-varying hybrid dynamics

$$\dot{x}(t) = Ax(t) + BFx(t_i)$$

might be problematic. As such, the “troublemaker” part, some function of  $BFx(t_i)$ , is embedded into a wider class of systems, whose

😊 analysis is simpler.

Typically, this wider class contains norm-bounded uncertainty, analyzed via **robust stability** tools. However, such an embedding

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as the infeasibility of the wider problem does not imply that of the original one. For that reason, the approach makes sense only for preliminary checks.

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## Toy problem: isolating the troublemaker

The trick is to rewrite the closed-loop dynamics as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BFx(t_i) = (A + BF)x(t) + BF(x(t_i) - x(t)) \\ &= A_F x(t) - B \int_{t_i}^t F \dot{x}(s) ds.\end{aligned}$$

Introduce two signals

$$e_1(t) := -Fx(t) \quad \text{and} \quad e_2(t) := -\int_{t_i}^t F\dot{x}(s)ds = \int_{t_i}^t e_1(s)ds$$

Because  $e_2$  satisfies  $\dot{e}_2(t) = e_1(t)$  with the reset  $e_2(t_i) = 0$ , the closed-loop dynamics read

$$\begin{cases} \dot{x}(t) = A_F x(t) + B e_2(t) \\ \dot{e}_2(t) = -F \dot{x}(t) = -F A_F x(t) - F B e_2(t), & e_2(t_i) = 0 \end{cases}$$

which is the feedback interconnection of  $T: e_2 \mapsto e_1$  and  $\Delta: e_1 \mapsto e_2$ .

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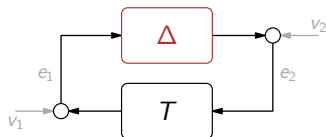
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## Toy problem: isolating the troublemaker (contd)



The system

$$T : \begin{cases} \dot{x}(t) = A_F x(t) + B e_2(t) \\ e_1(t) = -F A_F x(t) - F B e_2(t) \end{cases}$$

is LTI, with the proper transfer function

$$T(s) = - \left[ \begin{array}{c|c} A_F & B \\ \hline F A_F & F B \end{array} \right].$$

The system  $\Delta : e_1 \mapsto e_2$  is the time-varying integrator resetting at each  $t_i$ . Its interconnection with  $T$  is hard to analyze accurately. But  $\Delta$  happens to be stable, with a bounded gain, making it a good fit for **small-gain analyses**.

## Preliminary: $L_2$ -induced norm

A continuous-time signal  $f : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is said to belong to  $L_2$  if

$$\|f\|_2 := \left( \int_{\mathbb{R}_+} \|f(t)\|^2 dt \right)^{1/2} < \infty.$$

A system  $G : u \mapsto y$  is said to be  $L_2$ -stable if

$$u \in L_2 \implies y \in L_2 \quad \text{and} \quad \|y\|_2 \leq \gamma \|u\|_2$$

for all  $u$  and some  $\gamma \geq 0$  independent of  $u$ . The smallest such  $\gamma$ , i.e.

$$\sup_{u \in L_2, u \neq 0} \frac{\|Gu\|_2}{\|u\|_2} = \sup_{u \in L_2, \|u\|_2=1} \|Gu\|_2$$

is known as the  $L_2$ -induced norm of  $G$ , denoted  $\|G\|_{L_2 \rightarrow L_2}$ . If  $G$  is LTI, then

$$\|G\|_{L_2 \rightarrow L_2} = \|G\|_\infty := \sup_{s \in \mathbb{C}_0} \|G(s)\| = \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|,$$

provided  $G \in H_\infty$ , i.e.  $G(s)$  is holomorphic and bounded in  $\mathbb{C}_0$ .

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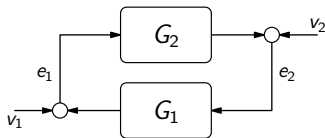
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## Preliminary: stability of feedback interconnection

The feedback interconnection



is said to be internally stable if all systems  $(v_1, v_2) \mapsto (e_1, e_2)$  are stable.

### Theorem (the Small Gain Theorem)

*If  $G_1$  and  $G_2$  are  $L_2$ -stable, then their feedback interconnection is internally stable if  $\|G_1\|_{L_2 \rightarrow L_2} \|G_2\|_{L_2 \rightarrow L_2} < 1$ .*

### Theorem (the Robust Stability Theorem)

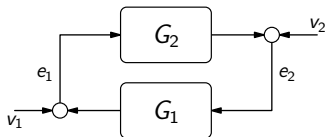
*The interconnection is stable for all stable  $G_2$  such that  $\|G_2\|_{L_2 \rightarrow L_2} \leq \alpha$  iff  $G_1$  is stable and such that  $\|G_1\|_{L_2 \rightarrow L_2} < 1/\alpha$ .*

### Proof



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*Proof:* Sufficiency follows by the SGT. Necessity—by constructing feasible destabilizing  $G_2$  for a given  $G_1$ . □

## Example of the RST

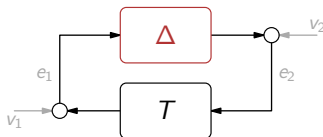
Consider a system  $P$ , whose frequency response is described as

$$P(j\omega) = P_0(j\omega)(1 + W(j\omega)\Delta(j\omega))$$

for given  $P_0$  (nominal plant) and stable  $W$  (weight) and a stable  $\Delta$  such that  $|\Delta(j\omega)| \leq 1$ . This description corresponds to the

- multiplicative uncertainty, where  $|P(j\omega) - P_0(j\omega)| \leq |P_0(j\omega)||W(j\omega)|$ .

If a controller  $C$  is used to stabilize this plant, the closed-loop system is



$$\text{where } T = -\frac{P_0 C}{1 + P_0 C} W.$$

The RST yields that the system is stable iff  $T$  is stable and  $\|T\|_\infty < 1$ , i.e.

$$\left| \frac{P_0(j\omega)C(j\omega)}{1 + P_0(j\omega)C(j\omega)} \right| < \frac{1}{|W(j\omega)|}, \quad \forall \omega.$$

## Toy problem: small-gain analysis

It can be shown that

- $\Delta$  is stable and its  $L_2$ -induced norm  $\|\Delta\|_{L_2 \rightarrow L_2} = 2\bar{h}/\pi$ ,  
where  $\bar{h} := \sup_i (t_{i+1} - t_i)$ . Hence, the interconnection

is stable if it is stable for all stable  $\Delta$  such that  $\|\Delta\|_{L_2 \rightarrow L_2} \leq 2\bar{h}/\pi$ . By the SGT, the later is equivalent to

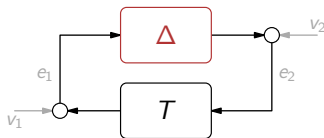
$$\|T\|_\infty \leq \frac{\pi}{2 \sup_i (t_{i+1} - t_i)},$$

where  $T(s) = -sF(sI - A_F)^{-1}B$ . This condition for  $T$  is readily verifiable.

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## Example

Consider the first-order plant  $\dot{x}(t) = x(t) + u(t)$  and the analog control law  $u(t) = -fx(t)$  with  $f > 1$ . The sampled-data controller

$$u(t) = -fx(t_i), \quad \forall t \in [t_i, t_{i+1})$$

stabilizes the closed-loop system if

$$\left\| \frac{sf}{s + (f - 1)} \right\|_{\infty} = f < \frac{\pi}{2 \sup_i (t_{i+1} - t_i)},$$

so we can guarantee the closed-loop stability for any sampling pattern such that

$$\sup_i (t_{i+1} - t_i) < \frac{\pi}{2f} < \frac{\pi}{2}.$$

This result

- is conservative
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## Extensions

There are more possibilities along the line outlined above. For example,

- $\Delta$  is also passive, so passivity analysis can be combined with the SGT, leading to less conservative results;
- even less conservative, asymptotically accurate, analysis results can be found in the literature, where
  - accuracy is gained by the increase of computational complexity
  - design of controllers becomes less tangible and not immediate

# Outline

Robustness to variable sampling times

Emulation-based architecture

Event-triggered control

## Extremis malis extrema remedia

Networked environments pose new challenges to sampled-data control. So

- why shall we confine ourselves to classical sampled-data architectures?

In networked control systems

- computations are assumed to be cheap
- communication is supposed to be expensive

As such, there seems to be

- no justification to keep using  $\mathcal{H}_{2\text{opt}}$  in the whole  $[0, t_{\text{e}}+1]$
- no justification to sample the output  $y$  by  $\mathcal{S}_{\text{id}}$
- no justification to limit processing only to one digital part

In other words,

- it's time to a paradigm shift.

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## Inspiration: sampled-data LQR with optimal hold

Given

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

and measured  $x(ih)$ , the optimal LQR control law is of the form

$$u(t) = Fe^{A_F(t-ih)}x(ih), \quad \forall t \in [ih, (i+1)h)$$

where the gain  $F$  (depends on CARE) is such that  $A_F := A + BF$  is Hurwitz.

This control law can be presented as the system with discrete jumps

$$\begin{aligned} \dot{x}_s(t) &= A_F x_s(t), & x_s(ih) &= x(ih), \\ u(t) &= Fx_s(t) \end{aligned}$$

This law may be interpreted as an endeavor to emulate the optimal analog closed-loop behavior. This logic appears to be universal, so

— could it be used under an intermittent sampling sequence  $\{t_i\}$ ?

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## Emulator-based intermittent state feedback

Consider

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and apply the control law  $u(t) = Fe^{A_F(t-t_i)}x(t_i)$  or, equivalently,

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under some  $F$  such that  $A_F = A + BF$  is Hurwitz. As  $x(t)$  is continuous for all bounded  $u$  and  $d$ , the closed-loop dynamics are

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or, denoting  $e := x - x_a$ ,

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which comprises two dynamically independent parts,

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## Emulator-based intermittent state feedback (contd)

The first part is completely decoupled from the second part and its solution

$$\varepsilon(t_i + \tau) = \int_0^\tau e^{A(\tau-\sigma)} B d(t_i + \sigma) d\sigma$$

for all  $\tau \in [t_i, t_{i+1})$  (as  $\varepsilon(t_i) = 0$ ). By arguments used in Lect. 6, Slide 18,

$$\|\varepsilon(t_i + \tau)\| \leq \tau e^{\|A\|\tau} \|B\| \sup_{\sigma \in [0, \tau]} \|d(t_i + \sigma)\|,$$

so  $\varepsilon$  is **bounded** whenever so are  $d$  and  $t_{i+1} - t_i$ . The second part has

$$\begin{aligned} x_0(t_j) &= e^{A_F(t_j - t_{j-1})} x_0(t_{j-1}) = e^{A_F(t_j - t_{j-1})} x_0(t_{j-1}) + e^{A_F(t_j - t_{j-1})} \varepsilon(t_{j-1}) \\ &= e^{A_F t_j} x_0 + \sum_{j=0}^{j-1} e^{A_F(t_j - t_j)} \varepsilon(t_j) \end{aligned}$$

(as  $e^{A_F t} e^{A_F s} = e^{A_F(t+s)}$ ). Since  $A_F$  is Hurwitz, sampled  $x_0(t_j)$  are bounded if so are  $d$  and  $t_{i+1} - t_i$ . Hence, the analog  $x_0(t)$  is bounded as well.

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$$\|\varepsilon(t_i + \tau)\| \leq \tau e^{\|A\|\tau} \|B\| \sup_{\sigma \in [0, \tau]} \|d(t_i + \sigma)\|,$$

so  $\varepsilon$  is **bounded** whenever so are  $d$  and  $t_{i+1} - t_i$ . The second part has

$$\begin{aligned} x_a(t_i^-) &= e^{A_F(t_i - t_{i-1})} x_a(t_{i-1}) = e^{A_F(t_i - t_{i-1})} x_a(t_{i-1}^-) + e^{A_F(t_i - t_{i-1})} \varepsilon(t_{i-1}^-) \\ &= e^{A_F t_i} x_0 + \sum_{j=0}^{i-1} e^{A_F(t_i - t_j)} \varepsilon(t_j^-) \end{aligned}$$

(as  $e^{A_F t} e^{A_F s} = e^{A_F(t+s)}$ ). Since  $A_F$  is Hurwitz, **sampled  $x_a(t_j^-)$  are bounded** if so are  $d$  and  $t_{i+1} - t_i$ . Hence, the analog  $x_a(t)$  is bounded as well.

## Emulator-based intermittent state feedback (contd)

Sumarizing, if  $A + BF$  is Hurwitz, then the control law

$$u(t) = Fe^{(A+BF)(t-t_i)}x(t_i)$$

stabilizes

$$\dot{x}(t) = Ax(t) + Bu(t) + Bd(t)$$

if the sequence of sampling intervals  $\{t_{i+1} - t_i\}$  is uniformly bounded.

**Remark:** In fact, remembering that (Lect. 9, Slide 52)

$$\|z\|_2^2 = x_0' X x_0 + \int_0^\infty \|u(t) - Fx(t)\|_{D_z' D_z}^2 dt$$

for all  $u$ , the choice of  $u$  above renders  $\|z\|_2^2 = x_0' X x_0$  if no disturbances are present. Thus, the control law above is *LQR optimal* for every  $\{t_i\}$ , provided  $F$  is chosen as the optimal gain for the corresponding continuous-time problem.

## Adding observer

If only

$$y(t) = Cx(t)$$

is measured, then some sort of observer is expected to be needed. But

- adding a discrete observer based on  $y(t_i)$  might not be easy.

Rather, let's

replace  $u(t) = Fe^{A_F(t-t_i)}x(t_i)$  with  $u(t) = Fe^{A_F(t-t_i)}\hat{x}(t)$ ,

where  $\hat{x}$  is generated by the (analog) observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)),$$

for which  $A_L := A + LC$  is Hurwitz. The dynamics of this controller are

$$\begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{x}_0(t) \end{bmatrix} = \begin{bmatrix} A_L & BF \\ 0 & A_F \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ x_0(t) \end{bmatrix} - \begin{bmatrix} L \\ 0 \end{bmatrix} y(t), \quad \begin{bmatrix} \hat{x}(t_i) \\ x_0(t_i) \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix} \hat{x}(t_i^-)$$

and they contain state jumps at each sampling instance  $t_i$ .

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and they contain state jumps at each sampling instance  $t_i$ .

## Adding observer: closed-loop dynamics

Combining the plant and the controller,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} A & 0 & BF \\ -LC & A_L & BF \\ 0 & 0 & A_F \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \\ x_a(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} d(t), \quad \begin{bmatrix} x(t_i) \\ \hat{x}(t_i) \\ x_a(t_i) \end{bmatrix} = \begin{bmatrix} x(t_i^-) \\ \hat{x}(t_i^-) \\ \hat{x}(t_i^-) \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} \dot{\epsilon}(t) \\ \dot{\varepsilon}(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} A_L & 0 & 0 \\ -LC & A & 0 \\ 0 & 0 & A_F \end{bmatrix} \begin{bmatrix} \epsilon(t) \\ \varepsilon(t) \\ x_a(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} d(t),$$

$$\begin{bmatrix} \epsilon(t_i) \\ \varepsilon(t_i) \\ x_a(t_i) \end{bmatrix} = \begin{bmatrix} \epsilon(t_i^-) \\ 0 \\ x_a(t_i^-) + \varepsilon(t_i^-) \end{bmatrix}$$

in terms of  $\begin{bmatrix} \epsilon \\ \varepsilon \\ x_a \end{bmatrix} := \begin{bmatrix} I & -I & 0 \\ 0 & I & -I \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ x_a \end{bmatrix}.$

## Adding observer: closed-loop dynamics (contd)

The observer error dynamics are stable and autonomous, with no jumps:

$$\dot{\epsilon}(t) = A_L \epsilon(t) + Bd(t),$$

and thus bounded whenever so is  $d$ . The signal  $\varepsilon$  resets to zero at each  $t_i$ ,

$$\dot{\varepsilon}(t) = A\varepsilon(t) - LC\epsilon(t), \quad \varepsilon(t_i) = 0$$

and thus bounded whenever so are  $d$  and  $\{t_{i+1} - t_i\}$ , similarly to the state-feedback case. The boundedness of  $x_a$ , satisfying

$$\dot{x}_a(t) = A_F x_a(t), \quad x_a(t_i) = x_a(t_i^-) + \varepsilon(t_i^-)$$

can be shown exactly as in the state-feedback case. Hence, the

- sampled-data observer-based controller stabilizes the system whenever the sequence of sampling intervals  $\{t_{i+1} - t_i\}$  is uniformly bounded.

## Controller architecture

The observer-base controller can be presented in the form



where the sensor-side observer acts as (here  $x_s := \hat{x}$ )

$$\dot{x}_s(t) = (A + LC)x_s(t) + Bu(t) - Ly(t)$$

and the actuator-side emulator acts as

$$\begin{aligned} \dot{x}_a(t) &= (A + BF)x_a(t), & x_a(t_i) &= x_s(t_i) \\ u(t) &= Fx_a(t) \end{aligned}$$

with **intermittent information flow** between them. One problem is that

the sensor-side part needs to know the control signal  $u(t)$ , which is generated at the other end of the network.

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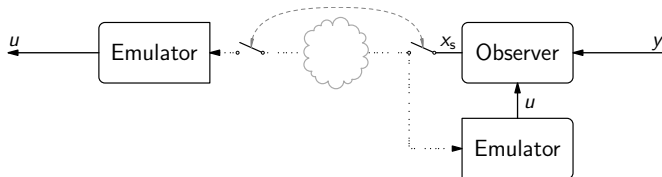
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with intermittent information flow between them. One problem is that

- the **sensor-side** part needs to know the control signal  $u(t)$ , which is generated at the other end of the network.

## Architecture I: generating $u$ at the sensor side



A way to resolve the problem is to

- run another copy of the emulator at the sensor side.

In this case the sensor-side part becomes

$$\dot{x}_s(t) = (A + LC)x_s(t) + BF\tilde{x}_a(t) - Ly(t)$$

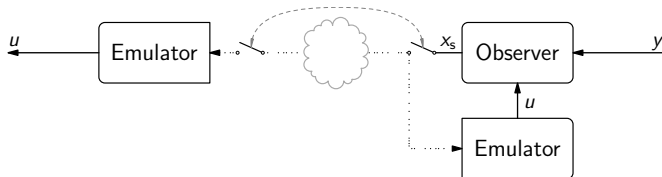
$$\dot{\tilde{x}}_a(t) = (A + BF)\tilde{x}_a(t), \quad \tilde{x}_a(t_i) = x_s(t_i)$$

and the actuator-side part remains unchanged. This works as long as

— two emulators remains synchronized,

which might not be trivial over an unreliable network.

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- two emulators remains synchronized,

which might not be trivial over an unreliable network.

## Architecture II: splitting observer

By linearity, the observation  $x_s = \tilde{x}_s + x_u$ , where

$$\dot{\tilde{x}}_s(t) = (A + LC)\tilde{x}_s(t) - Ly(t) \quad \text{and} \quad \dot{x}_u(t) = (A + LC)x_u(t) + Bu(t).$$

We need  $y$  to generate  $\tilde{x}_s$ , so it has to be implemented at the sensor side.

But  $x_u$ , driven by  $u$ , can be implemented at the actuator side, leading to

with the actuator-side part dynamics

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{x}_u(t) \end{bmatrix} = \begin{bmatrix} A + BF & 0 \\ BF & A + LC \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_u(t) \end{bmatrix}, \quad \begin{bmatrix} x_s(t) \\ x_u(t) \end{bmatrix} = \begin{bmatrix} \tilde{x}_s(t) + x_u(t) \\ x_u(t) \end{bmatrix}$$

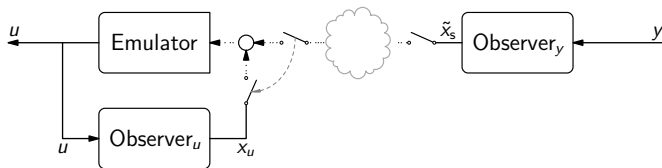
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and  $u(t) = Fx_a(t)$ .

# Outline

Robustness to variable sampling times

Emulation-based architecture

Event-triggered control

## ETC: idea

The reason for exchanging information between sensor(s) and actuator(s) in control systems is **uncertainty**, like disturbances, modeling mismatches, etc. If everything behaves (more-or-less) as expected, viz. in agreement with the model, control tasks can be accomplished in open loop. So,

- why not exchange information only when deemed necessary

from the control point of view? This can be translated as

- why not sample only when deemed necessary?

Sampled-data control configurations, in which

- sampling instances are generated on the basis of system behavior
- are dubbed **event-triggered control (ETC)** systems. They
- can potentially outperform time-triggered control algorithms (performance vs. sampling rate tradeoff, e.g. the same cost under slower sampling)
  - but might be way harder to analyze (intrinsically nonlinear)

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- ☹ but might be way harder to analyze  
(intrinsically nonlinear)

## Problem: continuous-time case

Consider the plant

$$\dot{x}(t) = u(t) + w(t)$$

where  $w(t)$  is a unit intensity white noise process. The goal is to minimize

$$\overline{\text{var}}(z) := \lim_{T \rightarrow \infty} \mathbb{E} \left[ \frac{1}{T} \int_0^T (|x(t)|^2 + r|u(t)|^2) dt \right]$$

for  $r > 0$  (i.e.  $z(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \sqrt{r} \end{bmatrix} u(t)$ ). It may not appear obvious, but this is an LQR problem, which is solved in continuous time by

$$u(t) = -(D_z' D_z)^{-1} (D_z' C_z + B' X) x(t) = -\frac{1}{\sqrt{r}} x(t),$$

where the CARE solution  $X = \sqrt{r}$  (cf. Lect. 9, Slide 9), and the optimal

$$\overline{\text{var}}(z) = \text{tr}(X) = \sqrt{r}.$$

## Problem: sampled-data case

If only  $x(ih)$  is measurable for some  $h > 0$ , then the optimal solution is

$$u(t) = -\frac{1}{\sqrt{r}} e^{-(t-ih)/\sqrt{r}} x(ih), \quad \forall t \in [ih, (i+1)h)$$

(cf. Lect. 9, Slide 53) and the optimal performance

$$\overline{\text{var}}(z) = \sqrt{r} + \frac{h}{2}$$

i.e.  $h/2$  is the cost of sampling.

**Remark:** As  $r \downarrow 0$ , the regulated variable becomes  $z(t) = x(t)$  and the optimal control law becomes  $u = -\mathcal{H}_{\text{imp}} \mathcal{S}_{\text{idl}} x$ , i.e.

$$u(t) = -\sum_{i \in \mathbb{Z}_+} \delta(t - ih) x(ih),$$

for which  $x(ih) = 0$  for every  $i$ . In other words, the impulse hold is optimal in that case.

## Lebesgue sampling

Under  $r \downarrow 0$ , Åström and Bernhardsson (1999) proposed to trigger sampling instances  $t_i$  according to

$$t_i = \min \{ t > t_{i-1} \mid |x(t)|^2 \geq h \}$$

for a given  $h > 0$  and to use the same  $u = -\mathcal{H}_{\text{imp}}\bar{x}$ . This scheme was called **Lebesgue sampling**, in analogy with the Lebesgue integral. This

has the average sampling period  $t_{\text{avg}} = h$  and improves the performance of

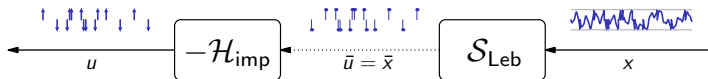
by a factor of 3, namely its  $\bar{\text{var}}(x) = h/6$ , vs.  $\bar{\text{var}}(x) = h/2$  in the periodic sampling case. Moreover, no other sampled-data law can do better.

## Lebesgue sampling

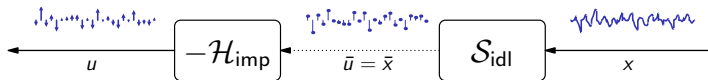
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## More general Lebesgue sampling

For  $r > 0$ , consider triggering sampling instances  $t_i$  according to

$$t_i = \min\{t > t_{i-1} \mid |x(t) + \sqrt{r} u(t)|^2 \geq h\},$$

still with

$$u(t) = -\frac{1}{\sqrt{r}} e^{-(t-t_i)/\sqrt{r}} x(t_i), \quad \forall t \in [t_i, t_{i+1}).$$

This can be interpreted as

- triggering on a “significant” deviation of  $u(t)$  from the analog optimal  $u_{\text{opt}}(t) = -x(t)/\sqrt{r}$ .

This control law still has  $h_{\text{avg}} = h$  and renders

$$\text{var}(z) = \sqrt{r} + \frac{h}{6},$$

i.e. the cost of sampling reduces by a factor of 3 vis-à-vis periodic sampling.  
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