

Sampled-Data Control Systems (036709)

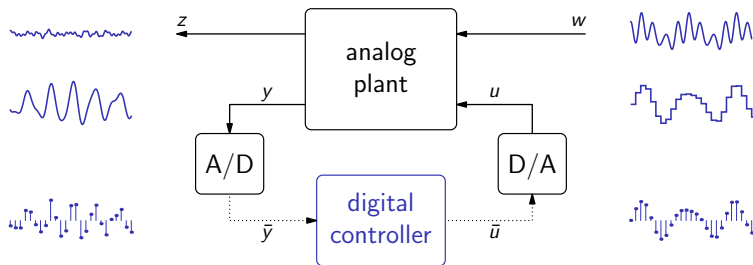
lecture no. 11

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Sampled-data control systems



Up to this point we assumed¹ that

- A/D and D/A conversions are carried out at the same rate.

This is not always the case. Control systems, in which

- control and measurements channels
- various control / measurement channels

act at different sampling rates are termed **multi-rate** (sometimes, dual-rate).

¹The synchronization assumption is implicitly abandoned by introducing analog delays.

Outline

Multi-rate control systems

Multi-rate discretization

Lifting

Control design in the lifted domain

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Control design in the lifted domain

Example 1: estimation via two sensors

Consider the system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + d(t) \\ y_1(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + n_1(t) \\ y_2(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + n_2(t) \end{cases}$$

It may be thought of as a rigid body, whose velocity (y_1) and position (y_2) are measured. There are situations where

- velocity measurements are easily accessible (say, on-board IMU)
- position measurements are scarce (from external source, say GPS)

This can be formalized as

$$\bar{y}_1[i] = y_1(ih_1) \quad \text{and} \quad \bar{y}_2[i] = y_2(ih_2)$$

under some $h_2 > h_1 > 0$.

Example 1: estimation via two sensors (contd)

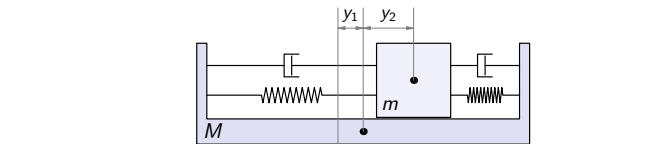
It should be emphasized that in many situations

- we cannot be satisfied with only fast velocity measurements, even if its noise level is low. This is because the dynamics are *not detectable* from y_1 ,

$$\text{rank} \left[\begin{array}{cc} -s & 1 \\ 0 & -s \\ \hline 0 & 1 \end{array} \right] \bigg|_{s=0} = 1,$$

Hence, velocity measurements must be complemented by (scarce) position measurements, naturally leading to a multi-rate setup.

Example 2: dual-stage actuator (DSA) system



We have two actuators with masses m and $M \gg m$, which are expected to coordinate in positioning m . The reason for using both:

- M has large operation range, but slow and not accurate (i.e. carrier)
- m has small operation range, but fast and accurate

so they complement each other. It is natural to

- use faster sampling rates in the “ m loop,”

e.g. by updating the control input acting on m faster than that acting on M .

Approaches to multi-rate control

Essentially, two

1. successive single-rate treatment

- loops are handled one-by-one, starting with that having the fastest rate
- well-justified if the effect of slow loops on fast ones is weak
- might be overly conservative otherwise

2. direct treatment

- all loops are handled at once
- technically more complicated
- potentially less conservative

Outline

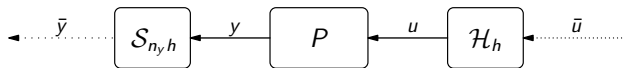
Multi-rate control systems

Multi-rate discretization

Lifting

Control design in the lifted domain

Fast hold



Assume that

- control signal $\bar{u}$ is synchronized with ih for some $h > 0$, i.e.

$$u(t) = \sum_{i \in \mathbb{Z}} \phi_{\text{ZOH},h}(t - ih) \bar{u}[ih], \quad \phi_{\text{ZOH},h}(t) := \mathbb{1}(t) - \mathbb{1}(t - h) = \begin{array}{c} 1 \\ \text{---} \\ 0 \quad h \quad t \end{array};$$

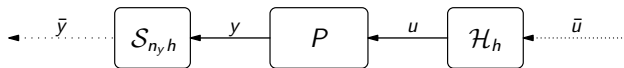
- measurement signal $\bar{y}$ is synchronized with $in_y h$ for some $n_y \in \mathbb{N}$, i.e.

$$\bar{y}[i] = y(in_y h).$$

Our goal is to understand the dynamics of

$$\bar{P}_{h,\{1,n_y\}} := \mathcal{S}_{n_y h} P \mathcal{H}_h : \bar{u} \mapsto \bar{y}.$$

Fast hold: shift invariance



If $\bar{u}$ is shifted by one step, then

1. u is shifted by h
2. y is shifted by h
3. $\bar{y}$ is *not* shifted by one step

because $\mathcal{H}_h$ is h -shift invariant

because P is time invariant

because $\mathcal{S}_{n_y h}$ is $n_y h$ -shift invariant

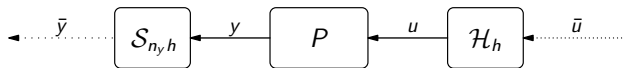
Therefore,

- the discretized $\bar{P}_{h,\{1,n_y\}} = \mathcal{S}_{n_y h} P \mathcal{H}_h$ is **not** necessarily **shift invariant**, with all consequences of that.

At the same time,

- $\bar{u}$ is shifted by n_y steps $\implies \bar{y}$ is shifted by one step,
which can be viewed as *kind of* n_y -periodic (or n_y -input periodic).

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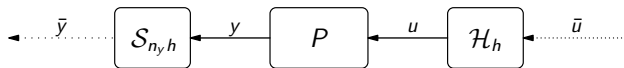
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- $\bar{u}$ is shifted by n_y steps $\implies \bar{y}$ is shifted by one step, which can be viewed as *kind of* n_y -periodic (or n_y -input periodic).

Fast hold: discretization



In this case, the response to $\bar{u}[i] = \delta[i - j]$ for $j \in \mathbb{Z}_{0..n_y-1}$ is

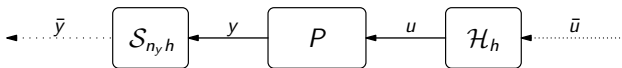
$$\begin{aligned}
 \bar{y}[i] &= \int_{\mathbb{R}} p(in_y h - t) u(t) dt = \int_{\mathbb{R}} p(in_y h - t) \sum_{k \in \mathbb{Z}} \phi_{\text{ZOH}, h}(t - kh) \bar{u}[k] dt \\
 &= \int_{\mathbb{R}} p(in_y h - t) \phi_{\text{ZOH}, h}(t - jh) dt = \int_{jh}^{(j+1)h} p(in_y h - t) dt
 \end{aligned}$$

Hence, the impulse response of $\bar{P}_{n_y}(1, n_y)$ is

$$\bar{P}_{n_y}(1, n_y)[i, j] = \int_0^h p((n_y i - j)h - t) dt = \bar{p}_{n_y}[n_y i - j],$$

where $\bar{P}_{n_y} = S_b P \mathcal{H}_h$. Note that $\bar{P}_{n_y}(1, n_y)[i, j] = \bar{P}_{n_y}(1, n_y)[i + 1, j + n_y]$

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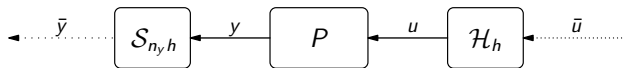
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 &= \int_{\mathbb{R}} p(in_y h - t) \phi_{\text{ZOH}, h}(t - jh) dt = \int_{jh}^{(j+1)h} p(in_y h - t) dt
 \end{aligned}$$

Hence, the impulse response of $\bar{P}_{h, \{1, n_y\}}$ is

$$\bar{\rho}_{h, \{1, n_y\}}[i, j] = \int_0^h p((n_y i - j)h - t) dt = \bar{\rho}_h[n_y i - j],$$

where $\bar{P}_h := \mathcal{S}_h P \mathcal{H}_h$. Note that $\bar{\rho}_{h, \{1, n_y\}}[i, j] = \bar{\rho}_{h, \{1, n_y\}}[i + 1, j + n_y]$.

Fast hold: remark about causality



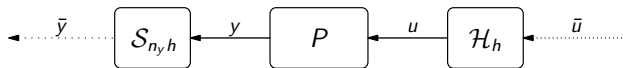
In

$$\bar{p}_{h,\{1,n_y\}}[i,j] = \int_0^h p((n_y i - j)h - t) dt$$

and $\bar{p}_{h,\{1,n_y\}}[i,j] = 0$ if $n_y i \leq j$, rather than $i < j$ as we expect from causal systems. This does not imply that the system is not causal, rather that the causality notion should be adjusted for multi-rate systems. Namely, we may say that

- $\bar{P}_{h,\{1,n_y\}}$ is causal if $\bar{y}[i]$ at every i can depend only on $\bar{u}[j]$ for $j \leq n_y i$ (equivalently, on $u(s)$ for $s \leq i n_y h$). In this sense $\bar{P}_{h,\{1,n_y\}}$ is causal.

Fast hold: system matrix

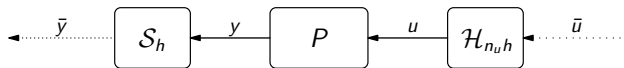


Remembering that $\bar{p}_{h,\{1,n_y\}}[i,j] = \bar{p}_h[n_y i - j]$, the system matrix is

$$\begin{aligned}
 \llbracket \bar{P}_{h,\{1,n_y\}} \rrbracket &= \begin{bmatrix} \bar{p}_{h,\{1,n_y\}}[0,0] & \cdots & \bar{p}_{h,\{1,n_y\}}[0,n_y-1] & \bar{p}_{h,\{1,n_y\}}[0,n_y] & \cdots \\ \bar{p}_{h,\{1,n_y\}}[1,0] & \cdots & \bar{p}_{h,\{1,n_y\}}[1,n_y-1] & \bar{p}_{h,\{1,n_y\}}[1,n_y] & \cdots \\ \bar{p}_{h,\{1,n_y\}}[2,0] & \cdots & \bar{p}_{h,\{1,n_y\}}[2,n_y-1] & \bar{p}_{h,\{1,n_y\}}[2,n_y] & \cdots \\ \vdots & & \vdots & \vdots & \end{bmatrix} \\
 &= \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \bar{p}_h[n_y] & \cdots & \bar{p}_h[1] & 0 & \cdots & 0 & 0 & \cdots \\ \bar{p}_h[2n_y] & \cdots & \bar{p}_h[n_y+1] & \bar{p}_h[n_y] & \cdots & \bar{p}_h[1] & 0 & \cdots \\ \bar{p}_h[3n_y] & \cdots & \bar{p}_h[2n_y+1] & \bar{p}_h[2n_y] & \cdots & \bar{p}_h[n_y+1] & \bar{p}_h[n_y] & \cdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \end{bmatrix}
 \end{aligned}$$

is not Toeplitz, as expected.

Fast sampler



Assume that

- measurement signal $\bar{y}$ is synchronized with ih for some $h > 0$, i.e.

$$\bar{y}[i] = y(ih);$$

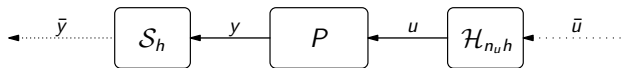
- control signal $\bar{u}$ is synchronized with $in_u h$ for some $n_u \in \mathbb{N}$, i.e.

$$u(t) = \sum_{i \in \mathbb{Z}} \phi_{\text{ZOH}, n_u h}(t - in_u h) \bar{u}[ih].$$

Our goal is to understand the dynamics of

$$\bar{P}_{h, \{n_u, 1\}} := S_h P \mathcal{H}_{n_u h} : \bar{u} \mapsto \bar{y}.$$

Fast sampler: shift invariance



If $\bar{u}$ is shifted by one step, then

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because $\mathcal{H}_{n_u h}$ is $n_u h$ -shift invariant

because P is time invariant

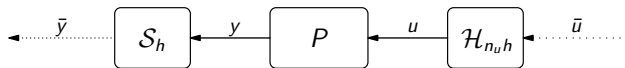
because $\mathcal{S}_h$ is h -shift invariant

Therefore,

- the discretized $\bar{P}_{h, \{n_u, 1\}} = \mathcal{S}_h P \mathcal{H}_{n_u h}$ is also **not shift invariant**,

with all consequences of that. Still, it is *kind of* n_u -periodic (or n_u -output periodic).

Fast sampler: discretization



In this case, the response to $\bar{u}[i] = \delta[i]$ is

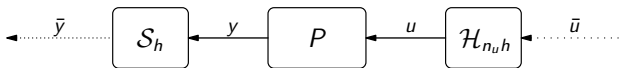
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 &= \int_0^{n_u h} p(ih - t)dt = \sum_{j=0}^{n_u-1} \int_{jh}^{(j+1)h} p(ih - t)dt
 \end{aligned}$$

Thus, the impulse response of $P_{h, \{n_u, 1\}}$ is (where, again, $\bar{P}_h := S_h P \mathcal{H}_h$)

$$\bar{P}_{h, \{n_u, 1\}}[i, 0] = \sum_{j=0}^{n_u-1} \int_0^h p((i-j)h - t)dt = \sum_{j=0}^{n_u-1} \bar{p}_h[i-j]$$

(mind that $\bar{p}_h[i] = 0$ if $i < 0$) and $\bar{P}_{h, \{n_u, 1\}}[i, j] = \bar{P}_{h, \{n_u, 1\}}[i + n_u, j + 1]$.

Fast sampler: discretization



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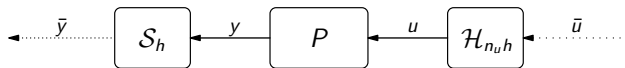
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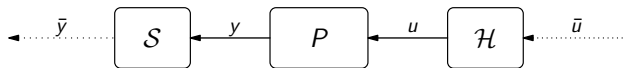
Fast sampler: system matrix



The system matrix, also not Toeplitz, is

$$\llbracket \bar{P}_{h,\{n_u,1\}} \rrbracket = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \bar{p}_h[1] & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \\ \sum_{j=0}^{n_u-2} \bar{p}_h[n_u-1-j] & 0 & 0 & \cdots \\ \sum_{j=0}^{n_u-1} \bar{p}_h[n_u-j] & 0 & 0 & \cdots \\ \sum_{j=0}^{n_u-1} \bar{p}_h[n_u+1-j] & \bar{p}_h[1] & 0 & \cdots \\ \vdots & \vdots & \vdots & \\ \sum_{j=0}^{n_u-1} \bar{p}_h[2n_u-1-j] & \sum_{j=0}^{n_u-2} \bar{p}_h[n_u-1-j] & 0 & \cdots \\ \sum_{j=0}^{n_u-1} \bar{p}_h[2n_u-j] & \sum_{j=0}^{n_u-1} \bar{p}_h[n_u-j] & 0 & \cdots \\ \sum_{j=0}^{n_u-1} \bar{p}_h[2n_u+1-j] & \sum_{j=0}^{n_u-1} \bar{p}_h[n_u+1-j] & \rho[1] & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \begin{matrix} 0 \\ 1 \\ \\ n_u \\ \\ 2n_u \end{matrix}$$

General multi-rate systems



where

$$\mathcal{H} = \text{diag}\{\mathcal{H}_{h_i}\} \quad \text{and} \quad \mathcal{S} = \text{diag}\{\mathcal{S}_{h_j}\}$$

for $h_k = n_k h$ under certain $n_k \in \mathbb{N}$. By linearity, this setup can be treated channel-wise, as the

- superposition of systems of the form $\bar{P}_{h,\{n_i,n_j\}} := \mathcal{S}_{n_j h} P \mathcal{H}_{n_i h}$.

This is similar to $\mathcal{S}_h P \mathcal{H}_{n_i h}$ and $\mathcal{S}_{n_j h} P \mathcal{H}_h$, but messier.

Remark: Even messier is the treatment of systems with irrational relations between h_i and h_j , in which case the system is not even periodic.

Outline

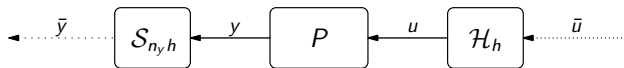
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Fast hold: system matrix again



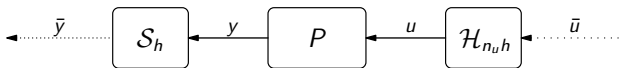
The system matrix

$$\llbracket \bar{P}_{h,\{1,n_y\}} \rrbracket = \left[\begin{array}{ccc|ccc|cc} 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \bar{p}_h[n_y] & \cdots & \bar{p}_h[1] & 0 & \cdots & 0 & 0 & \cdots \\ \bar{p}_h[2n_y] & \cdots & \bar{p}_h[n_y + 1] & \bar{p}_h[n_y] & \cdots & \bar{p}_h[1] & 0 & \cdots \\ \bar{p}_h[3n_y] & \cdots & \bar{p}_h[2n_y + 1] & \bar{p}_h[2n_y] & \cdots & \bar{p}_h[n_y + 1] & \bar{p}_h[n_y] & \cdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \end{array} \right]$$

is not Toeplitz (hence, not shift invariant), but

- can be made **block-Toeplitz** if partitioned appropriately.

Fast sampler: system matrix again



This is also block-Toeplitz under an appropriate partitioning:

$$\llbracket \bar{P}_{h, \{n_u, 1\}} \rrbracket = \left[\begin{array}{ccc|ccc} 0 & & & 0 & & 0 & \cdots \\ \bar{p}_h[1] & & & 0 & & 0 & \cdots \\ \vdots & & & \vdots & & \vdots & \\ \sum_{j=0}^{n_u-2} \bar{p}_h[n_u - 1 - j] & & & 0 & & 0 & \cdots \\ \hline \sum_{j=0}^{n_u-1} \bar{p}_h[n_u - j] & & & 0 & & 0 & \cdots \\ \sum_{j=0}^{n_u-1} \bar{p}_h[n_u + 1 - j] & & & \bar{p}_h[1] & & 0 & \cdots \\ \vdots & & & \vdots & & \vdots & \\ \sum_{j=0}^{n_u-1} \bar{p}_h[2n_u - 1 - j] & \sum_{j=0}^{n_u-2} \bar{p}_h[n_u - 1 - j] & & 0 & & 0 & \cdots \\ \hline \sum_{j=0}^{n_u-1} \bar{p}_h[2n_u - j] & \sum_{j=0}^{n_u-1} \bar{p}_h[n_u - j] & & 0 & & 0 & \cdots \\ \sum_{j=0}^{n_u-1} \bar{p}_h[2n_u + 1 - j] & \sum_{j=0}^{n_u-1} \bar{p}_h[n_u + 1 - j] & & p[1] & & \cdots \\ \vdots & \vdots & & \vdots & & \vdots & \end{array} \right]$$

Lifting: idea

If shift-invariant structure can be recovered by uniting inputs or outputs in appropriately dimensioned blocks, why not

- unite inputs or outputs signals in such blocks to start with?

This can indeed be done, just notice that the sequences

$$\{f[i]\} := \{f[0], f[1], \dots, f[r-1], f[r], f[r+1], \dots\}$$

and

$$\{\check{f}[j]\} := \left\{ \begin{bmatrix} f[0] \\ f[1] \\ \vdots \\ f[r-1] \end{bmatrix}, \begin{bmatrix} f[r] \\ f[r+1] \\ \vdots \\ f[2r-1] \end{bmatrix}, \begin{bmatrix} f[2r] \\ f[2r+1] \\ \vdots \\ f[3r-1] \end{bmatrix}, \dots \right\}$$

represent the same data for each $r \in \mathbb{N}$ (in fact, $\llbracket f \rrbracket = \llbracket \check{f} \rrbracket$). The mapping

$$\mathcal{L}_r : f \mapsto \check{f}$$

is called the (discrete) lifting transformation.

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is called the (discrete) **lifting** transformation.

Lifting: properties

Linearity: directly verifiable.

Invertibility: the inverse transformation

$$\mathcal{L}_r^{-1} : \left\{ \begin{bmatrix} \tilde{x}_1[0] \\ \vdots \\ \tilde{x}_r[0] \end{bmatrix}, \begin{bmatrix} \tilde{x}_1[1] \\ \vdots \\ \tilde{x}_r[1] \end{bmatrix}, \dots \right\} \mapsto \{ \tilde{x}_1[0], \dots, \tilde{x}_r[0], \tilde{x}_1[1], \dots, \tilde{x}_r[1], \dots \}$$

is well defined ($f[nr+j] = \tilde{x}_{j+1}[n]$) and satisfies $\mathcal{L}_r^{-1}\mathcal{L}_r = I$ and $\mathcal{L}_r\mathcal{L}_r^{-1} = I$.

Isometry / stability:

$$\begin{aligned} \|\tilde{f}\|_2^2 &= \sum_{i \in \mathbb{Z}} \|\tilde{f}[i]\|^2 = \sum_{i \in \mathbb{Z}} \sum_{j=0}^{r-1} \|\tilde{x}_{j+1}[i]\|^2 = \sum_{i \in \mathbb{Z}} \sum_{j=0}^{r-1} \|f[nr+j]\|^2 = \sum_{i \in \mathbb{Z}} \|f[i]\|^2 \\ &= \|f\|_2^2, \end{aligned}$$

i.e. the $\ell_2(\mathbb{Z}_+)$ -norm is preserved ($\ell_1(\mathbb{Z}_+)$ - and $\ell_\infty(\mathbb{Z}_+)$ -norms too).

Lifting: properties

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Invertibility: the inverse transformation

$$\mathcal{L}_r^{-1} : \left\{ \begin{bmatrix} \check{f}_1[0] \\ \vdots \\ \check{f}_r[0] \end{bmatrix}, \begin{bmatrix} \check{f}_1[1] \\ \vdots \\ \check{f}_r[1] \end{bmatrix}, \dots \right\} \mapsto \{ \check{f}_1[0], \dots, \check{f}_r[0], \check{f}_1[1], \dots, \check{f}_r[1], \dots \}$$

is well defined ($f[ri+j] = \check{f}_{j+1}[i]$) and satisfies $\mathcal{L}_r^{-1}\mathcal{L}_r = I$ and $\mathcal{L}_r\mathcal{L}_r^{-1} = I$.

Isometry / stability:

$$\begin{aligned} \|f\|_2^2 &= \sum_{i \in \mathbb{Z}} \|f[i]\|^2 = \sum_{i \in \mathbb{Z}} \sum_{j=0}^{r-1} \|\check{f}_{j+1}[i]\|^2 = \sum_{i \in \mathbb{Z}} \sum_{j=0}^{r-1} \|f[ri+j]\|^2 = \sum_{i \in \mathbb{Z}} \|f[i]\|^2 \\ &= \|f\|_2^2, \end{aligned}$$

i.e. the $\ell_2(\mathbb{Z}_+)$ -norm is preserved ($\ell_1(\mathbb{Z}_+)$ - and $\ell_\infty(\mathbb{Z}_+)$ -norms too).

Lifting: properties

Linearity: directly verifiable.

Invertibility: the inverse transformation

$$\mathcal{L}_r^{-1} : \left\{ \begin{bmatrix} \check{f}_1[0] \\ \vdots \\ \check{f}_r[0] \end{bmatrix}, \begin{bmatrix} \check{f}_1[1] \\ \vdots \\ \check{f}_r[1] \end{bmatrix}, \dots \right\} \mapsto \{ \check{f}_1[0], \dots, \check{f}_r[0], \check{f}_1[1], \dots, \check{f}_r[1], \dots \}$$

is well defined ($f[ri + j] = \check{f}_{j+1}[i]$) and satisfies $\mathcal{L}_r^{-1}\mathcal{L}_r = I$ and $\mathcal{L}_r\mathcal{L}_r^{-1} = I$.

Isometry / stability:

$$\begin{aligned} \|\check{f}\|_2^2 &= \sum_{i \in \mathbb{Z}} \|\check{f}[i]\|^2 = \sum_{i \in \mathbb{Z}} \sum_{j=0}^{r-1} \|\check{f}_{j+1}[i]\|^2 = \sum_{i \in \mathbb{Z}} \sum_{j=0}^{r-1} \|f[ri + j]\|^2 = \sum_{i \in \mathbb{Z}} \|f[i]\|^2 \\ &= \|f\|_2^2, \end{aligned}$$

i.e. the $\ell_2(\mathbb{Z}_+)$ -norm is preserved ($\ell_1(\mathbb{Z}_+)$ - and $\ell_\infty(\mathbb{Z}_+)$ -norms too).

Lifting: properties (contd)

Shift invariance: $\check{f} = \mathcal{L}_r f$ is shifted by one step iff f is shifted by r steps:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} f[0] \\ f[1] \\ \vdots \\ f[r-1] \end{bmatrix}, \begin{bmatrix} f[r] \\ f[r+1] \\ \vdots \\ f[2r-1] \end{bmatrix}, \dots \right\} \\ = \mathcal{L}_r \{0, 0, \dots, 0, f[0], f[1], \dots, f[r-1], f[r], f[r+1], \dots\}$$

Nesting: $\mathcal{L}_{n_2} \mathcal{L}_{n_1} = \mathcal{L}_{n_1 n_2}$ for all $n_1, n_2 \in \mathbb{N}$, e.g. if $n_1 = 3$ and $n_2 = 2$, then

$$\mathcal{L}_2 \mathcal{L}_3 \{f[i]\} = \mathcal{L}_6 \left\{ \begin{bmatrix} f[0] \\ f[1] \\ f[2] \end{bmatrix}, \begin{bmatrix} f[3] \\ f[4] \\ f[5] \end{bmatrix}, \dots \right\} = \left\{ \begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \\ f[4] \\ f[5] \end{bmatrix}, \begin{bmatrix} f[6] \\ f[7] \\ f[8] \\ f[9] \\ f[10] \\ f[11] \end{bmatrix}, \dots \right\}$$

Obviously, $\mathcal{L}_{n_2}^{-1} \mathcal{L}_{n_1}^{-1} = \mathcal{L}_{n_1 n_2}^{-1}$ as well.

Lifting: properties (contd)

Shift invariance: $\check{f} = \mathcal{L}_r f$ is shifted by one step iff f is shifted by r steps:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} f[0] \\ f[1] \\ \vdots \\ f[r-1] \end{bmatrix}, \begin{bmatrix} f[r] \\ f[r+1] \\ \vdots \\ f[2r-1] \end{bmatrix}, \dots \right\} \\ = \mathcal{L}_r \{0, 0, \dots, 0, f[0], f[1], \dots, f[r-1], f[r], f[r+1], \dots\}$$

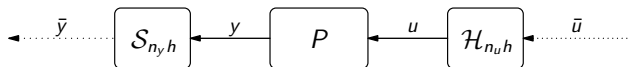
Nesting: $\mathcal{L}_{r_2} \mathcal{L}_{r_1} = \mathcal{L}_{r_2 r_1}$ for all $r_1, r_2 \in \mathbb{N}$, e.g. if $r_1 = 3$ and $r_2 = 2$, then

$$\mathcal{L}_{r_2} \mathcal{L}_{r_1} \{f[i]\} = \mathcal{L}_{r_2} \left\{ \begin{bmatrix} f[0] \\ f[1] \\ f[2] \end{bmatrix}, \begin{bmatrix} f[3] \\ f[4] \\ f[5] \end{bmatrix}, \dots \right\} = \left\{ \begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \\ f[4] \\ f[5] \end{bmatrix}, \begin{bmatrix} f[6] \\ f[7] \\ f[8] \\ f[9] \\ f[10] \\ f[11] \end{bmatrix}, \dots \right\}$$

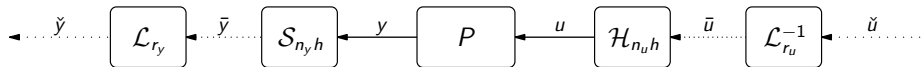
Obviously, $\mathcal{L}_{r_2}^{-1} \mathcal{L}_{r_1}^{-1} = \mathcal{L}_{r_2 r_1}^{-1}$ as well.

Lifted plant

Consider the systems $\bar{P}_{h,\{n_u,n_y\}} := S_{n_y h} P \mathcal{H}_{n_u h}$, i.e.



and define ν as the least common multiple of n_y and n_u , so that $\nu/n_u = r_u$ and $\nu/n_y = r_y$. The **lifted** system $\check{P}_{h,\{n_u,n_y\}} := \mathcal{L}_{r_y} \bar{P}_{h,\{n_u,n_y\}} \mathcal{L}_{r_u}^{-1}$, i.e.



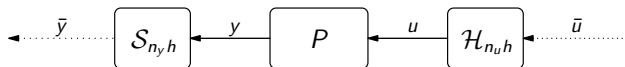
Because the signals $\check{u}$ and $\bar{u} / \bar{y}$ and $\check{y}$ convey the same information, these systems are equivalent.

1. $\bar{u}$ is shifted by r_u steps
2. u is shifted by $r_u n_u h = \nu h$
3. y is shifted by $\nu h = r_y n_y h$
4. $\bar{y}$ is shifted by r_y steps
5. $\check{y}$ is shifted by one step

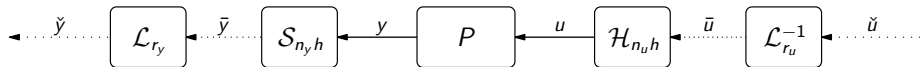
Therefore, lifted $\check{P}_{h,\{n_u,n_y\}}$ is shift invariant, with all consequences of that.

Lifted plant

Consider the systems $\bar{P}_{h,\{n_u,n_y\}} := S_{n_y h} P \mathcal{H}_{n_u h}$, i.e.



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Because the signals $\check{u}$ and $\bar{u} / \bar{y}$ and $\check{y}$ convey the same information, these systems are equivalent. If $\check{u}$ is shifted by one step, then

1. $\bar{u}$ is shifted by r_u steps
2. u is shifted by $r_u n_u h = \nu h$
3. y is shifted by $\nu h = r_y n_y h$
4. $\bar{y}$ is shifted by r_y steps
5. $\check{y}$ is shifted by one step

because $\mathcal{L}_{r_u}$ is shift invariant
 because $\mathcal{H}_{n_u h}$ is shift invariant
 because P is time invariant
 because $S_{n_y h}$ is shift invariant
 because $\mathcal{L}_{r_y}$ is shift invariant

Therefore, lifted $\check{P}_{h,\{n_u,n_y\}}$ is **shift invariant**, with all consequences of that.

Two properties

It is readily verifiable that

$$\mathcal{H}_{rh} = \mathcal{H}_h \mathcal{L}_r^{-1} \bar{H}_r, \quad \text{where } \bar{H}_r := \begin{bmatrix} I_m \\ \vdots \\ I_m \end{bmatrix} \in \mathbb{R}^{rm \times m}$$

and

$$\mathcal{S}_{rh} = \bar{S}_r \mathcal{L}_r \mathcal{S}_h, \quad \text{where } \bar{S}_r := \begin{bmatrix} I_p & 0_p & \cdots & 0_p \end{bmatrix} \in \mathbb{R}^{p \times rp}.$$

Consequently,

$$\mathcal{H}_{n_u h} \mathcal{L}_{r_u}^{-1} = \mathcal{H}_h \mathcal{L}_{n_u}^{-1} \bar{H}_{n_u} \mathcal{L}_{r_u}^{-1} = \mathcal{H}_h \mathcal{L}_v^{-1} \text{diag}\{\bar{H}_{n_u}, \dots, \bar{H}_{n_u}\} =: \mathcal{H}_h \mathcal{L}_v^{-1} \bar{H}_{n_u r_u}$$

and

$$\mathcal{L}_{r_y} \mathcal{S}_{n_y h} = \mathcal{L}_{r_y} \bar{S}_{n_y} \mathcal{L}_{n_y} \mathcal{S}_h = \text{diag}\{\bar{S}_{n_y}, \dots, \bar{S}_{n_y}\} \mathcal{L}_v \mathcal{S}_h =: \bar{S}_{n_y r_y} \mathcal{L}_v \mathcal{S}_h.$$

Two properties

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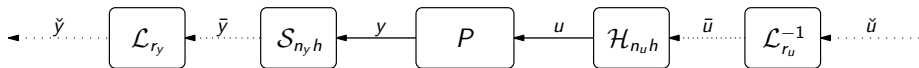
Consequently,

$$\mathcal{H}_{n_u h} \mathcal{L}_{r_u}^{-1} = \mathcal{H}_h \mathcal{L}_{n_u}^{-1} \bar{H}_{n_u} \mathcal{L}_{r_u}^{-1} = \mathcal{H}_h \mathcal{L}_v^{-1} \overbrace{\text{diag}\{\bar{H}_{n_u}, \dots, \bar{H}_{n_u}\}}^{r_u \text{ times}} =: \mathcal{H}_h \mathcal{L}_v^{-1} \bar{H}_{n_u, r_u}$$

and

$$\mathcal{L}_{r_y} \mathcal{S}_{n_y h} = \mathcal{L}_{r_y} \bar{S}_{n_y} \mathcal{L}_{n_y} \mathcal{S}_h = \underbrace{\text{diag}\{\bar{S}_{n_y}, \dots, \bar{S}_{n_y}\}}_{r_y \text{ times}} \mathcal{L}_v \mathcal{S}_h =: \bar{S}_{n_y, r_y} \mathcal{L}_v \mathcal{S}_h.$$

Lifted plant (contd)



Summarizing, the lifted plant $\check{P}_{h,\{n_u,n_y\}} : \check{u} \mapsto \check{y}$ is

$$\check{P}_{h,\{n_u,n_y\}} = \bar{S}_{n_y,r_y} \mathcal{L}_v S_h P \mathcal{H}_h \mathcal{L}_v^{-1} \bar{H}_{n_u,r_u}.$$

It is convenient to define the lifted system

$$\check{P}_v := \mathcal{L}_v \bar{P}_h \mathcal{L}_v^{-1},$$

where $\bar{P}_h = S_h P \mathcal{H}_h$ (the discretized plant for the baseline sampling period). Because

$$\bar{S}_{n_y,r_y} \in \mathbb{R}^{r_y p \times v p} \quad \text{and} \quad \bar{H}_{n_y,r_y} \in \mathbb{R}^{v m \times r_u m}$$

are static, we derive the model of $\check{P}_{h,\{n_u,n_y\}}$ via that of $\check{P}_v$, which is simpler.

System matrix and impulse response of $\check{P}_v$

The system matrix is the relation between vectorized input and output,

$$\begin{bmatrix} \check{y}[0] \\ \check{y}[1] \\ \vdots \end{bmatrix} = \llbracket \check{P}_v \rrbracket \begin{bmatrix} \check{u}[0] \\ \check{u}[1] \\ \vdots \end{bmatrix}$$

But this is the block form of the relation

$$\begin{bmatrix} \bar{y}[0] \\ \vdots \\ \bar{y}[v-1] \\ \hline \bar{y}[v] \\ \vdots \end{bmatrix} = \llbracket \bar{P}_h \rrbracket \begin{bmatrix} \bar{u}[0] \\ \vdots \\ \bar{u}[v-1] \\ \hline \bar{u}[v] \\ \vdots \end{bmatrix}$$

Hence,

- $\llbracket \check{P}_v \rrbracket$ is merely the repartitioned $\llbracket \bar{P}_h \rrbracket$.

System matrix and impulse response of $\check{P}_v$ (contd)

Thus,

$$\llbracket \check{P}_v \rrbracket = \left[\begin{array}{ccc|ccc|cc} \bar{\rho}_h[0] & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \\ \bar{\rho}_h[v-1] & \cdots & \bar{\rho}_h[0] & 0 & \cdots & 0 & 0 & \cdots \\ \hline \bar{\rho}_h[v] & \cdots & \bar{\rho}_h[1] & \bar{\rho}_h[0] & \cdots & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \\ \bar{\rho}_h[2v-1] & \cdots & \bar{\rho}_h[v] & \bar{\rho}_h[v-1] & \cdots & \bar{\rho}_h[0] & 0 & \cdots \\ \hline \bar{\rho}_h[2v] & \cdots & \bar{\rho}_h[v+1] & \bar{\rho}_h[v] & \cdots & \bar{\rho}_h[1] & \bar{\rho}_h[0] & \cdots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots \end{array} \right]$$

and the impulse response

$$\check{\rho}_v[0] = \begin{bmatrix} \bar{\rho}_h[0] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \bar{\rho}_h[v-1] & \cdots & \bar{\rho}_h[0] \end{bmatrix} \quad \text{and} \quad \check{\rho}_v[i] = \begin{bmatrix} \bar{\rho}_h[iv] & \cdots & \bar{\rho}_h[iv-v+1] \\ \vdots & & \vdots \\ \bar{\rho}_h[iv+v-1] & \cdots & \bar{\rho}_h[iv] \end{bmatrix}$$

for every $i \in \mathbb{N}$.

State-space realization of $\check{P}_v$

If $\bar{P}_h = C(zI - \bar{A})^{-1}\bar{B}$, then $\bar{p}_h[0] = 0$ and $\bar{p}_h[i] = C\bar{A}^{i-1}\bar{B}$ and we have:

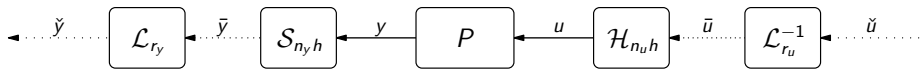
$$\begin{aligned}\check{p}_v[i] &= \begin{bmatrix} C\bar{A}^{iv-1}\bar{B} & \dots & C\bar{A}^{iv-v}\bar{B} \\ \vdots & & \vdots \\ C\bar{A}^{iv+v-2}\bar{B} & \dots & C\bar{A}^{iv}\bar{B} \end{bmatrix} \\ &= \begin{bmatrix} C \\ \vdots \\ C\bar{A}^{v-1} \end{bmatrix} \bar{A}^{v(i-1)} [\bar{A}^{v-1}\bar{B} \quad \dots \quad \bar{B}] \end{aligned}$$

Hence,

$$\check{P}_v(z) = \left[\begin{array}{c|cccccc} \bar{A}^v & \bar{A}^{v-1}\bar{B} & \bar{A}^{v-2}\bar{B} & \dots & \bar{A}\bar{B} & \bar{B} \\ \hline C & 0 & 0 & \dots & 0 & 0 \\ C\bar{A} & C\bar{B} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C\bar{A}^{v-2} & C\bar{A}^{v-3}\bar{B} & C\bar{A}^{v-4}\bar{B} & \dots & 0 & 0 \\ C\bar{A}^{v-1} & C\bar{A}^{v-2}\bar{B} & C\bar{A}^{v-3}\bar{B} & \dots & C\bar{B} & 0 \end{array} \right] =: \left[\begin{array}{c|c} \bar{A}^v & \acute{B} \\ \hline \grave{C} & \check{D} \end{array} \right].$$

Note that the feedthrough term of $\check{P}_v$, i.e. $\check{D}$, is block lower triangular.

State-space realization of $\check{P}_{h,\{n_u,n_y\}}$



Because $\check{P}_{h,\{n_u,n_y\}} = \bar{\mathcal{S}}_{n_y,r_y} \check{P}_v \bar{H}_{n_u,r_u}$, we have that

$$\check{P}_{h,\{n_u,n_y\}}(z) = \left[\begin{array}{c|c} \bar{A}^v & \acute{B} \bar{H}_{n_u,r_u} \\ \hline \bar{\mathcal{S}}_{n_y,r_y} \acute{C} & \bar{\mathcal{S}}_{n_y,r_y} \acute{D} \bar{H}_{n_u,r_u} \end{array} \right].$$

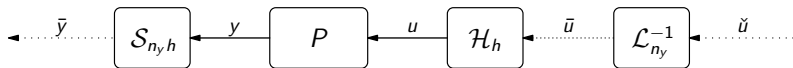
In fact,

- pre-multiplication by $\bar{\mathcal{S}}_{n_y,r_y}$ takes every r_y th block row
- post-multiplication by $\bar{H}_{n_u,r_u}$ sums up every r_u -tuple columns

e.g.

$$\bar{\mathcal{S}}_{n_y,r_y} \acute{C} = \begin{bmatrix} C \\ C \bar{A}^{r_y-1} \\ \vdots \end{bmatrix} \quad \text{and} \quad \acute{B} \bar{H}_{n_u,r_u} = \left[\cdots \quad \bar{A}^{r_u} \sum_{j=0}^{r_u-1} \bar{A}^j \bar{B} \quad \sum_{j=0}^{r_u-1} \bar{A}^j \bar{B} \right]$$

Fast hold: lifted model



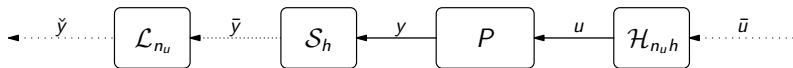
This is the special case for $n_u = 1$ and $\nu = n_y$, so that $r_u = n_y$ and $r_y = 1$. Hence, we have that

$$\bar{\mathcal{S}}_{n_y,1} = \bar{\mathcal{S}}_{n_y} = \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \bar{H}_{1,n_y} = I_{n_y m}$$

and then

$$\check{P}_{h,\{1,n_y\}}(z) = \left[\begin{array}{c|ccc} \bar{A}^{n_y} & \bar{A}^{n_y-1}\bar{B} & \bar{A}^{n_y-2}\bar{B} & \cdots & \bar{B} \\ \hline C & 0 & 0 & \cdots & 0 \end{array} \right].$$

Fast sampler: lifted model



This is the special case for $n_y = 1$ and $\nu = n_u$, so that $r_y = n_u$ and $r_u = 1$. Hence, we have that

$$\bar{S}_{1,n_u} = I_{n_u p} \quad \text{and} \quad \bar{H}_{n_u,1} = \bar{H}_{n_u} = \begin{bmatrix} I_m \\ \vdots \\ I_m \end{bmatrix}$$

and then

$$\check{P}_{h,\{n_u,1\}}(z) = \left[\begin{array}{c|c} \bar{A}^{n_u} & \bar{B}_{n_u} \\ \hline C & 0 \\ C\bar{A} & C\bar{B}_1 \\ \vdots & \vdots \\ C\bar{A}^{n_u-1} & C\bar{B}_{n_u-1} \end{array} \right],$$

where $\bar{B}_i := \sum_{j=0}^{i-1} \bar{A}^j \bar{B} = \int_0^{ih} e^{At} dt B$ for each $i \in \mathbb{Z}_{1..n_u}$.

Outline

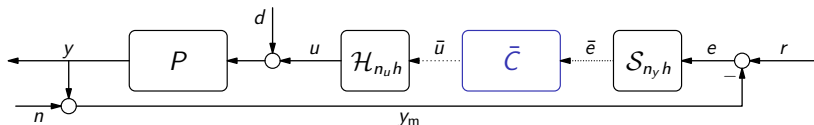
Multi-rate control systems

Multi-rate discretization

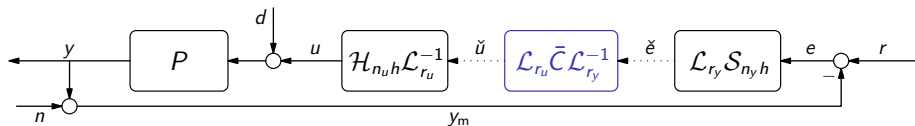
Lifting

Control design in the lifted domain

Lifting dual-rate control problems



This system is equivalent to



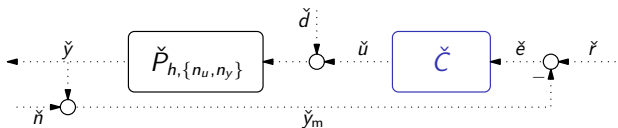
where digital signals $\check{e}$ and $\check{u}$ are synchronized with the very same instances

$$t_i = i r_u n_u h = i r_y n_y h = i v h.$$

It is then approximated by a pure discrete problem by

- approximating all analog exogenous signals by their lifted counterparts.

Lifting dual-rate control problems (contd)



This is the discrete problem for the LSI plant $\check{P}_{h, \{n_u, n_y\}} = \mathcal{L}_{r_y} \bar{P}_{h, \{n_u, n_y\}} \mathcal{L}_{r_u}^{-1}$ and a controller

$$\check{C} := \mathcal{L}_{r_u} \bar{C} \mathcal{L}_{r_y}^{-1}.$$

Having designed the latter, the controller is implemented as

$$\bar{C} = \mathcal{L}_{r_u}^{-1} \check{C} \mathcal{L}_{r_y}.$$

Note that even if $\check{C}$ is LSI, the actual $\bar{C}$ is not necessarily shift invariant, we already saw that non LSI $\bar{P}$ can result in an LSI $\mathcal{L}_{r_u} \bar{P} \mathcal{L}_{r_y}^{-1}$.

Causality constraints

More far-reaching, and less obvious, is the

- relation between causality properties of $\bar{C}$ and $\check{C}$.

To understand it, remember the following facts:

1. $\check{c}[i - j]$ defines the relation between $\check{e}[j]$ and $\check{u}[i]$
2. $\check{e}[j]$ contains information about $e(t)$ for $t \in [j\nu h, (j + 1)\nu h)$
3. $\check{u}[i]$ contains information about $u(t)$ for $t \in [i\nu h, (i + 1)\nu h)$

Hence,

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Hence,

- the causality of $\check{C}$ is necessary for that of $\bar{C}$
(otherwise $u(t)$ necessarily depends on future $e(s)$)
- the strict causality of $\check{C}$ is sufficient for that of $\bar{C}$
(meaning $u(t)$ for $t \in [i\nu h, (i + 1)\nu h)$ may only depend on $e(s)$ for $s < i\nu h$)

the causality of $\bar{C}$ is not sufficient for that of $\check{C}$

(the dependence of $\check{u}[i]$ on $\check{e}[i]$ permits $u(t)$ for $t \in [i\nu h, (i + 1)\nu h)$ to depend on $e(s)$ for $s \in [i\nu h, (i + 1)\nu h)$, which might not be causal)

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(the dependence of $\check{u}[i]$ on $\check{e}[i]$ permits $u(t)$ for $t \in [i\nu h, (i + 1)\nu h)$ to depend on $e(s)$ for $s \in [i\nu h, (i + 1)\nu h)$, which might not be causal)

Causality constraints (contd)

Thus, the causality of $\bar{C}$ requires additional constraints on the feedthrough term $\check{c}[0]$ of $\check{C}$. To understand it, look at the relation $\check{u}[k] = \check{c}[0]\check{e}[k] + \text{past}$ from the perspective of the underlying signals $\bar{e} = \mathcal{L}_{r_y}^{-1}\check{e}$ and $\bar{u} = \mathcal{L}_{r_u}^{-1}\check{u}$:

$$\begin{bmatrix} \bar{u}[r_u k] \\ \bar{u}[r_u k + 1] \\ \vdots \\ \bar{u}[r_u k + r_u - 1] \end{bmatrix} = \begin{bmatrix} \check{c}_{11}[0] & \check{c}_{12}[0] & \cdots & \check{c}_{1r_y}[0] \\ \check{c}_{21}[0] & \check{c}_{22}[0] & \cdots & \check{c}_{2r_y}[0] \\ \vdots & \vdots & & \vdots \\ \check{c}_{r_u1}[0] & \check{c}_{r_u2}[0] & \cdots & \check{c}_{r_ur_y}[0] \end{bmatrix} \begin{bmatrix} \bar{e}[r_y k] \\ \bar{e}[r_y k + 1] \\ \vdots \\ \bar{e}[r_y k + r_y - 1] \end{bmatrix} + \text{past}$$

Also remember that for all $i \in \mathbb{Z}_{1..r_u}$ and $j \in \mathbb{Z}_{1..r_y}$

1. $\bar{u}[r_u k + i - 1]$ defines $u(t)$ for $t \in [vkh + n_u(i - 1)h, vkh + n_u i h)$
2. $\bar{e}[r_y k + j - 1] = e(vkh + n_y(j - 1)h)$

Therefore, we have to guarantee that

$$\check{c}_j[0] = 0 \text{ whenever } j > r_y$$

Causality constraints (contd)

Thus, the causality of $\bar{C}$ requires additional constraints on the feedthrough term $\check{c}[0]$ of $\check{C}$. To understand it, look at the relation $\check{u}[k] = \check{c}[0]\check{e}[k] + \text{past}$ from the perspective of the underlying signals $\bar{e} = \mathcal{L}_{r_y}^{-1}\check{e}$ and $\bar{u} = \mathcal{L}_{r_u}^{-1}\check{u}$:

$$\begin{bmatrix} \bar{u}[r_u k] \\ \bar{u}[r_u k + 1] \\ \vdots \\ \bar{u}[r_u k + r_u - 1] \end{bmatrix} = \begin{bmatrix} \check{c}_{11}[0] & \check{c}_{12}[0] & \cdots & \check{c}_{1r_y}[0] \\ \check{c}_{21}[0] & \check{c}_{22}[0] & \cdots & \check{c}_{2r_y}[0] \\ \vdots & \vdots & & \vdots \\ \check{c}_{r_u 1}[0] & \check{c}_{r_u 2}[0] & \cdots & \check{c}_{r_u r_y}[0] \end{bmatrix} \begin{bmatrix} \bar{e}[r_y k] \\ \bar{e}[r_y k + 1] \\ \vdots \\ \bar{e}[r_y k + r_y - 1] \end{bmatrix} + \text{past}$$

Also remember that for all $i \in \mathbb{Z}_{1..r_u}$ and $j \in \mathbb{Z}_{1..r_y}$

1. $\bar{u}[r_u k + i - 1]$ defines $u(t)$ for $t \in [\textcolor{teal}{vkh} + n_u(i - 1)h, \textcolor{teal}{vkh} + n_u i h)$
2. $\bar{e}[r_y k + j - 1] = e(\textcolor{teal}{vkh} + n_y(j - 1)h)$

Therefore, we have to guarantee that

- $\check{c}_{ij}[0] = 0$ whenever $\textcolor{teal}{vkh} + n_u(i - 1)h < \textcolor{teal}{vkh} + n_y(j - 1)h$.

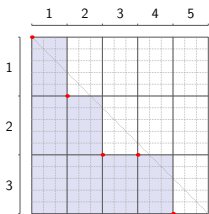
Causality constraints (contd)

The condition, to be verified for $i \in \mathbb{Z}_{1..r_u}$ and $j \in \mathbb{Z}_{1..r_y}$,

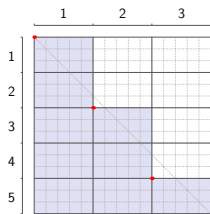
$$\check{c}_{ij}[0] = 0 \text{ whenever } n_u(i-1) < n_y(j-1)$$

requires $\check{c}[0]$ to have a

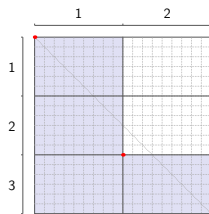
- (generalized) block lower-triangular structure:



$n_u = 5$ and $n_y = 3$



$n_u = 3$ and $n_y = 5$

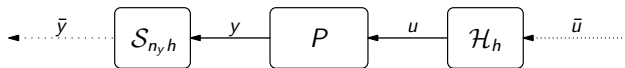


$n_u = 6$ and $n_y = 9$

This constraint has to be incorporated into the design of $\check{C}$, rendering the

- design more involved.

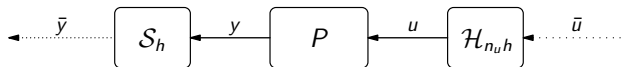
Causality constraints: simple special cases



Fast hold ($r_u = n_y$ & $r_y = 1$): in this case $\check{c}_{ij}[0] = 0$ whenever $i < 1$, which never holds, i.e.

- $\check{c}[0]$ is unconstrained

and the design of $\check{C}$ is similar to the single-rate case (dimensions).



Fast sampler ($r_u = 1$ & $r_y = n_u$): in this case $\check{c}_{ij}[0] = 0$ whenever $1 < j$, i.e.

- $\check{c}[0] = \begin{bmatrix} \check{c}_{11}[0] & 0 & \cdots & 0 \end{bmatrix}$ for an arbitrary $\check{c}_{11}[0]$

and the design of $\check{C}$ is nontrivially different from the single-rate case.

Strictly causal $\check{C}$

Handling algebraic constraints on $\check{c}[0]$ might be nontrivial. It is way simpler

- to design strictly causal $\check{C}$, with $\check{c}[0] = 0$, instead.

This

- ☺ substantially simplifies the design
- ☹ restricts attainable performance

as $\check{c}[0] = 0$ should not be normally expected to be the “best” possible choice of $\check{c}[0]$

Below we outline the structure of the resulting controller $\bar{C} = \mathcal{L}_{r_u}^{-1} \check{C} \mathcal{L}_{r_y}$.

Strictly causal $\check{C}$ in state space

Any strictly causal LSI finite-dimensional $\check{C} = \mathcal{L}_{r_u} \bar{C} \mathcal{L}_{r_y}^{-1}$ can be presented in terms of its realization as

$$\check{C}(z) = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{r_u} \end{bmatrix} (zI - A)^{-1} \begin{bmatrix} B_1 & B_2 & \cdots & B_{r_y} \end{bmatrix}.$$

The corresponding state equation, whose time is synchronized with νkh , is

$$\begin{cases} x[k+1] = Ax[k] + \sum_{j=1}^{r_y} B_j \check{e}_j[k] =: Ax[k] + \bar{e}_{eq}[t+1] \\ \check{u}_i[k] = C_i x[k], \quad i \in \mathbb{Z}_{1..n_u} \end{cases}$$

But

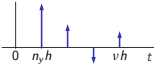
- $\check{e}_j[k] = e(\nu kh + n_y jh - h) = e((k+1)\nu h - n_y(r_y - j)h - h)$
- $u(t) = \check{u}_i[k]$ for all $t \in [\nu kh + n_u i h - h, \nu kh + n_u h)$

Strictly causal $\check{C}$: interpretation

Sampler: the signal

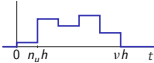
$$\bar{e}_{\text{eq}}[k+1] := \sum_{j=1}^{r_y} B_j \check{e}_j[k] = \sum_{j=1}^{r_y} B_j e((k+1)\nu h - n_y(r_y - j)h - h)$$

can be viewed as generated by the static sampler $\mathcal{S}_{\text{eq}, \nu h} : e \mapsto \bar{e}_{\text{eq}}$, whose

$$\psi_{\text{eq}}(t) = \sum_{j=1}^{r_y} B_j \delta(t - n_y(r_y - j)h - h)$$


and the sampling period is νh .

Hold: the control signal $u(t)$ can be viewed as generated by the static hold $\mathcal{H}_{\text{eq}, \nu h} : x \mapsto u$, whose

$$\phi_{\text{eq}}(t) = \sum_{i=1}^{r_u} \phi_{\text{ZOH}, n_u h}(t - (i-1)n_u h) C_i$$


and the sampling period is νh , again.

Strictly causal $\check{C}$: interpretation (contd)

Summarizing, the resulted controller

$$\mathcal{H}_{nuh} \bar{C} \mathcal{S}_{nyh} = \mathcal{H}_{eq, \nu h} \bar{C}_x \mathcal{S}_{eq, \nu h},$$

where

$$\bar{C}_x(z) = z \left[\begin{array}{c|c} A & I \\ \hline I & 0 \end{array} \right] = \left[\begin{array}{c|c} A & A \\ \hline I & I \end{array} \right].$$

The design of $\check{C}$ can then be interpreted as

- the **joint** design of $\mathcal{H}_{eq, \nu h}$, $\bar{C}_x$, and $\mathcal{S}_{eq, \nu h}$ under waveform constraints on A/D (piecewise impulse) and D/A (piecewise constant) converters.