

Sampled-Data Control Systems (036709)

lecture no. 10

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Outline

Time delays in sampled-data systems

State-feedback for time-delay systems

State-feedback for time-delay systems: closed-loop dynamics

LQR for time-delay systems

Outline

Time delays in sampled-data systems

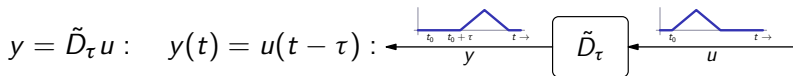
State-feedback for time-delay systems

State-feedback for time-delay systems: closed-loop dynamics

LQR for time-delay systems

Continuous-time delay element

Given $\tau > 0$,



These dynamics are

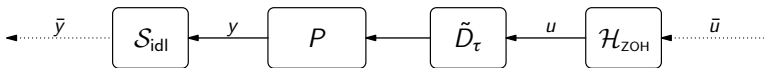
- linear time invariant
- stable
- infinite dimensional

The transfer function of the delay element,

$$\tilde{D}_\tau(s) = e^{-\tau s}$$

is non-rational and belongs to H_∞ (holomorphic and bounded in $\mathbb{C}_0$).

Discretization of input-delay systems



Assume that P is a finite dimensional LTI system, whose impulse response

$$p(t) = Ce^{At}B\mathbb{1}(t).$$

The impulse response of $P_\tau = P\tilde{D}_\tau$ is

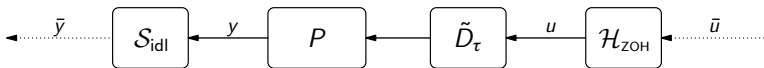
$$p_\tau(t) = p(t - \tau) = Ce^{A(t-\tau)}B\mathbb{1}(t - \tau).$$

The impulse response of the discretized system

$$\bar{p}_h[i] = \int_0^h p_\tau(ih - t)dt = \int_0^h p(ih - t - \tau)dt$$

and it can be nonzero only if $ih - \tau > 0$.

Discretization of input-delay systems (contd)



Assume that $\tau = \nu h - \theta \leq \nu h$ for some $\nu \in \mathbb{N}$ and $\theta \in [0, h)$. In this case

$$\bar{p}_h[i] = \int_0^h p(ih - t - \nu h + \theta) dt = \int_0^h C e^{A(ih - \nu h - t + \theta)} B \mathbb{1}(ih - \nu h - t + \theta) dt$$

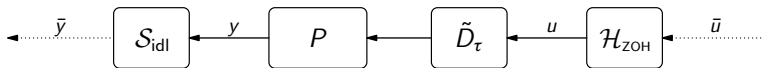
and we have that if

$$i < \nu: \bar{p}_h[i] = 0$$

$$i = \nu: \bar{p}_h[i] = C \int_0^\theta e^{A(\theta-t)} dt B = C \overbrace{\int_0^\theta e^{At} dt}^{\bar{D}} B$$

$$i > \nu: \bar{p}_h[i] = C e^{A(i-\nu-1)h+\theta} \int_0^h e^{A(h-t)} dt B = C \underbrace{(e^{Ah})^{i-\nu-1}}_{\bar{A}} e^{A\theta} \underbrace{\int_0^h e^{At} dt}_{\bar{B}} B$$

State-space realization



Thus,

$$P_h : \begin{cases} x[t+1] = \bar{A}x[t] + \bar{B}\bar{u}[t-\nu] \\ \bar{y}[t] = Cx[t] + \bar{D}\bar{u}[t-\nu] \end{cases}$$

under

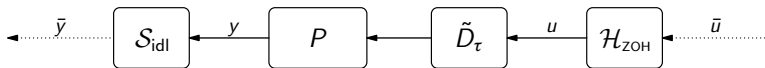
$$\begin{bmatrix} \bar{A} & \bar{B} \\ C & \bar{D} \end{bmatrix} := \begin{bmatrix} A_h & A_\theta B_h \\ C & CB_\theta \end{bmatrix},$$

where

$$\begin{bmatrix} A_t & B_t \end{bmatrix} := \begin{bmatrix} e^{At} & \int_0^t e^{As} ds B \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \exp \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} t \right).$$

This is the cascade of the ν -delay and delay-free system with the realization $(\bar{A}, \bar{B}, C, \bar{D})$.

“Physical” realization



Note that $x[i] \neq \bar{x}[i] := x(ih)$ if $\theta \neq 0$. Indeed, if $C = I$, then $\bar{y} = \bar{x}$ and

$$\bar{x}[t] = x[t] + B_\theta \bar{u}[t - \nu].$$

Hence,

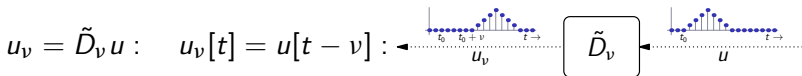
$$x[t] = \bar{x}[t] - B_\theta \bar{u}[t - \nu] = \bar{x}[t] - \int_0^\theta e^{At} dt B \bar{u}[t - \nu]$$

and

$$\begin{aligned} \bar{x}[t+1] &= \bar{A}\bar{x}[t] + B_\theta \bar{u}[t - \nu + 1] + (A_\theta B_h - A_h B_\theta) \bar{u}[t - \nu] \\ &= \bar{A}\bar{x}[t] + \underbrace{\int_0^\theta e^{At} dt B}_{\bar{B}_1} \bar{u}[t - \nu + 1] + \underbrace{\int_\theta^h e^{At} dt B}_{\bar{B}_0} \bar{u}[t - \nu]. \end{aligned}$$

Dimension of discrete delay

Given $\nu \in \mathbb{N}$,



Its impulse response is $\delta[t - \nu]I_m$, the transfer function $\tilde{D}_\nu(z) = z^{-\nu}I_m$, and the Hankel operator has the matrix representation

$$\llbracket H_{\tilde{D}_\nu} \rrbracket = \begin{matrix} & \text{\textit{\nu}th column} \\ \begin{bmatrix} 0 & \cdots & 0 & I_m & 0 & \cdots \\ 0 & \cdots & I_m & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \\ I_m & \cdots & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \end{bmatrix} & \text{\textit{\nu}th row} \end{matrix}$$

Its rank is $m\nu$, so $\tilde{D}_\nu$ is $m\nu$ -dimensional, where m is the dimension of u .

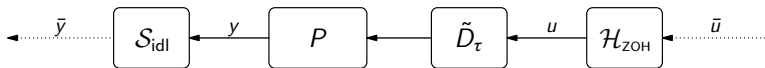
State-space realization of discrete delay

A natural one:

$$\tilde{D}_v : \left\{ \begin{aligned} \begin{bmatrix} u[t - v + 1] \\ \vdots \\ u[t - 1] \\ u[t] \end{bmatrix} &= \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u[t - v] \\ u[t - v + 1] \\ \vdots \\ u[t - 1] \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} u[t] \\ \\ u_v[t] &= [I \ 0 \ \cdots \ 0] \begin{bmatrix} u[t - v] \\ u[t - v + 1] \\ \vdots \\ u[t - 1] \end{bmatrix}, \end{aligned} \right.$$

which is both reachable and observable.

Realization of discretized input-delay systems

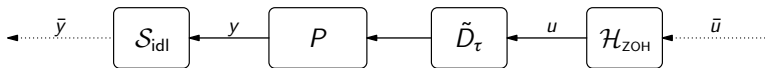


Constructing the realization of the cascade interconnection,

$$\bar{P}_h : \begin{cases} x_v[t+1] = \begin{bmatrix} \bar{A} & \bar{B} & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} x_v[t] + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \bar{u}[t] \\ \bar{y}[t] = [C \mid \bar{D} \mid 0 \mid \cdots \mid 0] x_v[t], \end{cases}$$

where $x_v[t] = [x'[t] \quad \bar{u}'[t-\nu] \quad \bar{u}'[t-\nu+1] \quad \cdots \quad \bar{u}'[t-1]]'$ and the spectrum of the “A” matrix comprises $\text{spec}(\bar{A})$ and a mode at the origin of multiplicity $m\nu$.

“Physical” realization, again



Because now $\bar{x}[t] = x_{v,1}[t] + \bar{B}_1 x_{v,2}$, a similarity transformation yields

$$\bar{P}_h : \begin{cases} \bar{x}_v[t+1] = \underbrace{\begin{bmatrix} \bar{A} & \bar{B}_0 & \bar{B}_1 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\bar{A}_v} \bar{x}_v[t] + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}}_{\bar{B}_v} \bar{u}[t] \\ \bar{y}[t] = \underbrace{\begin{bmatrix} C & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\bar{C}_v} \bar{x}_v[t], \end{cases}$$

where $\bar{x}_v[t] = [\bar{x}'[t] \ \bar{u}'[t-\nu] \ \bar{u}'[t-\nu+1] \ \cdots \ \bar{u}'[t-1]]'$ and $\bar{x}[i] = x(ih)$.

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State-feedback for time-delay systems: closed-loop dynamics

LQR for time-delay systems

Setup

Assume for simplicity that $\tau = \nu h$ (i.e. $\theta = 0$) and consider

$$\bar{x}_\nu[t+1] = \bar{A}_\nu \bar{x}_\nu[t] + \bar{B}_\nu \bar{u}[t]$$

where

$$\bar{x}_\nu[t] := \begin{bmatrix} \bar{x}[t] \\ \bar{u}[t-\nu] \\ \vdots \\ \bar{u}[t-2] \\ \bar{u}[t-1] \end{bmatrix} \quad \text{and} \quad [\bar{A}_\nu \mid \bar{B}_\nu] := \left[\begin{array}{cc|cccc|c} \bar{A} & \bar{B} & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \end{array} \right]$$

under

$$[\bar{A} \quad \bar{B}] := \left[e^{Ah} \int_0^h e^{At} dt B \right].$$

The goal is to

- design state feedback with pole placement for this system.

State feedback structure

The state feedback control law

$$\bar{u}[t] = F\bar{x}_v[t] + F_r r[t] = \begin{bmatrix} F_x & F_v & \cdots & F_1 \end{bmatrix} \begin{bmatrix} \bar{x}[t] \\ \bar{u}[t-v] \\ \vdots \\ \bar{u}[t-1] \end{bmatrix} + F_r r[t]$$

reads

$$\bar{u}[t] - \sum_{i=1}^v F_i \bar{u}[t-i] = F_x \bar{x}[t] + F_r r[t].$$

This is a **dynamic** controller $(\bar{x}, r) \mapsto \bar{u}$. In the z -domain it reads

$$\left(I - \sum_{i=1}^v F_i z^{-i} \right) \bar{U}(z) = F_x \bar{X}(z) + F_r R(z)$$

and its transfer function

$$K(z) = \left(I - \sum_{i=1}^v F_i z^{-i} \right)^{-1} \begin{bmatrix} F_x & F_r \end{bmatrix}.$$

is even not necessarily stable.

Reachability of $(\bar{A}_v, \bar{B}_v)$

Theorem

$\lambda \in \mathbb{C}$ is an unreachable mode of $(\bar{A}_v, \bar{B}_v)$ iff it is an unreachable mode of $(\bar{A}, \bar{B})$, where

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} := \begin{bmatrix} e^{Ah} & \int_0^h e^{At} dt B \end{bmatrix}.$$

Proof: Follows by

$$\begin{bmatrix} \bar{A}_v - \lambda I & \bar{B}_v \end{bmatrix} = \begin{bmatrix} \bar{A} - \lambda I & \bar{B} & 0 & \cdots & 0 & | & 0 \\ 0 & -\lambda I & I & \cdots & 0 & | & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & | & \vdots \\ 0 & 0 & 0 & \cdots & I & | & 0 \\ 0 & 0 & 0 & \cdots & -\lambda I & | & I \end{bmatrix},$$

which is block lower-triangular, with nonsingular $(2, 2)$ block.



Reachability of $(\bar{A}_v, \bar{B}_v)$ (contd)

Thus,

- $(\bar{A}_v, \bar{B}_v)$ is reachable provided the delay-free (A, B) is controllable and sampling is not pathological.

Hence,

- arbitrary pole placement is possible.

But,

- does it make sense to move all open-loop poles?

After all,

- mv out of $n + mv$ of them are already dead-beat,

which would make sense to keep. This prompts to rethink the procedure.

Reachability of $(\bar{A}_v, \bar{B}_v)$ (contd)

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Hence,

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But,

- does it make sense to move all open-loop poles?

After all,

- $m\nu$ out of $n + m\nu$ of them are already dead-beat,

which would make sense to keep. This prompts to rethink the procedure.

Preliminary: partial pole placement

Consider

$$x[t+1] = Ax[t] + Bu[t]$$

for $x[t] \in \mathbb{R}^n$ and assume that only $\tilde{n} < n$ eigenvalues of A need be shifted by feedback. Suppose there is T such that

$$TAT^{-1} = \begin{bmatrix} \tilde{A} & 0 \\ \times & A_{\text{stay}} \end{bmatrix} \quad \text{and} \quad TB = \begin{bmatrix} \tilde{B} \\ B_{\text{stay}} \end{bmatrix}$$

such that $A_{\text{stay}} \in \mathbb{R}^{(n-\tilde{n}) \times (n-\tilde{n})}$ contains eigenvalues to be kept untouched. If we choose $u[t] = \tilde{F} \begin{bmatrix} I & 0 \end{bmatrix} T x[t]$, then the closed-loop “ A ” matrix is

$$A_{\text{cl}} = T^{-1} \begin{bmatrix} \tilde{A} & 0 \\ \times & A_{\text{stay}} \end{bmatrix} T = T^{-1} \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{F} & 0 \\ \times & A_{\text{stay}} \end{bmatrix} T$$

always keeps $\text{spec}(A_{\text{stay}}) \subset \text{spec}(A_{\text{cl}})$. In fact

we only need $\tilde{T} := \begin{bmatrix} I & 0 \end{bmatrix} T \in \mathbb{R}^{\tilde{n} \times n}$ to implement this law.

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- we only need $\tilde{T} := \begin{bmatrix} I & 0 \end{bmatrix} T \in \mathbb{R}^{\tilde{n} \times n}$ to implement this law.

Partial pole placement: equation for $\tilde{T}$

Because

$$TA = \begin{bmatrix} \tilde{A} & 0 \\ \times & A_{\text{stay}} \end{bmatrix} T$$

we have that

$$\tilde{T}A = [\tilde{A} \ 0] T = \tilde{A} [I \ 0] T = \tilde{A}\tilde{T}.$$

So all we need is to solve for $\tilde{T} \in \mathbb{R}^{\tilde{n} \times n}$

$$\tilde{T}A = \tilde{A}\tilde{T} \quad \text{and} \quad \text{rank } \tilde{T} = \tilde{n}$$

for any $\tilde{A}$ such that $\text{spec}(\tilde{A}) \subset \text{spec}(A)$. Then

$$u[t] = \tilde{F}\tilde{T}x[t]$$

guarantees that

$$\text{spec}(A + B\tilde{F}\tilde{T}) = \text{spec}(\tilde{A} + \tilde{T}B\tilde{F}) \cup (\text{spec}(A) \setminus \text{spec}(\tilde{A})).$$

Partial pole placement: equation for $\tilde{T}$

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Partial pole placement for $(\bar{A}_v, \bar{B}_v)$

Partitioning $\tilde{T} = \begin{bmatrix} T_x & T_v & \cdots & T_2 & T_1 \end{bmatrix}$ and choosing $\tilde{A} = \bar{A}$, we have:

$$\begin{aligned} \tilde{T} \begin{bmatrix} \bar{A} & \bar{B} & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} &= \underbrace{\begin{bmatrix} T_x \bar{A} & T_x \bar{B} & T_v & \cdots & T_3 & T_2 \end{bmatrix}}_{\tilde{T} \bar{A}_v} \\ &= \underbrace{\begin{bmatrix} \bar{A} T_x & \bar{A} T_v & \bar{A} T_{v-1} & \cdots & \bar{A} T_2 & \bar{A} T_1 \end{bmatrix}}_{\bar{A} \tilde{T}} \end{aligned}$$

This yields $T_2 = \bar{A} T_1$, $T_3 = \bar{A} T_2 = \bar{A}^2 T_1$, $\dots$, $T_v = \bar{A} T_{v-1} = \bar{A}^{v-1} T_1$,

$$\bar{A} T_v = T_x \bar{B}, \quad \text{and} \quad \bar{A} T_x = T_x \bar{A}.$$

Possible choices are $T_x = \bar{A}^v$, $T_v = \bar{A}^{v-1} \bar{B}$, and $T_1 = \bar{B}$.

State feedback solution

Thus,

$$\tilde{T} = \begin{bmatrix} \bar{A}^v & \bar{A}^{v-1}\bar{B} & \cdots & \bar{A}\bar{B} & \bar{B} \end{bmatrix}$$

and the resulting feedback gain

$$F = \begin{bmatrix} F_x & F_v & \cdots & F_2 & F_1 \end{bmatrix} = \bar{F} \begin{bmatrix} \bar{A}^v & \bar{A}^{v-1}\bar{B} & \cdots & \bar{A}\bar{B} & \bar{B} \end{bmatrix},$$

where $\bar{F}$ assigns the spectrum of

$$\bar{A} + \tilde{T}\bar{B}_v\bar{F} = \bar{A} + \bar{B}\bar{F},$$

which is a delay-free problem, solvable if $(\bar{A}, \bar{B})$ is reachable. The resulting

$$x[k] = F \left(\bar{A}^v x[k] + \sum_{i=1}^v \bar{A}^{v-i} \bar{B} u[k-i] \right) + F v[k]$$

assigns the $n + mv$ closed-loop eigenvalues to $\text{spec}(\bar{A} + \bar{B}\bar{F}) \cup \{0\}^{mv}$ then.

State feedback solution

Thus,

$$\tilde{T} = \begin{bmatrix} \bar{A}^v & \bar{A}^{v-1}\bar{B} & \cdots & \bar{A}\bar{B} & \bar{B} \end{bmatrix}$$

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$$\bar{u}[i] = \bar{F} \left(\bar{A}^v \bar{x}[i] + \sum_{j=1}^v \bar{A}^{j-1} \bar{B} \bar{u}[i-j] \right) + F_r r[i]$$

assigns the $n + mv$ closed-loop eigenvalues to $\text{spec}(\bar{A} + \bar{B}\bar{F}) \cup \{0\}^{mv}$ then.

Control law: interpretation

The signal

$$\bar{x}_{\text{pr}}[i] := \bar{A}^{\nu} \bar{x}[i] + \sum_{j=1}^{\nu} \bar{A}^{j-1} \bar{B} \bar{u}[i-j] = \bar{A}^{\nu} \bar{x}[i] + \sum_{j=i}^{i+\nu-1} \bar{A}^{i+\nu-1-j} \bar{B} \bar{u}[j-\nu]$$

is the ν step ahead **prediction of $\bar{x}[i + \nu]$** for the dynamics

$$\bar{x}[i+1] = \bar{A} \bar{x}[i] + \bar{B} \bar{u}[i-\nu],$$

i.e. under ideal conditions, w/o disturbances / uncertainty, $\bar{x}_{\text{pr}}[i] = \bar{x}[i + \nu]$.

The control law

$$\bar{u}[i] = \bar{F} \left(\bar{A}^{\nu} \bar{x}[i] + \sum_{j=1}^{\nu} \bar{A}^{j-1} \bar{B} \bar{u}[i-j] \right) + F_r r[i] = \bar{F} \bar{x}_{\text{pr}}[i] + F_r r[i]$$

is then known as predictor based.

Remark: The result remains qualitatively the same if $\theta \neq 0$, except for more cumbersome derivations and the fact that $\bar{x}_{\text{pr}}[i]$ predicts $x(ih + \tau)$, between sampling instances ih .

Outline

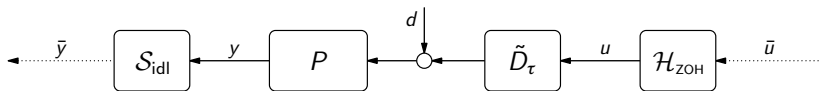
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LQR for time-delay systems

Adding load disturbance



Assuming that $\tau = \nu h$ and $d = \mathcal{H}_{\text{ZOH}} \bar{d}$, the system $(\bar{u}, \bar{d}) \mapsto \bar{y}$ is

$$\left\{ \begin{array}{l} \bar{x}_\nu[t+1] = \begin{bmatrix} \bar{A} & \bar{B} & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \bar{x}_\nu[t] + \underbrace{\begin{bmatrix} \bar{B} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{\bar{B}_d} \bar{d}[t] + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \bar{u}[t] \\ \bar{y}[t] = [C \mid 0 \ 0 \ \cdots \ 0] \bar{x}_\nu[t], \end{array} \right.$$

where $\bar{x}_\nu[t] = [\bar{x}'[t] \ \bar{u}'[t-\nu] \ \bar{u}'[t-\nu+1] \ \cdots \ \bar{u}'[t-1]]'$, as previously (the first row reads $\bar{x}[t+1] = \bar{A}\bar{x}[t] + \bar{B}\bar{d}[t] + \bar{B}\bar{u}[t-\nu]$, as expected).

Similarity transformation

Introduce the matrix

$$T = \begin{bmatrix} \bar{A}^\nu & \bar{A}^{\nu-1}\bar{B} & \cdots & \bar{A}\bar{B} & \bar{B} \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix} = \begin{bmatrix} \bar{A}^{-\nu} & -\bar{A}^{-1}\bar{B} & \cdots & -\bar{A}^{1-\nu}\bar{B} & -\bar{A}^{-\nu}\bar{B} \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix}^{-1}$$

(mind that $\bar{A} = e^{Ah}$ is non-singular). In this case

$$\tilde{x}[t] := T\bar{x}_\nu[t] = \begin{bmatrix} \bar{x}_{\text{pr}}[t] \\ \bar{u}[t - \nu + 1] \\ \vdots \\ \bar{u}[t - 2] \\ \bar{u}[t - 1] \end{bmatrix},$$

where $\bar{x}_{\text{pr}}[i] := \bar{A}^\nu \bar{x}[i] + \sum_{j=1}^\nu \bar{A}^{i-1} \bar{B} \bar{u}[i-j]$ is the predicted $\bar{x}[t + \nu]$.

Similarity transformation (contd)

Moreover, it can be verified that

$$T\bar{A}_v T^{-1} = \left[\begin{array}{c|cccc} \bar{A} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{array} \right], \quad T\bar{B}_d = \left[\begin{array}{c} \bar{A}^\nu \bar{B} \\ \hline 0 \\ \vdots \\ 0 \\ 0 \end{array} \right], \quad T\bar{B}_v = \left[\begin{array}{c} \bar{B} \\ \hline 0 \\ \vdots \\ 0 \\ I \end{array} \right],$$

and

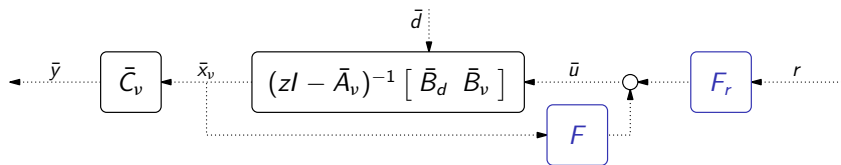
$$\left[C \mid 0 \quad \cdots \quad 0 \right] T^{-1} = \left[C\bar{A}^{-\nu} \mid -C\bar{A}^{-1}\bar{B} \quad \cdots \quad -C\bar{A}^{-\nu}\bar{B} \right].$$

Also,

$$FT^{-1} = \bar{F} \left[I \quad 0 \quad \cdots \quad 0 \right].$$

which results in $\bar{u}[t] = F\bar{x}_v[t] = \bar{F}\bar{x}_{pr}[t]$, as expected.

Closed-loop dynamics

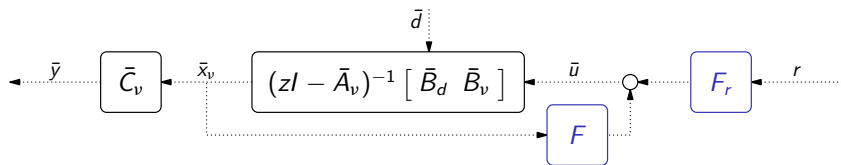


Thus, the closed-loop dynamics comprises

$$\left\{ \begin{array}{l} \tilde{x}[t+1] = \begin{bmatrix} \bar{A} & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tilde{x}[t] + \begin{bmatrix} \bar{A}^v \bar{B} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \bar{d}[t] + \begin{bmatrix} \bar{B} \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \bar{u}[t] \\ \tilde{y}[t] = C [\bar{A}^{-v} \mid -\bar{A}^{-1} \bar{B} \quad -\bar{A}^{-2} \bar{B} \quad \cdots \quad -\bar{A}^{-v} \bar{B}] \tilde{x}[t] \\ \bar{u}[t] = [\bar{F} \mid 0 \ 0 \ \cdots \ 0] \tilde{x}[t] + F_r r[t], \end{array} \right.$$

where components of $\tilde{x}$ from 2 to $v+1$ are unobservable from $\bar{u}$.

Closed-loop dynamics (contd)

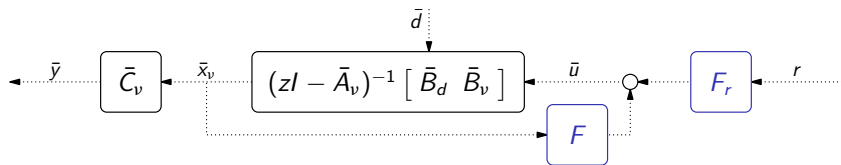


Plugging $\bar{u}$ from the last equation as the input to the first, we end up with

$$\left\{ \begin{array}{l} \tilde{x}[t+1] = \begin{bmatrix} \bar{A} + \bar{B}\bar{F} & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ \bar{F} & 0 & 0 & \cdots & 0 \end{bmatrix} \tilde{x}[t] + \begin{bmatrix} \bar{A}^v \bar{B} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \bar{d}[t] + \begin{bmatrix} \bar{B}F_r \\ 0 \\ \vdots \\ 0 \\ F_r \end{bmatrix} r[t] \\ \bar{y}[t] = C [\bar{A}^{-v} \mid -\bar{A}^{-1}\bar{B} \quad -\bar{A}^{-2}\bar{B} \quad \cdots \quad -\bar{A}^{-v}\bar{B}] \tilde{x}[t] \\ \bar{u}[t] = [\bar{F} \mid 0 \ 0 \ \cdots \ 0] \tilde{x}[t] + F_r r[t], \end{array} \right.$$

whose modes are those of $\text{spec}(\bar{A} + \bar{B}\bar{F})$ and mv dead-beat ones.

Closed-loop transfer functions



By direct substitutions,

$$T_{\bar{u}r}(z) = \left[\begin{array}{c|c} \bar{A} + \bar{B}\bar{F} & \bar{B}F_r \\ \hline \bar{F} & F_r \end{array} \right] \quad \text{and} \quad T_{\bar{u}\bar{d}}(z) = \left[\begin{array}{c|c} \bar{A} + \bar{B}\bar{F} & \bar{A}^v \bar{B} \\ \hline \bar{F} & 0 \end{array} \right].$$

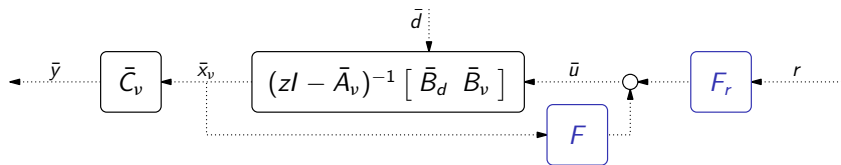
With more hard labor,

$$T_{\bar{y}r}(z) = \left[\begin{array}{c|c} \bar{A} + \bar{B}\bar{F} & \bar{B}F_r \\ \hline \bar{C} & 0 \end{array} \right] z^{-v}$$

and

$$T_{\bar{y}\bar{d}}(z) = \sum_{i=1}^v \bar{C}\bar{A}^{i-1}\bar{B}z^{-i} + \left[\begin{array}{c|c} \bar{A} + \bar{B}\bar{F} & \bar{A}^v \bar{B} \\ \hline \bar{C} & 0 \end{array} \right] z^{-v}.$$

Closed-loop transfer functions



By direct substitutions,

$$T_{\bar{u}r}(z) = \left[\frac{\bar{A} + \bar{B}\bar{F}}{\bar{F}} \mid \frac{\bar{B}F_r}{F_r} \right] \quad \text{and} \quad T_{\bar{u}\bar{d}}(z) = \left[\frac{\bar{A} + \bar{B}\bar{F}}{\bar{F}} \mid \frac{\bar{A}^v \bar{B}}{0} \right].$$

With more hard labor,

$$T_{\bar{y}r}(z) = \left[\frac{\bar{A} + \bar{B}\bar{F}}{C} \mid \frac{\bar{B}F_r}{0} \right] z^{-v}$$

and

$$T_{\bar{y}\bar{d}}(z) = \sum_{i=1}^v C \bar{A}^{i-1} \bar{B} z^{-i} + \left[\frac{\bar{A} + \bar{B}\bar{F}}{C} \mid \frac{\bar{A}^v \bar{B}}{0} \right] z^{-v}.$$

Example: DC motor with rigid mechanical load

Consider by now familiar

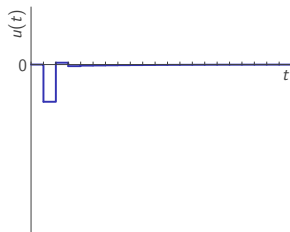
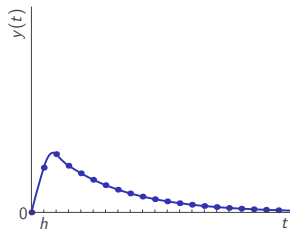
$$P(s) = \frac{1}{s(s+1)}.$$

The design of $\bar{F}$ is the pole placement to the roots of $\chi_{\text{cl}}(z) = z(z - \alpha)$ for the discretized delay-free plant, like in Lect. 6. Below we study

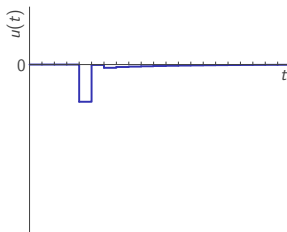
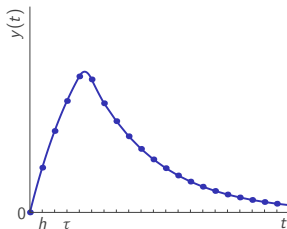
- disturbance response ($d(t) = \delta(t)$) under various delays $\tau - \nu h$ and α .

Example 1: disturbance response simulations

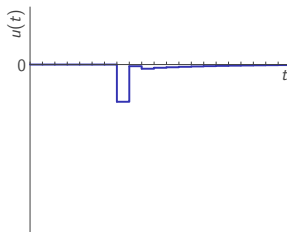
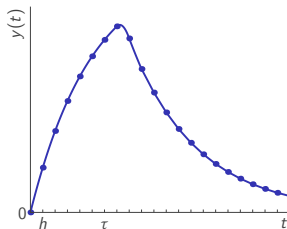
Let $\alpha = e^{-h} \approx 0.819$ (keeps the open-loop stable pole untouched). Then:



$\nu = 0$



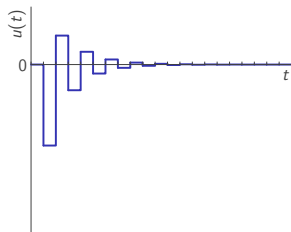
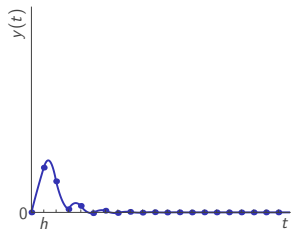
$\nu = 3$



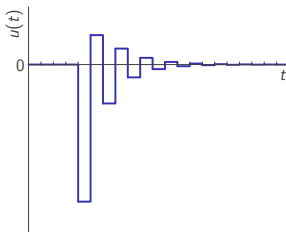
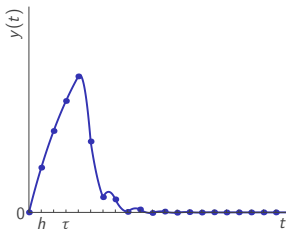
$\nu = 6$

Example 1: disturbance response simulations

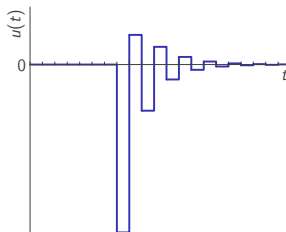
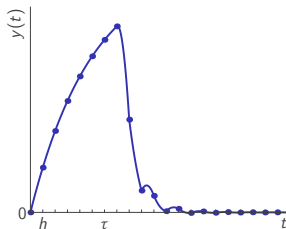
Let $\alpha = 0$ (dead-beat strategy). Then:



$\nu = 0$



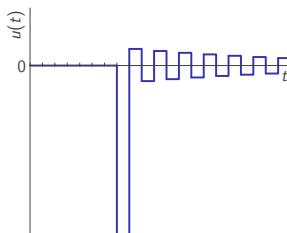
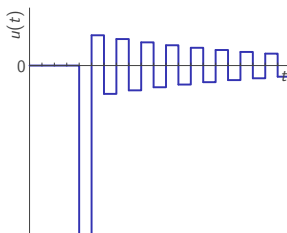
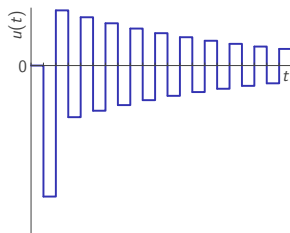
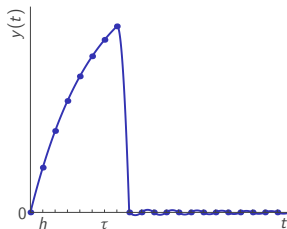
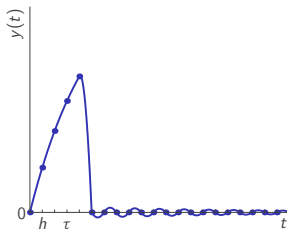
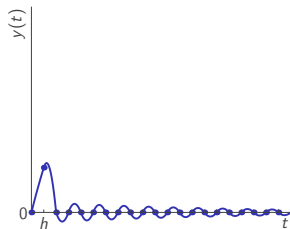
$\nu = 3$



$\nu = 6$

Example 1: disturbance response simulations

Let $\alpha = z_s \approx -0.936$ (canceling lightly-damped *sampling zero*). Then:



$\nu = 0$

$\nu = 3$

$\nu = 6$

Outline

Time delays in sampled-data systems

State-feedback for time-delay systems

State-feedback for time-delay systems: closed-loop dynamics

LQR for time-delay systems

Sampled-data LQR problems

Consider the input-delay dynamics

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t - \tau), & x(0) = x_0 \\ z(t) = C_z x(t) + D_z u(t - \tau) \end{cases}$$

where $z(t)$ is the regulated signal, whose “smallness” quantifies the control performance. The sampled-data LQR problems are still:

- supposing that the signal $x(t)$ is measurable **at $t = ih$** ,
- under whatever appropriate constraint on the waveform of $u(t)$,
- design $u(t)$, asymptotically stabilizing the system and minimizing

$$\|z\|_2^2 = \int_0^\infty \|z(t)\|^2 dt.$$

Note that

- we start measuring at $t = 0$, so that $u(t) = 0$ for $t < 0$.

Decomposition of the regulated signal

If $t < \tau$, then $x(t) = e^{At}x_0$, so that

$$z(t) = C_z e^{At} x_0$$

is independent of the control law.

If $t \geq \tau$, then

$$z(t) = C_z x(t) + D_z u(t - \tau) = C_z x_{\text{pr}}(t - \tau) + D_z u(t - \tau),$$

where

$$x_{\text{pr}}(t) := x(t + \tau) = e^{A\tau} x(t) + \int_{t-\tau}^t e^{A(t-s)} B u(s) ds.$$

Thus, the cost function

$$\|z\|_2^2 = \int_0^\tau \|C_z e^{At} x_0\|^2 dt + \int_0^\infty \|z(t + \tau)\|^2 dt = \int_0^\tau \|C_z e^{At} x_0\|^2 dt + \|z_{\text{pr}}\|_2^2,$$

where $z_{\text{pr}}(t) = C_z x_{\text{pr}}(t) + D_z u(t)$ depends on $u(t)$. The first question is
can the evolution of x_{pr} be described in the standard state-space form?

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- can the evolution of x_{pr} be described in the standard state-space form?

Evolution of $x_{\text{pr}}(t)$

By Leibniz integral rule

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(s, t) ds = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(s, t) ds + \frac{db(t)}{dt} f(b(t), t) - \frac{da(t)}{dt} f(a(t), t)$$

we have that

$$\begin{aligned} \dot{x}_{\text{pr}}(t) &= \frac{d}{dt} \left(e^{A\tau} x(t) + \int_{t-\tau}^t e^{A(t-s)} B u(s) ds \right) \\ &= e^{A\tau} \dot{x}(t) + \int_{t-\tau}^t A e^{A(t-s)} B u(s) ds + B u(t) - e^{A\tau} B u(t - \tau) \\ &= A \left(e^{A\tau} x(t) + \int_{t-\tau}^t e^{A(t-s)} B u(s) ds \right) + B u(t) \\ &= A x_{\text{pr}}(t) + B u(t) \end{aligned}$$

under the initial condition $x_{\text{pr}}(0) = e^{A\tau} x_0$. This is a

- standard (delay-free) state equation.

Equivalent delay-free LQR

Thus, the problem reduces to the sampled-data LQR for

$$\begin{cases} \dot{x}_{\text{pr}}(t) = Ax_{\text{pr}}(t) + Bu(t), & x_{\text{pr}}(0) = e^{A\tau}x_0 \\ z_{\text{pr}}(t) = C_z x_{\text{pr}}(t) + D_z u(t) \end{cases}$$

which is a delay-free problem, whose solutions we already know.

Whatever sampled-data problem is solved for this setup, we then need to

- replace $x_{\text{pr}}(ih) \rightarrow e^{A\tau}x(ih) + \int_{ih-\tau}^{ih} e^{A(ih-t)}Bu(s)dt$
- calculate the optimal cost for the original problem as

$$x_0' \left(\int_0^\tau e^{A'\tau} C_z' C_z e^{A\tau} dt + e^{A'\tau} X e^{A\tau} \right) x_0,$$

where X is the corresponding Riccati solution (depends on the setup).

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where X is the corresponding Riccati solution (depends on the setup).

Sampled-data LQR with $\mathcal{H}_{\text{ZOH}}$

If $u(t) = \bar{u}[i] = \text{const}$ for all $t \in [ih, (i+1)h]$ and $\tau = \nu h - \theta$, then

$$\begin{aligned}
 x_{\text{pr}}(ih) &= e^{A\tau} x(ih) + \int_{ih-\tau}^{ih} e^{A(ih-t)} B u(t) dt \\
 &= e^{A\tau} x(ih) + \sum_{j=1}^{\nu-1} \int_{(i-j)h}^{(i-j)h+h} e^{A(ih-t)} B u(t) dt \\
 &\quad + \int_{ih-\tau}^{(i-\nu)h+h} e^{A(ih-t)} B u(t) dt \\
 &= e^{A\tau} \bar{x}[i] + \sum_{j=1}^{\nu-1} \int_0^h e^{A((j-1)h+t)} dt B \bar{u}[i-j] \\
 &\quad + \int_{\theta}^h e^{A(\tau-h+t)} dt B \bar{u}[i-\nu]
 \end{aligned}$$

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 &= e^{A\tau} \bar{x}[i] + \sum_{j=1}^{\nu-1} \int_0^h e^{A((j-1)h+t)} dt B \bar{u}[i-j] \\
 &\quad + \int_{\theta}^h e^{A(\tau-h+t)} dt B \bar{u}[i-\nu] \\
 &= e^{A\tau} \bar{x}[i] + \sum_{j=1}^{\nu-1} \bar{A}^{j-1} \bar{B} \bar{u}[i-j] + e^{A(\tau-h)} \bar{B}_0 \bar{u}[i-\nu].
 \end{aligned}$$

Sampled-data LQR with optimal hold

In this case $u(t) = Fe^{(A+BF)(t-ih)}\bar{x}[i]$ for all $t \in [ih, (i+1)h]$. Hence,

$$\begin{aligned}
 x_{\text{pr}}(ih) &= e^{A\tau}x(ih) + \sum_{j=1}^{v-1} \int_{(i-j)h}^{(i-j)h+h} e^{A(ih-t)} Bu(t) dt \\
 &\quad + \int_{ih-\tau}^{(i-v)h+h} e^{A(ih-t)} Bu(t) dt \\
 &= e^{A\tau}\bar{x}[i] + \sum_{j=1}^{v-1} \int_{(i-j)h}^{(i-j)h+h} e^{A(ih-t)} BF e^{(A+BF)(t-(i-j)h)} dt \bar{x}[i-j] \\
 &\quad + \int_{ih-\tau}^{(i-v)h+h} e^{A(ih-t)} BF e^{(A+BF)(t-(i-v)h)} dt \bar{x}[i-v] \\
 &= e^{A\tau}\bar{x}[i] + \sum_{j=1}^{v-1} e^{A(j-1)h} \int_0^h e^{A(h-t)} BF e^{(A+BF)t} dt \bar{x}[i-j] \\
 &\quad + e^{A(\tau-h)} \int_{\theta}^h e^{A(h+\theta-t)} BF e^{(A+BF)t} dt \bar{x}[i-v].
 \end{aligned}$$

Sampled-data LQR with optimal hold (contd)

By the logic of deriving Van Loan's formulae,

$$\begin{aligned}
 \int_0^h e^{A(h-t)} B F e^{(A+BF)t} dt &= \begin{bmatrix} I & 0 \end{bmatrix} \exp \left(\begin{bmatrix} A & BF \\ 0 & A+BF \end{bmatrix} h \right) \begin{bmatrix} 0 \\ I \end{bmatrix} \\
 &= \begin{bmatrix} I & I \end{bmatrix} \exp \left(\begin{bmatrix} A & 0 \\ 0 & A+BF \end{bmatrix} h \right) \begin{bmatrix} -I \\ I \end{bmatrix} \\
 &= e^{(A+BF)h} - e^{Ah}
 \end{aligned}$$

and we end up with

$$\begin{aligned}
 x_{\text{pr}}(ih) &= e^{A\tau} \bar{x}[i] + \sum_{j=1}^{v-1} \bar{A}^{j-1} (e^{(A+BF)h} - \bar{A}) \bar{x}[i-j] \\
 &\quad + \bar{A}^{v-1} (e^{(A+BF)h} - e^{A(h-\theta)} e^{(A+BF)\theta}) \bar{x}[i-v].
 \end{aligned}$$