

Sampled-Data Control Systems (036709)

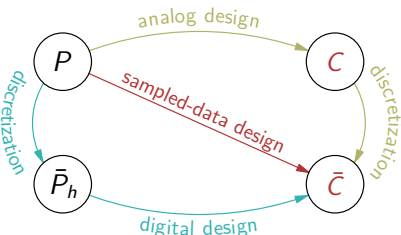
lecture no. 9

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Approaches to computer control design



1. Discrete-time design

(discretize the problem first, then do your favorite discrete design)

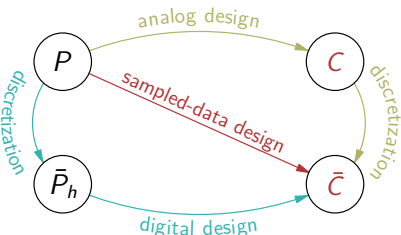
2. Digital redesign of analog controllers

(do your favorite analog design first, then discretize the resulting controller)

3. Direct digital (sampled-data) design

(design $\bar{C}$, and possibly $\mathcal{S}$ and $\mathcal{H}$, directly for analog plant and specs)

Approaches to computer control design



1. Discrete-time design

(discretize the problem first, then do your favorite discrete design)

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Outline

Mathematical preliminaries

Continuous-time linear-quadratic regulator (LQR) problem

Discrete-time linear-quadratic regulator (LQR) problem

Sampled-data linear-quadratic regulator (LQR) problem: ZOH

Sampled-data linear-quadratic regulator (LQR) problem: optimal hold

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Schur complement

Consider the block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square, and assume that $\det A_{22} \neq 0$. In this case

$$A = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}.$$

The matrix

$$\Delta_{22} := A_{11} - A_{12}A_{22}^{-1}A_{21}$$

is known as the **Schur complement** of A_{22} in A .

Remark: If $\det A_{11} \neq 0$, then we can also define the Schur complement of A_{11} in A as $\Delta_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$. All its properties / uses are similar to those of Δ_{22} .

Schur complement: completing the square

If $A = A'$, then

$$\begin{aligned}
 x'Ax &= \begin{bmatrix} x'_1 & x'_2 \end{bmatrix} \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta_{22} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
 &= \begin{bmatrix} x'_1 & x'_2 + x'_1A_{12}A_{22}^{-1} \end{bmatrix} \begin{bmatrix} \Delta_{22} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 + A_{22}^{-1}A_{21}x_1 \end{bmatrix} \\
 &= x'_1\Delta_{22}x_1 + (x_2 + A_{22}^{-1}A_{21}x_1)'A_{22}(x_2 + A_{22}^{-1}A_{21}x_1).
 \end{aligned}$$

This is effectively the matrix form of **completing the square** procedure.

Consequences:

$$A > 0 \iff (A_{22} > 0) \wedge (\Delta_{22} > 0).$$

$$\text{if } \det A_{22} \neq 0, \text{ then } A \geq 0 \iff (A_{22} > 0) \wedge (\Delta_{22} \geq 0).$$

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- if $\det A_{22} \neq 0$, then $A \geq 0 \iff (A_{22} > 0) \wedge (\Delta_{22} \geq 0).$

Matrix square root

Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive semi-definite, i.e. $A = A' \geq 0$. The square root $B = A^{1/2}$ is the unique $n \times n$ matrix, such that¹

$$B = B' \geq 0 \quad \text{and} \quad B^2 = A.$$

Moreover, $A^{1/2} > 0 \iff A > 0$.

Example

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \geq 0 \quad \implies \quad A^{1/2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \geq 0.$$

Example

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} > 0 \quad \implies \quad A^{1/2} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} > 0.$$

¹MATLAB: `sqrtm(A)`. Mathematica: `MatrixPower[A,1/2]`.

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CARE

The matrix equation (we assume that D has full column rank)

$$A'X + XA + C'C - (C'D + XB)(D'D)^{-1}(D'C + B'X) = 0$$

is called the **continuous-time algebraic Riccati equation** (CARE). Its solution $X \in \mathbb{R}^{n \times n}$ is said to be stabilizing if the matrix

$$A_F := A - B(D'D)^{-1}(D'C + B'X)$$

is Hurwitz. The stabilizing X , if exists, is unique and satisfies $X = X'$. If

1. (A, B) is stabilizable,
2. $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full column rank $\forall \omega \in \mathbb{R}$,

then the stabilizing X exists² and is such that $X \geq 0$.

Remark: If (C, A) has no pure imaginary unobservable modes, then condition 2. equals the full column rank of $D + C(j\omega I - A)^{-1}B$ for all $\omega \in \mathbb{R}$.

²MATLAB: `icare(A,B,C'*C,D'*D,C'*D)`, since R2019a.

CARE: alternative form

Let

$$\begin{aligned} \begin{bmatrix} \Phi_{11}(X) & \Phi_{12}(X) \\ \Phi_{21}(X) & \Phi_{22}(X) \end{bmatrix} &:= \begin{bmatrix} C'C + A'X + XA & C'D + XB \\ D'C + B'X & D'D \end{bmatrix} \\ &= \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} X & 0 \end{bmatrix} + \begin{bmatrix} X \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix}. \end{aligned}$$

The CARE reads then

$$\Phi_{11}(X) - \Phi_{12}(X)\Phi_{22}^{-1}(X)\Phi_{21}(X) = 0,$$

i.e. CARE solutions zero the Schur complement of $\Phi_{22}(X)$. If $X = X'$ solves the CARE, then for all compatibly dimensioned x and u

$$\begin{bmatrix} x' & u' \end{bmatrix} \begin{bmatrix} \Phi_{11}(X) & \Phi_{12}(X) \\ \Phi_{21}(X) & \Phi_{22}(X) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = (u - Fx)' \Phi_{22}(X) (u - Fx),$$

where $F := -\Phi_{22}^{-1}(X)\Phi_{21}(X) = -(D'D)^{-1}(D'C + B'X)$.

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CARE: example

Let $A = a$, $B = 1$, $C = [\sqrt{q}]$, and $D = [\frac{0}{\sqrt{r}}]$ for $r > 0$. With these,

– $(A, B) = (a, 1)$ is controllable,

$$- \begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix} = \begin{bmatrix} a - j\omega & 1 \\ \sqrt{q} & 0 \\ 0 & \sqrt{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/\sqrt{r} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a - j\omega & 0 \\ \sqrt{q} & 0 \\ 0 & \sqrt{r} \end{bmatrix},$$

so the assumptions hold iff $(a \neq 0) \vee (q \neq 0)$. The CARE reads in this case

$$2aX + q - X^2/r = 0 \iff X = ar \pm \sqrt{(a^2 r + q)r}.$$

Hence,

$$A_F = A - B(D'D)^{-1}(B'X + D'C) = a - X/r = \mp \sqrt{a^2 + q/r}$$

and it is Hurwitz for the "+" solution (provided $(a \neq 0) \vee (q \neq 0)$). Thus, the stabilizing

$$X = ar + \sqrt{(a^2 r + q)r} \geq 0.$$

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DARE

The matrix equation

$$X = A'XA + C'C - (C'D + A'XB)(D'D + B'XB)^{-1}(D'C + B'XA).$$

is dubbed the **discrete-time algebraic Riccati equation** (DARE). Its solution $X \in \mathbb{R}^{n \times n}$ is said to be stabilizing if $\det(D'D + B'XB) \neq 0$ and

$$A_F := A - B(D'D + B'XB)^{-1}(D'C + B'XA)$$

is Schur. The stabilizing X , if exists, is unique and satisfies $X = X'$. If

- (A, B) is stabilizable,
- $\begin{bmatrix} A - e^{j\theta}I & B \\ C & D \end{bmatrix}$ has full column rank $\forall \theta \in [-\pi, \pi]$,

then the stabilizing X exists³ and is such that $X \geq 0$.

Remark: Unlike CARE, the DARE may be well posed for singular, or even zero, D .

³MATLAB: `idare(A,B,C'*C,D'*D,C'*D)`, since R2019a.

DARE: alternative form

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The DARE reads then

$$\Phi_{11}(X) - \Phi_{12}(X)\Phi_{22}^{-1}(X)\Phi_{21}(X) = 0,$$

i.e. DARE solutions zero the Schur complement of $\Phi_{22}(X)$. If $X = X'$ solves the DARE, then for all compatibly dimensioned x and u

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DARE: example

Let $A = a$, $B = 1$, $C = \begin{bmatrix} \sqrt{q} \\ 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 \\ \sqrt{r} \end{bmatrix}$. With these,

– $(A, B) = (a, 1)$ is controllable,

$$- \begin{bmatrix} A - e^{j\theta} I & B \\ C & D \end{bmatrix} = \begin{bmatrix} a - e^{j\theta} & 1 \\ \sqrt{q} & 0 \\ 0 & \sqrt{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sqrt{r} & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} a - e^{j\theta} & 1 \\ \hline \sqrt{q} & 0 \\ \sqrt{r}(e^{j\theta} - a) & 0 \end{array} \right],$$

so the assumptions hold iff $(r(|a| - 1) \neq 0) \vee (q \neq 0)$. The DARE reads in this case

$$X = a^2 X + q - \frac{a^2 X^2}{r + X} = \frac{ra^2 X}{r + X} + q$$

and is

$r = 0$: linear, with $X = q$ (stabilizing iff $q \neq 0$)

$$r > 0: \text{quadratic, with } X = \frac{r(a^2 - 1) + q \pm \sqrt{(r(a^2 - 1) + q)^2 + 4qr}}{2} + qr$$

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and is

$r = 0$: linear, with $X = q$ (stabilizing iff $q \neq 0$)

$r > 0$: quadratic, with $X = \frac{r(a^2 - 1) + q}{2} \pm \sqrt{\frac{(r(a^2 - 1) + q)^2}{4} + qr}$

DARE: example (contd)

If $r > 0$, then

$$r + X = \frac{r(a^2 + 1) + q}{2} \pm \sqrt{\frac{(r(a^2 - 1) + q)^2}{4} + qr}$$

is zero only for its “−” part for $a = 0$. Furthermore,

$$A_F = a - \frac{aX}{r + X} = \frac{r(a^2 + 1) + q}{2ar} \mp \frac{\sqrt{(r(a^2 + 1) + q)^2 - (2ar)^2}}{2ar}$$

and it is Schur for the “+” solution of X . Hence, the stabilizing

$$X = \frac{r(a^2 - 1) + q}{2} + \sqrt{\frac{(r(a^2 - 1) + q)^2}{4} + qr}.$$

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Optimization-based design

The underlying idea:

- express requirements in terms of a size of a **cost function** to be minimized by the control law.

Important to remember that

- no performance index can reflect all our requirements.

We therefore shall use

- optimization as design tool,
- rather than the control goal per se. Every controller is optimal with respect to some cost function. But not every optimal controller makes sense. Thus, optimization methods should be judged by
- simplicity of their solutions
 - simplicity of tuning their properties via weighting functions
 - byproducts (what do we get for granted)

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Continuous-time LQR problem

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ z(t) = C_z x(t) + D_z u(t) \end{cases}$$

where $z(t)$ is the **regulated signal**, whose “smallness” quantifies the control performance. The Linear-Quadratic Regulator (LQR) problem:

- supposing that the state vector $x(t)$ is measurable,
- design $u(t)$, asymptotically stabilizing the system and minimizing

$$\|z\|_2^2 = \int_0^\infty \|z(t)\|^2 dt.$$

Frequently,

$$z(t) = \begin{bmatrix} Q^{1/2} x(t) \\ R^{1/2} u(t) \end{bmatrix} \Rightarrow \|z(t)\|^2 = x'(t) Q x(t) + u'(t) R u(t)$$

for some $Q \geq 0$ and $R > 0$. But other choices are possible.

Continuous-time LQR problem

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$$z(t) = \begin{bmatrix} Q^{1/2} x(t) \\ R^{1/2} u(t) \end{bmatrix} \implies \|z(t)\|^2 = z'(t)z(t) = x'(t)Qx(t) + u'(t)Ru(t)$$

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Continuous-time LQR problem: assumptions

Standard assumptions:

$\mathcal{A}_1$: (A, B) is stabilizable

$\mathcal{A}_2$: D_z has full column rank

$\mathcal{A}_3$: $\begin{bmatrix} A - j\omega I & B \\ C_z & D_z \end{bmatrix}$ has full column rank $\forall \omega \in \mathbb{R}$

Assumption

- $\mathcal{A}_1$ is necessary for the existence of stabilizing control law.

Assumptions

- $\mathcal{A}_2$ and $\mathcal{A}_3$ guarantee the existence and uniqueness of the solution and are technical. If $\mathcal{A}_{1-3}$ hold, then the CARE

$$A'X + XA + C_z' C_z - (C_z' D_z + X B)(D_z' D_z)^{-1} (D_z' C_z + B' X) = 0$$

has a stabilizing solution $X = X' \geq 0$.

Continuous-time LQR problem: solution

Introduce the matrix

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} := \begin{bmatrix} C_z' \\ D_z' \end{bmatrix} \begin{bmatrix} C_z & D_z \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} X & 0 \end{bmatrix} + \begin{bmatrix} X \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix}$$

and consider the quadratic form

$$\gamma(t) := \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

By direct substitution,

$$\gamma(t) = \|z(t)\|^2 + \dot{x}'(t)Xx(t) + x'(t)X\dot{x}(t) = \|z(t)\|^2 + \frac{d}{dt}x'(t)Xx(t)$$

By completing the square, and the fact that X zeroes $R_{11} - R_{12}R_{22}^{-1}R_{21}$,

$$\gamma(t) = (u(t) - Fx(t))'R_{22}(u(t) - Fx(t)),$$

where $F := -R_{22}^{-1}R_{21} = -(D_z'D_z)^{-1}(D_z'C_z + B'X)$.

Continuous-time LQR problem: solution (contd)

Thus, regardless the choice of the control signal u , we have that

$$\|z(t)\|^2 = -\frac{d}{dt}x'(t)Xx(t) + (u(t) - Fx(t))'R_{22}(u(t) - Fx(t)).$$

Using the convention $\|x\|_M := \sqrt{x'Mx}$,

$$\begin{aligned}\|z\|_2^2 &= -\int_0^\infty \frac{d}{dt}x'(t)Xx(t)dt + \int_0^\infty \|u(t) - Fx(t)\|_{R_{22}}^2 dt \\ &= x_0'Xx_0 - \lim_{t \rightarrow \infty} x'(t)Xx(t) + \int_0^\infty \|u(t) - Fx(t)\|_{R_{22}}^2 dt\end{aligned}$$

Therefore, whenever the system is asymptotically stable,

$$\|z\|_2^2 = x_0'Xx_0 + \int_0^\infty \|u(t) - Fx(t)\|_{R_{22}}^2 dt \geq x_0'Xx_0$$

and the bound is attained iff

$$u(t) - Fx(t) = -(D_z'D_z)^{-1}(D_z'C_z + B'X)x(t).$$

Continuous-time LQR problem: solution (contd)

Thus, regardless the choice of the control signal u , we have that

$$\|z(t)\|^2 = -\frac{d}{dt}x'(t)Xx(t) + (u(t) - Fx(t))'R_{22}(u(t) - Fx(t)).$$

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and the bound is attained iff

$$u(t) = Fx(t) = -(D_z'D_z)^{-1}(D_z'C_z + B'X)x(t).$$

Continuous-time LQR problem: solution (contd)

Theorem

If $\mathcal{A}_{1-3}$ hold, then the unique solution of the continuous-time LQR problem is the state feedback

$$u(t) = -(D_z' D_z)^{-1} (D_z' C_z + B' X) x(t),$$

which stabilizes the system and attains $\|z\|_2^2 = x_0' X x_0$.

Proof: It only remains to prove that $u(t)$ above is indeed stabilizing. This follows from the fact that X is the stabilizing solution to the corresponding CARE, so that the closed-loop matrix

$$A + BF = A - B(D_z' D_z)^{-1} (D_z' C_z + B' X)$$

is Hurwitz. □

Example: DC motor with rigid mechanical load

Consider an analog plant having the transfer function

$$P(s) = \frac{1}{s(s+1)},$$

which can be thought of as a voltage controlled DC motor with normalized time constant. Bring in its companion form

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

in which case the state comprises $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$.

Example: regulated signal

We would like to have

- fast convergence of $y(t)$ $\implies$ penalize $\|y\|_2$
- smooth convergence of $y(t)$ $\implies$ penalize $\|\dot{y}\|_2$
- small control signal $u(t)$ $\implies$ penalize $\|u\|_2$

Thus, choose

$$z(t) = \begin{bmatrix} \sqrt{\lambda} y(t) \\ \sqrt{1-\lambda} \dot{y}(t) \\ \sqrt{\rho} u(t) \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda} C \\ \sqrt{1-\lambda} CA \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \sqrt{\rho} \end{bmatrix} u(t)$$

for some weights $\lambda \in [0, 1]$ and $\rho > 0$ ($\lambda \uparrow 1$ encourages faster convergence, $\lambda \downarrow 0$ encourages smoother convergence, and the increase of ρ discourages large control energy). For this choice

$$\|z\|_2^2 = \lambda \|y\|_2^2 + (1-\lambda) \|\dot{y}\|_2^2 + \rho \|u\|_2^2.$$

Example: regulated signal

We would like to have

- fast convergence of $y(t)$ $\implies$ penalize $\|y\|_2$
- smooth convergence of $y(t)$ $\implies$ penalize $\|\dot{y}\|_2$
- small control signal $u(t)$ $\implies$ penalize $\|u\|_2$

Thus, choose⁴

$$z(t) = \begin{bmatrix} \sqrt{\lambda} y(t) \\ \sqrt{1-\lambda} \dot{y}(t) \\ \sqrt{\rho} u(t) \end{bmatrix} = \overbrace{\begin{bmatrix} \sqrt{\lambda} C \\ \sqrt{1-\lambda} CA \\ 0 \end{bmatrix}}^{C_z} x(t) + \overbrace{\begin{bmatrix} 0 \\ 0 \\ \sqrt{\rho} \end{bmatrix}}^{D_z} u(t)$$

for some weights $\lambda \in [0, 1]$ and $\rho > 0$ ($\lambda \uparrow 1$ encourages faster convergence, $\lambda \downarrow 0$ encourages smoother convergence, and the increase of ρ discourages large control energy). For this choice

$$\|z\|_2^2 = \lambda \|y\|_2^2 + (1 - \lambda) \|\dot{y}\|_2^2 + \rho \|u\|_2^2.$$

⁴Because $y = Cx$ and $\dot{y} = C\dot{x} = C(Ax + Bu) = CAx$ (for this system $CB = 0$).

Example: assumptions

$\mathcal{A}_1$: The realization is minimal, hence (A, B) is controllable.

$\mathcal{A}_3$: The matrix

$$\left[\begin{array}{cc|c} -j\omega & 1 & 0 \\ 0 & -1 - j\omega & 1 \\ \hline \sqrt{\lambda} & 0 & 0 \\ 0 & \sqrt{1 - \lambda} & 0 \\ 0 & 0 & \sqrt{\rho} \end{array} \right]$$

has full column rank for all ω iff $\lambda \neq 0$. In fact, if $\lambda = 0$, then the pair

$$(C_z, A) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right)$$

has an unobservable mode at the origin.

Thus,

- we have to keep $\lambda \in (0, 1]$.

Example: some analytic formulae

The stabilizing

$$X = \begin{bmatrix} \sqrt{\lambda} \alpha & \sqrt{\lambda \rho} \\ \sqrt{\lambda \rho} & \sqrt{\rho}(\alpha - \sqrt{\rho}) \end{bmatrix} > 0,$$

where $\alpha := \sqrt{1 - \lambda + 2\sqrt{\lambda \rho} + \rho} > \sqrt{\rho}$. The gain (PD):

$$F = -\sqrt{\frac{\lambda}{\rho}} \begin{bmatrix} 1 & (\alpha - \sqrt{\rho})/\sqrt{\lambda} \end{bmatrix}.$$

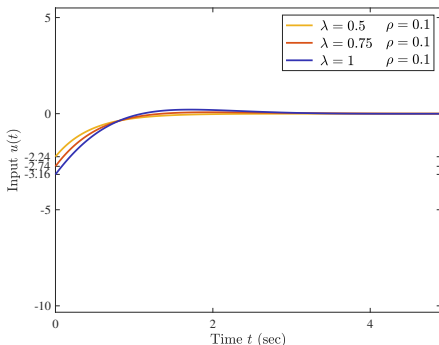
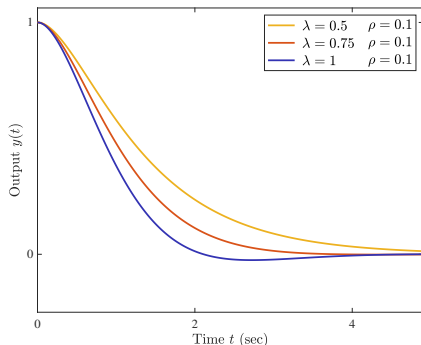
If $\dot{y}(0) = 0$, then $u(0) = -\sqrt{\lambda/\rho}$, increases as λ increases and ρ decreases.
Closed-loop eigenvalues:

$$\frac{-\alpha \pm \sqrt{1 - \lambda - 2\sqrt{\lambda \rho} + \rho}}{2\sqrt{\rho}}.$$

They are real iff $\lambda \leq 1 + 3\rho - 2\sqrt{2\rho^2 + \rho}$, which is always true if $\rho \geq 4$ or $\lambda \leq 0.5$. If $\lambda = 1$, then the damping factor $\zeta = \sqrt{2 + \sqrt{\rho}}/2 > 1/\sqrt{2}$.

Example: simulations

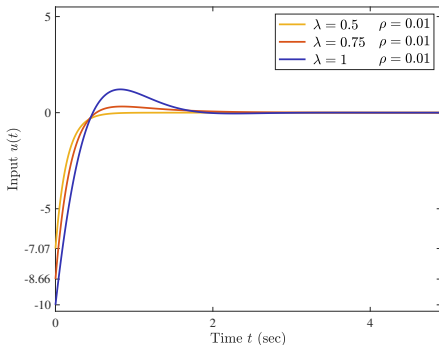
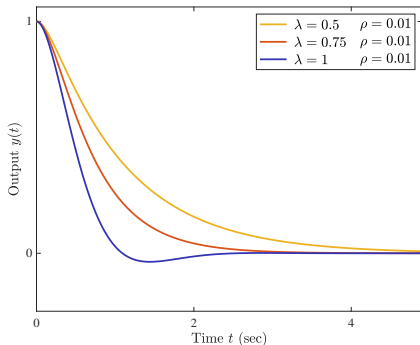
With (large) $\rho = 0.1$:



- under $\lambda = 1$ the response is faster, but overshoots (complex poles)
- as λ decreases, the response slows down, but smooths
- control effort decreases as λ decreases (always peaks at $-\sqrt{\lambda/\rho}$)

Example: simulations (contd)

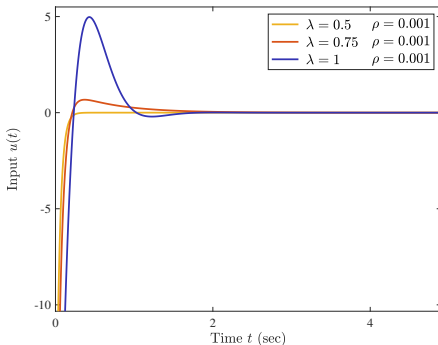
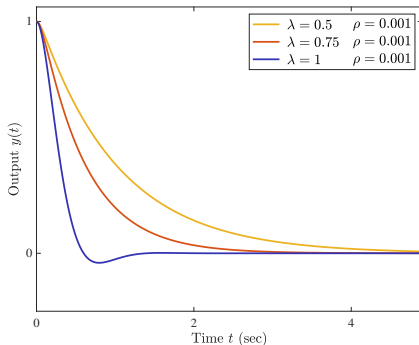
With (middle) $\rho = 0.01$:



- under $\lambda = 1$ the response is faster, but overshoots (complex poles)
- as λ decreases, the response slows down, but smooths
- control effort decreases as λ decreases (always peaks at $-\sqrt{\lambda/\rho}$)

Example: simulations (contd)

With (small) $\rho = 0.001$:



- under $\lambda = 1$ the response is faster, but overshoots (complex poles)
- as λ decreases, the response slows down, but smooths
- control effort decreases as λ decreases (always peaks at $-\sqrt{\lambda/\rho}$)

Outline

Mathematical preliminaries

Continuous-time linear-quadratic regulator (LQR) problem

Discrete-time linear-quadratic regulator (LQR) problem

Sampled-data linear-quadratic regulator (LQR) problem: ZOH

Sampled-data linear-quadratic regulator (LQR) problem: optimal hold

Discrete-time LQR problem

Consider

$$\begin{cases} x[t+1] = Ax[t] + Bu[t], & x[0] = x_0 \\ z[t] = C_z x[t] + D_z u[t] \end{cases}$$

where $z[t]$ is the regulated signal, whose “smallness” quantifies the control performance. The discrete-time LQR problem:

- supposing that the state vector $x[t]$ is measurable,
- design $u[t]$, asymptotically stabilizing the system and minimizing

$$\|z\|_2^2 = \sum_{t \in \mathbb{Z}_+} \|z[t]\|^2.$$

Frequently,

$$z[t] = \begin{bmatrix} Q^{1/2} x[t] \\ R^{1/2} u[t] \end{bmatrix} \Rightarrow \|z[t]\|^2 = x'[t] Q x[t] + u'[t] R u[t]$$

for some $Q \succeq 0$ and $R \succeq 0$. But other choices are possible.

Discrete-time LQR problem

Consider

$$\begin{cases} x[t+1] = Ax[t] + Bu[t], & x[0] = x_0 \\ z[t] = C_z x[t] + D_z u[t] \end{cases}$$

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- supposing that the state vector $x[t]$ is measurable,
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$$\|z\|_2^2 = \sum_{t \in \mathbb{Z}_+} \|z[t]\|^2.$$

Frequently,

$$z[t] = \begin{bmatrix} Q^{1/2} x[t] \\ R^{1/2} u[t] \end{bmatrix} \implies \|z[t]\|^2 = z'[t]z[t] = x'[t]Qx[t] + u'[t]Ru[t]$$

for some $Q \geq 0$ and $R \geq 0$. But other choices are possible.

Discrete-time LQR problem: assumptions

Standard assumptions:

$\mathcal{A}_4$: (A, B) is stabilizable

$\mathcal{A}_5$: $\begin{bmatrix} A - e^{j\theta}I & B \\ C_z & D_z \end{bmatrix}$ has full column rank $\forall \theta \in [-\pi, \pi]$

Assumption

- $\mathcal{A}_4$ is necessary for the existence of stabilizing control law.

Assumption

- $\mathcal{A}_5$ guarantees the existence and uniqueness of the solution and is technical. If $\mathcal{A}_{4-5}$ hold, then the DARE

$$X = A'XA + C_z'C_z - (C_z'D_z + A'XB)(D_z'D_z + B'XB)^{-1}(D_z'C_z + B'XA)$$

has a stabilizing solution $X = X' \geq 0$ (in this case $D_z'D_z + B'XB > 0$).

Discrete-time LQR problem: solution

Introduce the matrix

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} := \begin{bmatrix} C_z' \\ D_z' \end{bmatrix} \begin{bmatrix} C_z & D_z \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} X \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$$

and consider the quadratic form

$$\gamma[t] := \begin{bmatrix} x'[t] & u'[t] \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x[t] \\ u[t] \end{bmatrix}.$$

By direct substitution,

$$\gamma[t] = \|z[t]\|^2 + x'[t+1]Xx[t+1] - x'[t]Xx[t]$$

By completing the square, and the fact that X zeroes $R_{11} - R_{12}R_{22}^{-1}R_{21}$,

$$\gamma[t] = (u[t] - Fx[t])'R_{22}(u[t] - Fx[t]),$$

where $F := -R_{22}^{-1}R_{21} = -(D_z'D_z + B'XB)^{-1}(D_z'C_z + B'XA)$.

Discrete-time LQR problem: solution (contd)

Thus, regardless the choice of the control signal u , we have that

$$\|z[t]\|^2 = x'[t]Xx[t] - x'[t+1]Xx[t+1] + \|u[t] - Fx[t]\|_{R_{22}}^2,$$

where, again, $\|x\|_M := \sqrt{x'Mx}$, and

$$\begin{aligned}\|z\|_2^2 &= \sum_{t \in \mathbb{Z}_+} (x'[t]Xx[t] - x'[t+1]Xx[t+1]) + \sum_{t \in \mathbb{Z}_+} \|u[t] - Fx[t]\|_{R_{22}}^2 \\ &= x'_0 X x_0 - \lim_{t \rightarrow \infty} x'[t]Xx[t] + \sum_{t \in \mathbb{Z}_+} \|u[t] - Fx[t]\|_{R_{22}}^2\end{aligned}$$

Therefore, whenever the system is asymptotically stable,

$$\|z\|_2^2 = x'_0 X x_0 + \sum_{t \in \mathbb{Z}_+} \|u[t] - Fx[t]\|_{R_{22}}^2 \geq x'_0 X x_0$$

and the bound is attained iff

$$u[t] - Fx[t] = -(D'_2 D_2 + B'XB)^{-1}(D'_2 C_2 + B'XA)x[t].$$

Discrete-time LQR problem: solution (contd)

Thus, regardless the choice of the control signal u , we have that

$$\|z[t]\|^2 = x'[t]Xx[t] - x'[t+1]Xx[t+1] + \|u[t] - Fx[t]\|_{R_{22}}^2,$$

where, again, $\|x\|_M := \sqrt{x'Mx}$, and

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and the bound is attained iff

$$u[t] = Fx[t] = -(D'_zD_z + B'XB)^{-1}(D'_zC_z + B'XA)x[t].$$

Discrete-time LQR problem: solution (contd)

Theorem

If $\mathcal{A}_{4-5}$ hold, then the unique solution of the discrete-time LQR problem is the state feedback

$$u[t] = -(D_z' D_z + B' X B)^{-1} (D_z' C_z + B' X A) x[t],$$

which stabilizes the system and attains $\|z\|_2^2 = x_0' X x_0$.

Proof: It only remains to prove that $u[t]$ above is indeed stabilizing. This follows from the fact that X is the stabilizing solution to the corresponding CARE, so that the closed-loop matrix

$$A + BF = A - B(D_z' D_z + B' X B)^{-1} (D_z' C_z + B' X A)$$

is Schur. □

Discrete-time LQR problem: $D_z = 0$

In this case the closed-loop matrix reads

$$A + BF = (I - B(B'XB)^{-1}B'X)A$$

Since $B'XB > 0$, the matrix $B'X$ has full row rank⁵. Hence, the equality

$$B'X(A + BF) = B'X(I - B(B'XB)^{-1}B'X)A = 0$$

implies that

- at least m eigenvalues of $A + BF$ are at the origin (dead-beat).

⁵Otherwise, $\exists \eta \neq 0$ such that $\eta'B'X = 0 \implies \eta'B'XB\eta = 0 \implies B'XB \not> 0$.

Example: DC motor with rigid mechanical load

The same DC motor has the discretized realization (Lect. 6, Slide 37)

$$\begin{cases} x[t+1] = \begin{bmatrix} 1 & 1 - e^{-h} \\ 0 & e^{-h} \end{bmatrix} x[t] + \begin{bmatrix} h - (1 - e^{-h}) \\ 1 - e^{-h} \end{bmatrix} u[t] \\ y[t] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[t] \end{cases}$$

and the state still comprises $x_1[i] = y(ih)$ and $x_2[i] = \dot{y}(ih)$. Let us keep

$$z[t] = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{1-\lambda} \\ 0 & 0 \end{bmatrix} x[t] + \begin{bmatrix} 0 \\ 0 \\ \sqrt{\rho} \end{bmatrix} u[t]$$

for some weights $\lambda \in [0, 1]$ and $\rho \geq 0$. For this choice still

$$\|z[i]\|^2 = \lambda \|y(ih)\|^2 + (1 - \lambda) \|\dot{y}(ih)\|^2 + \rho \|u[i]\|_2^2,$$

which has the advantage of being physical.

Example: assumptions

$\mathcal{A}_4$: Sampling is not pathological, hence (A, B) is controllable.

$\mathcal{A}_5$: The matrix

$$\begin{bmatrix} 1 - e^{j\theta} & 1 - e^{-h} & h - (1 - e^{-h}) \\ 0 & e^{-h} - e^{j\theta} & 1 - e^{-h} \\ \hline \sqrt{\lambda} & 0 & 0 \\ 0 & \sqrt{1 - \lambda} & 0 \\ 0 & 0 & \sqrt{\rho} \end{bmatrix}$$

has full column rank for all θ iff $\lambda \neq 0$ for all $h > 0$. If $\lambda = 0$, then

$$(C_z, A) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 - e^{-h} \\ 0 & e^{-h} \end{bmatrix} \right)$$

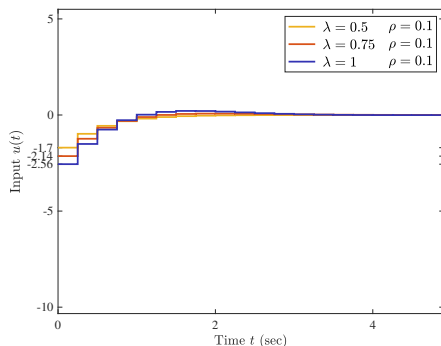
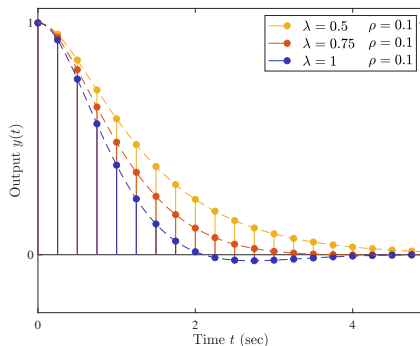
has an unobservable mode at $z = 1$.

Thus,

- we have to keep $\lambda \in (0, 1]$ in the discrete-time case as well.

Example: simulations

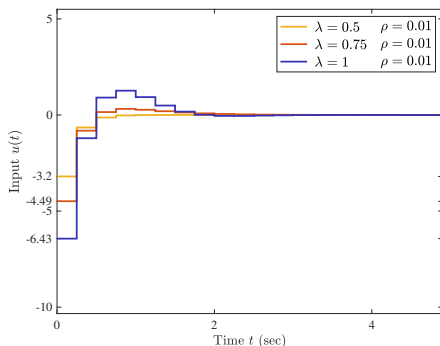
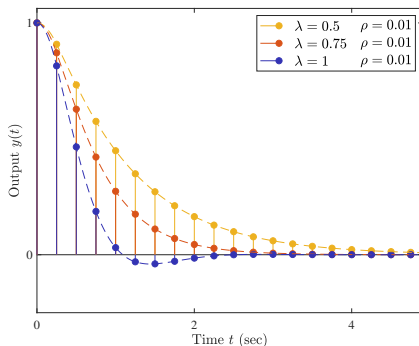
With (large) $\rho = 0.1$:



- under $\lambda = 1$ the response is faster, but overshoots (complex poles)
- as λ decreases, the response slows down, but smooths
- control effort decreases as λ decreases
- intersample behavior is as expected

Example: simulations (contd)

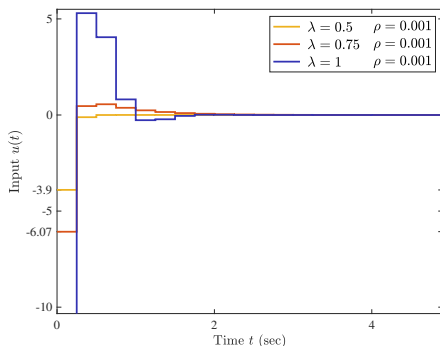
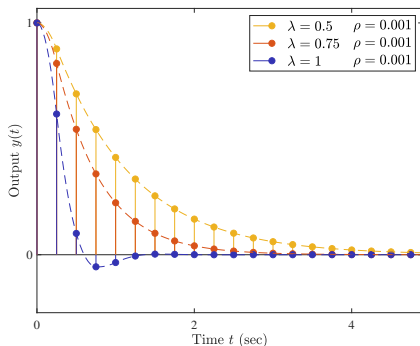
With (middle) $\rho = 0.01$:



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- as λ decreases, the response slows down, but smooths
- control effort decreases as λ decreases
- intersample behavior is as expected

Example: simulations (contd)

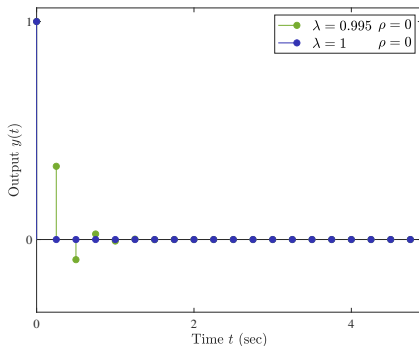
With (small) $\rho = 0.001$:



- under $\lambda = 1$ the response is faster, but overshoots (complex poles)
- as λ decreases, the response slows down, but smooths
- control effort decreases as λ decreases
- intersample behavior is as expected

Example: simulations (contd)

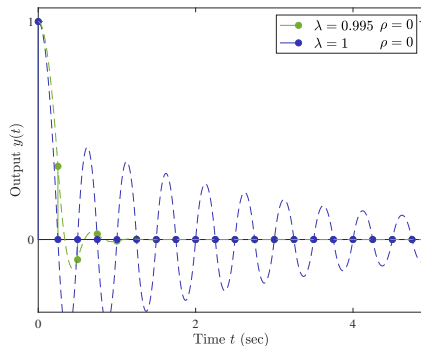
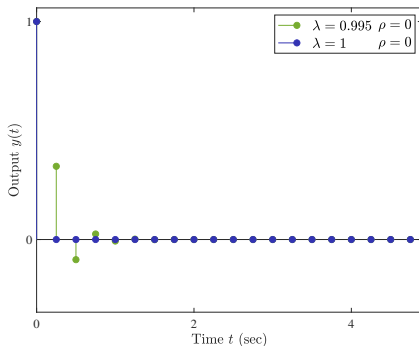
With $\rho = 0$:



- if $\dot{y}$ is not penalized, we have the dead-beat discrete response
 - the intersample behavior is poor (oscillations are invisible in z , so that it pays off to place pole at the sampling zero, cf. Lect. 6, Slide 44)
 - even a tiny penalty of $\dot{y}$ changes that (the dead-beat $\dot{y}(ih)$ is huge)

Example: simulations (contd)

With $\rho = 0$:



- if $\dot{y}$ is not penalized, we have the dead-beat discrete response
- the intersample behavior is poor (oscillations are invisible in z , so that it pays off to place pole at the sampling zero, cf. Lect. 6, Slide 44)
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Sampled-data linear-quadratic regulator (LQR) problem: optimal hold

Sampled-data LQR problem with $\mathcal{H}_{\text{ZOH}}$

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ z(t) = C_z x(t) + D_z u(t) \end{cases}$$

where $z(t)$ is the regulated signal, whose “smallness” quantifies the control performance. The sampled-data LQR problem with fixed $\mathcal{H}_{\text{ZOH}}$:

- supposing that the state vector $x(t)$ is measurable **at $t = ih$** ,
- and that the control signal **$u(t) = \bar{u}[i] = \text{const}$** for all $t \in [ih, (i+1)h]$,
- design $u(t)$, asymptotically stabilizing the system and minimizing

$$\|z\|_2^2 = \int_0^\infty \|z(t)\|^2 dt.$$

Lossless discretization

We know that the sampled state $\bar{x}[i] := x(ih)$ satisfies

$$\bar{x}[i+1] = \bar{A}\bar{x}[i] + \bar{B}\bar{u}[i] := e^{Ah}\bar{x}[i] + \int_0^h e^{At}dtB\bar{u}[i]$$

and that (Lect. 6, Slides 14 and 6)

$$x(ih + \tau) = e^{A\tau}\bar{x}[i] + \int_0^\tau e^{A\sigma}d\sigma B\bar{u}[i] = \begin{bmatrix} I & 0 \end{bmatrix} \exp\left(\overbrace{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^\tau}^{:=\tilde{A}}\right) \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix}$$

Hence, it is enough to consider $\bar{x}[i]$ as the state of the system. Moreover,

$$\begin{aligned} x(ih + \tau) &= C_x \begin{bmatrix} I & 0 \end{bmatrix} \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tau\right) \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix} + D_x \bar{u}[i] \\ &= \begin{bmatrix} C_x & D_x \end{bmatrix} \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tau\right) \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix} = \tilde{C} e^{\tilde{A}\tau} \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix} \\ &=: \tilde{z} \end{aligned}$$

for all $\tau \in [0, h]$ (as the $(2, 2)$ block of the matrix exponent equals I_m).

Lossless discretization

We know that the sampled state $\bar{x}[i] := x(ih)$ satisfies

$$\bar{x}[i+1] = \bar{A}\bar{x}[i] + \bar{B}\bar{u}[i] := e^{Ah}\bar{x}[i] + \int_0^h e^{At}dtB\bar{u}[i]$$

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$$x(ih + \tau) = e^{A\tau}\bar{x}[i] + \int_0^\tau e^{A\sigma}d\sigma B\bar{u}[i] = \begin{bmatrix} I & 0 \end{bmatrix} \exp\left(\overbrace{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tau}^{:=\tilde{A}}\right) \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix}$$

Hence, it is enough to consider $\bar{x}[i]$ as the state of the system. Moreover,

$$\begin{aligned} z(ih + \tau) &= C_z \begin{bmatrix} I & 0 \end{bmatrix} \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tau\right) \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix} + D_z \bar{u}[i] \\ &= \underbrace{\begin{bmatrix} C_z & D_z \end{bmatrix}}_{:=\tilde{C}} \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \tau\right) \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix} = \tilde{C}e^{\tilde{A}\tau} \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix} \end{aligned}$$

for all $\tau \in [0, h]$ (as the $(2, 2)$ block of the matrix exponent equals I_m).

Lossless discretization: cost

Now

$$\begin{aligned}
 \|z\|_2^2 &= \int_0^\infty \|z(t)\|^2 dt = \sum_{i \in \mathbb{Z}_+} \int_0^h \|z(ih + \tau)\|^2 d\tau \\
 &= \sum_{i \in \mathbb{Z}_+} \int_0^h \begin{bmatrix} \bar{x}'[i] & \bar{u}'[i] \end{bmatrix} e^{\tilde{A}'\tau} \tilde{C}' \tilde{C} e^{\tilde{A}\tau} \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix} d\tau \\
 &= \sum_{i \in \mathbb{Z}_+} \begin{bmatrix} \bar{x}'[i] & \bar{u}'[i] \end{bmatrix} \int_0^h e^{\tilde{A}'\tau} \tilde{C}' \tilde{C} e^{\tilde{A}\tau} d\tau \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix} \\
 &= \sum_{i \in \mathbb{Z}_+} \begin{bmatrix} \bar{x}'[i] & \bar{u}'[i] \end{bmatrix} \begin{bmatrix} \tilde{C}'_z \\ \tilde{D}'_z \end{bmatrix} \begin{bmatrix} \tilde{C}_z & \tilde{D}_z \end{bmatrix} \begin{bmatrix} \bar{x}[i] \\ \bar{u}[i] \end{bmatrix} = \|\tilde{z}\|_2^2,
 \end{aligned}$$

where $\tilde{z}[i] := \tilde{C}_z \bar{x}[i] + \tilde{D}_z \bar{u}[i]$ is an *equivalent* discrete regulated signal and

$$\begin{bmatrix} \tilde{C}_z & \tilde{D}_z \end{bmatrix} := \left(\int_0^h e^{\tilde{A}'\tau} \tilde{C}' \tilde{C} e^{\tilde{A}\tau} d\tau \right)^{1/2} \in \mathbb{R}^{(n+m) \times (n+m)}.$$

Sampled-data LQR problem with $\mathcal{H}_{\text{ZOH}}$: solution

Theorem

The solution $\bar{u}$ to the sampled-data LQR problem is equivalent to that to the discrete LQR problem with

$$\begin{cases} \bar{x}[t+1] = \bar{A}\bar{x}[t] + \bar{B}\bar{u}[t] & \bar{x}[0] = x_0 \\ \tilde{z}[t] = \tilde{C}_z\bar{x}[t] + \tilde{D}_z\bar{u}[t] \end{cases}$$

where

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} = \begin{bmatrix} e^{Ah} & \int_0^h e^{At} dt B \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{C}_z & \tilde{D}_z \end{bmatrix} = \left(\int_0^h \exp\left(\begin{bmatrix} A' & 0 \\ B' & 0 \end{bmatrix} t\right) \begin{bmatrix} C_z' \\ D_z' \end{bmatrix} \begin{bmatrix} C_z & D_z \end{bmatrix} \exp\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} t\right) dt \right)^{1/2}.$$

Computations

By (a slight alteration of) Van Loan's formula

$$\int_0^h e^{\tilde{A}'\tau} \tilde{C}' \tilde{C} e^{\tilde{A}\tau} d\tau = \tilde{\Gamma}'_{22} \tilde{\Gamma}_{12}, \quad \begin{bmatrix} \tilde{\Gamma}_{11} & \tilde{\Gamma}_{12} \\ 0 & \tilde{\Gamma}_{22} \end{bmatrix} := \exp \left(\begin{bmatrix} -\tilde{A}' & \tilde{C}' \tilde{C} \\ 0 & \tilde{A} \end{bmatrix} h \right).$$

Hence

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \tilde{C}'_z \tilde{C}_z & \tilde{C}'_z \tilde{D}_z \\ \tilde{D}'_z \tilde{C}_z & \tilde{D}'_z \tilde{D}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_n \\ \Gamma'_{33} & 0 & 0 \\ \Gamma'_{34} & I_m & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{13} & \Gamma_{14} \\ \Gamma_{23} & \Gamma_{24} \\ \Gamma_{33} & \Gamma_{34} \end{bmatrix},$$

where

$$\begin{bmatrix} \Gamma_{11} & 0 & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & I_m & \Gamma_{23} & \Gamma_{24} \\ 0 & 0 & \Gamma_{33} & \Gamma_{34} \\ 0 & 0 & 0 & I_m \end{bmatrix} := \exp \left(\begin{bmatrix} -A' & 0 & C'_z C_z & C'_z D'_z \\ -B' & 0 & D'_z C_z & D'_z D_z \\ 0 & 0 & A & B \\ 0 & 0 & 0 & 0 \end{bmatrix} h \right)$$

(the DARE required only $\tilde{C}'_z \tilde{C}_z$, $\tilde{C}'_z \tilde{D}_z$, etc, not separately $\tilde{C}_z$ or $\tilde{D}_z$).

Example: DC motor with rigid mechanical load

The same DC motor,

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

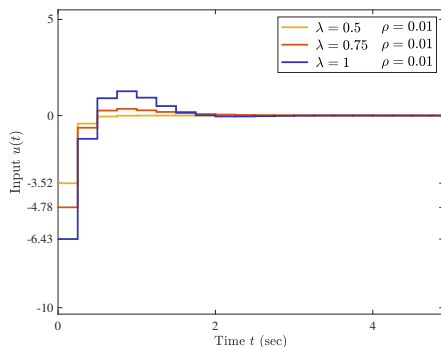
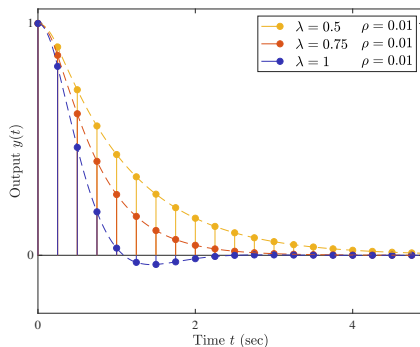
and the same regulated signal

$$z(t) = \begin{bmatrix} \sqrt{\lambda} y(t) \\ \sqrt{1-\lambda} \dot{y}(t) \\ \sqrt{\rho} u(t) \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{1-\lambda} \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \sqrt{\rho} \end{bmatrix} u(t)$$

for some weights $\lambda \in (0, 1]$ and $\rho \geq 0$ (we still need to keep $\lambda \neq 0$, but can have $\rho = 0$).

Example: simulations

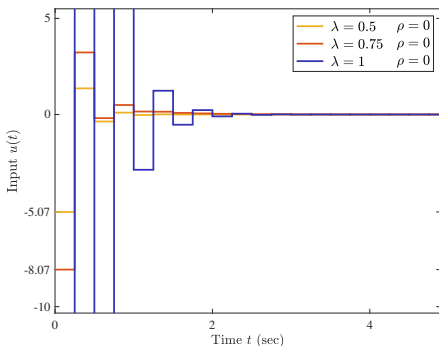
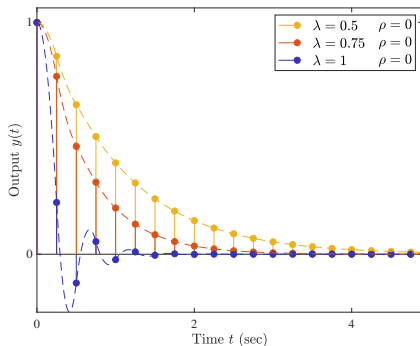
With (middle) $\rho = 0.01$:



which is quite close to the response in the discrete design.

Example: simulations (contd)

With $\rho = 0$:



it is qualitatively different from the response in the discrete design. This is because the sampled-data design

- sees the whole picture, rather than only its stroboscopic model.

Outline

Mathematical preliminaries

Continuous-time linear-quadratic regulator (LQR) problem

Discrete-time linear-quadratic regulator (LQR) problem

Sampled-data linear-quadratic regulator (LQR) problem: ZOH

Sampled-data linear-quadratic regulator (LQR) problem: optimal hold

Sampled-data LQR problem with free hold

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ z(t) = C_z x(t) + D_z u(t) \end{cases}$$

where $z(t)$ is the regulated signal, whose “smallness” quantifies the control performance. The sampled-data LQR problem with free $\mathcal{H}$:

- supposing that the state vector $x(t)$ is measurable **at $t = ih$** ,
- design $u(t)$, asymptotically stabilizing the system and minimizing

$$\|z\|_2^2 = \int_0^\infty \|z(t)\|^2 dt.$$

This formulation does not restrict the waveform of $u(t)$, it only restricts the information available to the controller.

Solution

Assume that $\mathcal{A}_{1-3}$ hold and return to the relation

$$\|z\|_2^2 = x_0' X x_0 + \int_0^\infty \|u(t) - Fx(t)\|_{R_{22}}^2 dt \geq 0$$

where $R_{22} = D_z' D_z > 0$, X is the stabilizing solution to the CARE, and F is the corresponding continuous-time LQR gain. The question is

- can we render $u(t) = Fx(t)$, $\forall t \in \mathbb{R}_+$, w/o access to the analog $x(t)$?

The answer is affirmative and the required control algorithm can be derived by reverse engineering. Just note that if the condition above holds, then

$$\dot{x}(t) = (A + BF)x(t), \quad x(0) = x_0.$$

Hence, for every $i \in \mathbb{Z}_+$ and all $\tau \in [0, h]$,

$$x(ih + \tau) = e^{(A+BF)\tau} x(ih) \quad \implies \quad Fx(ih + \tau) = Fe^{(A+BF)\tau} x(ih),$$

which is a function of $x(ih)$, as required for implementing u .

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Solution (contd)

Theorem

If $\mathcal{A}_{1-3}$ hold, then the unique solution of the sampled-data LQR problem is

$$u(t) = Fe^{(A+BF)(t-ih)}x(ih), \quad \forall t \in [ih, (i+1)h)$$

where $F = -(D_z' D_z)^{-1}(D_z' C_z + B' X)$ and $X \geq 0$ is the stabilizing solution to the associated CARE. The optimal cost is $\|z\|_2^2 = x_0' X x_0$.

Proof: It's only left to prove stability. The closed-loop discrete dynamics are

$$\begin{aligned} \bar{x}[i+1] &= e^{Ah} \bar{x}[i] + \int_0^h e^{A(h-\tau)} B F e^{(A+BF)\tau} d\tau \bar{x}[i] \\ &= \begin{bmatrix} I & 0 \end{bmatrix} \exp\left(\begin{bmatrix} A & BF \\ 0 & A+BF \end{bmatrix} h\right) \begin{bmatrix} I \\ I \end{bmatrix} \bar{x}[i] \\ &= \begin{bmatrix} I & I \end{bmatrix} \exp\left(\begin{bmatrix} A & 0 \\ 0 & A+BF \end{bmatrix} h\right) \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{x}[i] = e^{(A+BF)h} \bar{x}[i] \end{aligned}$$

(cf. Lect. 6, Slide 14, Van Loan, and use similarity with $\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$).



Yet another miracle of the generalized hold?

The fact that the optimal cost, $x_0' X x_0$, doesn't depend on the sampling rate appears surprising and counterintuitive. Even if $h \rightarrow \infty$, i.e. we sample only once, at $t = 0$, the analog cost is recovered.

How is this possible?

To answer this question, bear in mind that LQR copes with nonzero initial conditions only, no interactions with the outside world are assumed. Hence, all controllable dynamics can be produced in open loop from sheer knowledge of initial conditions. This is not too restrictive in the continuous-time case, because

1. the effect of $x(0) = x_0$ is equivalent to that of the disturbance $x_0 \delta(t)$,
2. by time invariance, $Fx(t)$ is also optimal under $x_0 \delta(t - \tau)$ for all $\tau > 0$ (the latter is equivalent to the condition $x(\tau) = x_0$). But this is no longer the case in SD systems, where the controller is time varying.

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More realistic disturbance model

A possible way to cope with the situation is to consider the response to the state jump to x_0 at every $t = \tau$. Denoting the regulated signal for a given τ by z_τ , we have that

$$\|z_\tau\|_2^2 = \int_\tau^\infty \|z_\tau(t)\|^2 dt = x_0' X x_0 + \int_\tau^\infty \|u(t) - Fx(t)\|_{R_{22}}^2 dt \geq 0$$

If $\tau \in ((k-1)h, kh)$ for some $k \in \mathbb{N}$, then

$$\|z_\tau\|_2^2 = x_0' X x_0 + \int_\tau^{kh} \|Fx(t)\|_{R_{22}}^2 dt + \sum_{i=k}^\infty \int_0^h \|u(ih+t) - Fx(ih+t)\|_{R_{22}}^2 dt$$

We still have the optimal $u(t) = Fe^{(A+BF)\tau} x(ih)$, but now the optimal cost

$$\|z_\tau\|_2^2 = x_0' X x_0 + \int_\tau^{kh} \|Fe^{A(t-\tau)} x_0\|_{R_{22}}^2 dt = x_0' X x_0 + \int_0^{kh-\tau} \|Fe^{At} x_0\|_{R_{22}}^2 dt,$$

because the state for $t \in [\tau, kh]$ is $x(t) = e^{A(t-\tau)} x_0$.

The optimal cost

It may be reasonable to consider the average cost over all $\tau \in [0, h]$, so that

$$\begin{aligned}\mathcal{J} &= \frac{1}{h} \int_0^h \|z_\tau\|_2^2 d\tau = \frac{1}{h} \int_0^h \left(x_0' X x_0 + \int_0^{h-\tau} \|F e^{A t} x_0\|_{R_{22}}^2 dt \right) d\tau \\ &= x_0' \left(X + \frac{1}{h} \int_0^h \int_0^\tau e^{A' t} F' D_z' D_z F e^{A t} dt d\tau \right) x_0.\end{aligned}$$

It does increase with h . For the DC motor example above, with $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$$\mathcal{J} = \sqrt{\lambda} \sqrt{1 - \lambda + 2\sqrt{\lambda\rho} + \rho} + \frac{\lambda}{2} h.$$

i.e. it grows linearly with h .

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