

Sampled-Data Control Systems (036709)

lecture no. 8

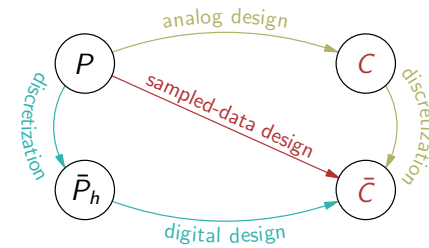
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Approaches to computer control design



1. Discrete-time design

(discretize the problem first, then do your favorite discrete design)

2. Digital redesign of analog controllers

(do your favorite analog design first, then discretize the resulting controller)

3. Direct digital (sampled-data) design

(design $\bar{C}$, and possibly S and $\mathcal{H}$, directly for analog plant and specs)

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Outline

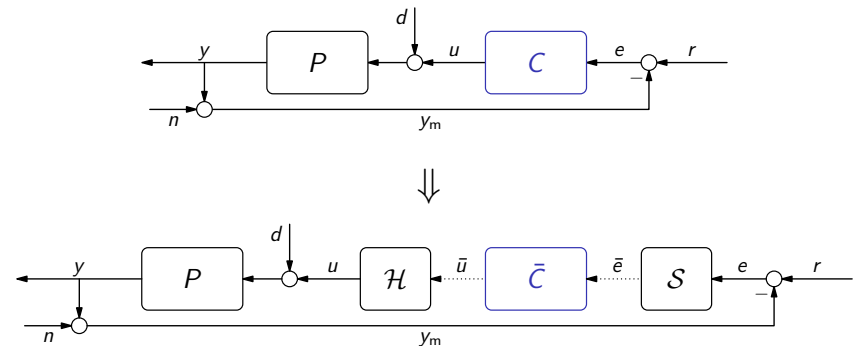
Digital redesign: general considerations

Choice of $\bar{C}$

Effects on analog design

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The redesign problem



Starting point:

- “good” analog controller C (designed by whatever method)

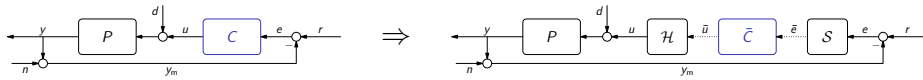
Goal:

- find $\bar{C}$ such that $\mathcal{H}\bar{C}S \approx C$

(conventionally, $\mathcal{H} = \mathcal{H}_{\text{ZOH}}$).

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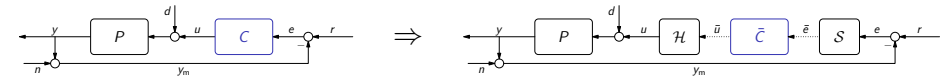
Why?



- plant properties
physical in CT; artificial, with messy parameters (cf. sampling zeros), in DT
- exogenous signals properties
physical in CT; artificial, with distorted spectra due to folding and cloning, in DT
- design insight
especially, for frequency-domain methods
- quick prototyping
to decide whether to invest in more accurate methods

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When not?



Don't use this method if

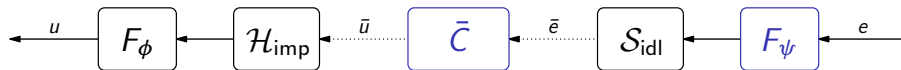
- your intuition in DT is good
- discrete design repeats corresponding analog design literally
e.g. in state-space methods, optimization-based design approaches, et cetera
- sampling rate is not “sufficiently high,” with margins

Reasons:

- ☹ normally no guarantees (even stability)
- ☹ analytic takes on digital redesign produce higher-order controllers
(and use advanced sampled-data tools anyway, so why bother with an intermediate step)

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How sampled-data controllers process data



In the frequency domain,

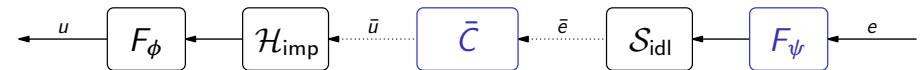
- (low-pass) antialiasing filter F_ψ filters out spectral parts above ω_N
- the ideal sampler **folds** analog frequency range to $\theta/h \in [-\omega_N, \omega_N]$
- digital part processes the signal in $\theta/h \in [-\omega_N, \omega_N]$
- the impulse hold **clones** digital spectrum to all $[(2i-1)\omega_N, (2i+1)\omega_N]$
- the ZOH filter, with $F_\phi(s) = (1 - e^{-sh})/s$, filters out high frequencies

Potential problems:

1. F_ψ might not filter out ultra-Nyquist components of $E(j\omega)$ well enough
(hence, high frequencies of $E(j\omega)$ might still infiltrate low frequencies of $\bar{E}(e^{j\theta})$)
2. both F_ψ and F_ϕ are low pass $\implies$ extra phase lag in the loop
3. $\mathcal{H}_{\text{imp}}$ injects high-frequency components

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Anti-aliasing filter



Ideally, F_ψ should

- filter out ultra-Nyquist components of $E(j\omega)$ completely
- not introduce phase lag

This suggest the use of

- the ideal low-pass filter, with $F_\psi(j\omega) = \mathbb{1}(\omega + \omega_N) - \mathbb{1}(\omega - \omega_N)$
so that $|F_\psi(j\omega)| = 0$ for all $\omega \notin [-\omega_N, \omega_N]$ and $\arg F_\psi(j\omega) = 0$

But this filter is non-causal, its impulse response is

$$f_\psi(t) = \frac{1}{h} \frac{\sin(\omega_N t)}{\omega_N t} =: \frac{1}{h} \text{sinc}_h(\omega_N t) = \frac{1}{h} \cdot \text{plot of sinc function}$$

Causality is an **inescapable** constraint in the **feedback** control setting.

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Anti-aliasing filter (contd)



In feedback applications we have to resort to causal filters. Consequently,

- aliased frequencies cannot be avoided completely
- the choice of the Nyquist frequency should be conservative (conventional wisdom has it that $\omega_N \geq 10 \div 30 \omega_c$, where ω_c is the analog crossover)

The name of the game is then to select a low-pass $F_\psi(s)$ which

- is strictly proper (otherwise S_{idl} is unstable)
- has its stopband edge frequency $\omega_{sb} \leq \omega_N$ (ω_{sb} is the smallest frequency such that $|F_\psi(j\omega)| \leq \epsilon$ in all $\omega \geq \omega_{sb}$, for a sufficiently small ϵ ; typically, $\epsilon \in [0.01, 0.1]$)

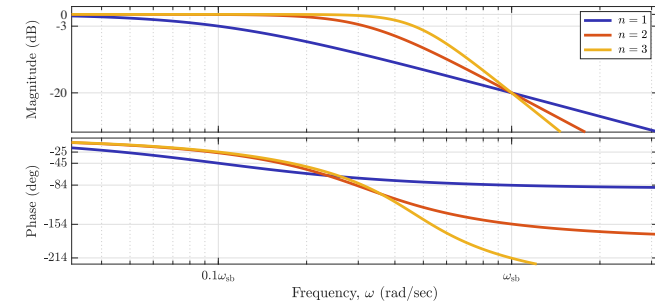
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Butterworth filters

The Butterworth filter of order n with the stopband edge frequency ω_{sb} is

$$F_\psi(s) = \frac{1}{B_n(s/(\omega_{sb} \sqrt[n]{\epsilon^2/(1-\epsilon^2)}}),$$

$B_n(s)$ is the unique n -order Hurwitz polynomial¹ with $|B_n(j\omega)| = \sqrt{1 + \omega^{2n}}$. For example, with $\epsilon = 0.1$ we have:



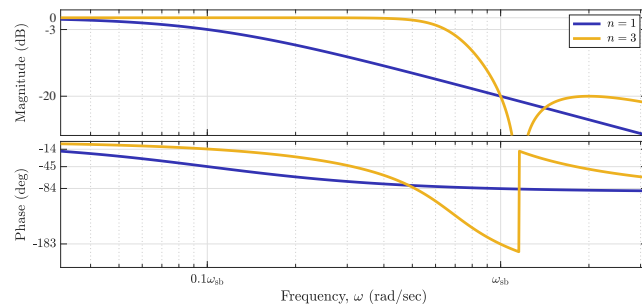
¹MATLAB: `[num,den] = butter(n,wsb*(eps^2/(1-eps^2))^(1/(2*n)),'s');`

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Type II (inverse) Chebyshev filters

The Type II Chebyshev filters² of order n with the same ω_{sb}

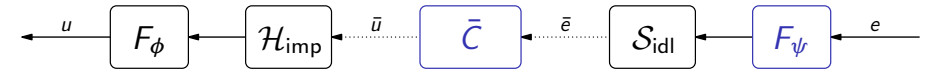
- ☺ have shorter decay region
- ☺ introduce smaller phase lags at infra- ω_{sb} frequencies
- ☹ have smaller asymptotic magnitude rolloff (strictly proper iff n is odd) than comparable Butterworth filters. For example, with $\epsilon = 0.1$ we have:



²MATLAB: `[num,den] = cheby2(n,mag2db(1/eps),wsb,'s');`

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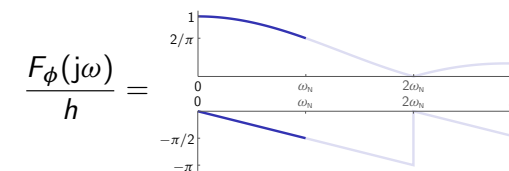
Dynamics of $\mathcal{H}_{ZOH}$



The transfer function

$$F_\phi(s) = \frac{1 - e^{-sh}}{s}$$

filters out ultra-Nyquist components of $\mathcal{H}_{imp}\bar{u}$, but also introduces a phase lag of $\omega h/2$ for $\omega \in [0, \omega_N]$:



In the range $\omega \in [0, 0.1\omega_N]$, the effect of F_ϕ is less harmful than that of F_ψ :

- the magnitude is in $[0.996, 1]$ and the phase is in $[-9^\circ, 0]$.

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Outline

Digital redesign: general considerations

Choice of $\bar{C}$

Effects on analog design

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Logic



The choice of $\bar{C}$ is heavily dominated by ad hoc methods, where

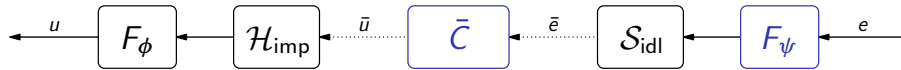
- accuracy is often sacrificed for simplicity and, sometimes, transparency.

Translation procedures may be motivated by

- approximating derivatives by shifts
(differential equations vs. difference equations; s vs. z)
- discretizing analog plants in $\bar{P}_h = SPH$
(but beware that the logic is completely different, $\bar{P}_h$ aims at reflecting stroboscopic properties of P , whereas $\bar{C}$ is supposed to imitate C via $\mathcal{H}\bar{C}S$)

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Approximating derivatives



Often based on numerical differentiation rules:

forward Euler: $\dot{x}(ih) \approx \frac{x(ih+h) - x(ih)}{h} \Rightarrow s \approx \frac{z-1}{h}$

backward Euler: $\dot{x}(ih) \approx \frac{x(ih) - x(ih-h)}{h} \Rightarrow s \approx \frac{z-1}{hz}$

Tustin³: $\frac{\dot{x}(ih+h) + \dot{x}(ih)}{2} \approx \frac{x(ih+h) - x(ih)}{h} \Rightarrow s \approx \frac{2}{h} \frac{z-1}{z+1}$

Example

If $C(s) = 1/(s+1)$, then

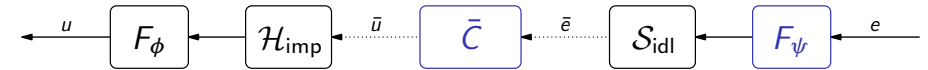
$$\bar{C}(z) = C(s) \Big|_{s=\frac{2}{h} \frac{z-1}{z+1}} = \frac{1}{2/h \cdot (z-1)/(z+1) + 1} = \frac{h(z+1)}{(h+1)z + h - 1}.$$



³MATLAB: `c2d(C,h,'tustin')`, where C is a continuous-time system.

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Inspirations by stroboscopic models



mechanical use: $\bar{C} = \mathcal{S}CH$

matched⁴: poles of $\bar{C}(z)$ are $e^{s_p h}$, zeros of $\bar{C}(z)$ are $e^{s_z h}$, often scaled to match the static gains.

Example

If $C(s) = 1/(s+1)$, then

$$\bar{C}(z) = \frac{1 - e^{-h}}{z - e^{-h}},$$

so that $C(0) = \bar{C}(1)$.



⁴MATLAB: `c2d(C,h,'matched')`, where C is a continuous-time system.

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General bilinear transformation

Given $\gamma > 0$, consider the mapping (Tustin corresponds to $\gamma = 2/h$)

$$s \rightarrow \gamma \frac{z-1}{z+1} \iff z \rightarrow \frac{\gamma+s}{\gamma-s}$$

between s and z complex planes. Every $s = \sigma + j\omega$ is mapped to

$$z = \frac{\gamma + (\sigma + j\omega)}{\gamma - (\sigma + j\omega)} \implies |z|^2 = \frac{(\gamma + \sigma)^2 + \omega^2}{(\gamma - \sigma)^2 + \omega^2}.$$

Hence,

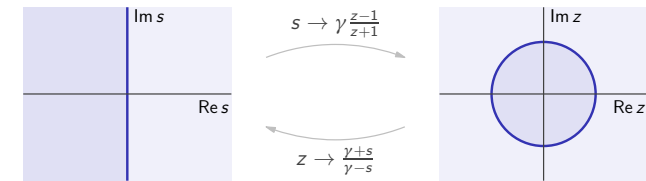
$$|z|^2 - 1 = \frac{2\gamma\sigma}{(\gamma - \sigma)^2 + \omega^2}$$

and we end up with the relations

$$|z| < 1 \iff \sigma < 0, \quad |z| > 1 \iff \sigma > 0, \quad |z| = 1 \iff \sigma = 0.$$

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General bilinear transformation (contd)



Thus,

- any “stable” s is mapped to a “stable” z
- any “unstable” s is mapped to a “unstable” z
- any “borderline” s is mapped to a “borderline” z

Moreover,

- any CT frequency ω is mapped to the DT frequency $\theta = 2 \arctan(\omega/\gamma)$ (i.e. bilinear transformations squeeze the whole $j\mathbb{R}$ to $\mathbb{T}$, with no folding effects)
- the lowest $\omega = 0$ is mapped to the lowest $\theta = 0$
- the highest $\omega = \pm\infty$ is mapped to the highest $\theta = \pm\pi$

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More about Tustin

If

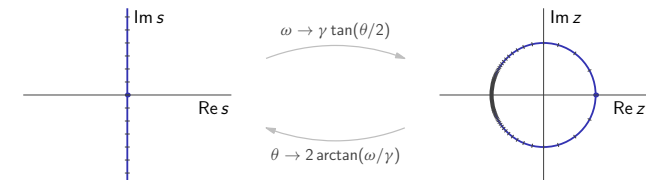
$$\bar{C}(z) = C(s) \Big|_{s=\frac{2}{h} \frac{z-1}{z+1}},$$

then

- $\bar{C}(z)$ is (un) stable iff $C(s)$ is (un) stable,
- the number of integrators in $\bar{C}(z)$ equals that in $C(s)$,
- $\bar{C}(z)$ is bi-proper, unless $C(s)$ has either poles or zeros at $s = 2/h$
 - zero multiplicity of $C(s)$ at $s = 2/h$ equals the pole excess of $\bar{C}(z)$
 - pole multiplicity of $C(s)$ at $s = 2/h$ equals the zero excess of $\bar{C}(z)$
- $\bar{C}(z)$ has a zero at $z = -1$ of multiplicity m iff $C(s)$ is strictly proper and its pole excess is m ,
- $\bar{C}(z)$ has a pole at $z = -1$ of multiplicity m iff $C(s)$ is non-proper and its zero excess is m ,

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Tustin with pre-warping



The nonlinear **warping** of the frequency mapping is not ideal. We would be happier with $\theta = \omega h$ in low frequencies (if folding effects are insignificant). This happens only at $\omega = 0$ and the frequency $\omega_{\text{nowarp}} \in (0, \omega_N)$, at which

$$2 \arctan \frac{\omega_{\text{nowarp}}}{\gamma} = \omega_{\text{nowarp}} h \iff \gamma = \omega_{\text{nowarp}} \cot \frac{\omega_{\text{nowarp}} h}{2} \in \left(0, \frac{2}{h}\right).$$

The bilinear transformation with γ as above for a given $\omega_{\text{nowarp}} \in (0, \omega_N)$ is known⁵ as **Tustin with pre-warping**. As $\omega_{\text{nowarp}} \rightarrow 0$, the ordinary Tustin for $\gamma = 2/h$ is recovered, for which $d\bar{C}(e^{j\theta})/d\theta|_{\theta=0} = dC(\omega)/d\omega|_{\omega=0}$ as well.

⁵MATLAB: `c2d(C,h,c2dOptions('Method','tustin','PrewarpFrequency',w0))`.

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Tustin with pre-warping (contd)

Rule of thumb:

- choose ω_{nowarp} slightly below ω_c of the analog design.

Perhaps the only exception is the integral action, or a PI module, for which the regular Tustin may be preferable to keep its velocity gain unchanged. A possible sequence in this case:

1. split the analog $C = C_{\text{PI}} C_{\text{rem}}$, where C_{rem} contains no integral actions,
2. approximate C_{PI} by the standard Tustin, to end up with $\bar{C}_{\text{PI}}$,
3. approximate C_{rem} by a Tustin with pre-warping, to end up with $\bar{C}_{\text{rem}}$,
4. construct $\bar{C} = \bar{C}_{\text{PI}} \bar{C}_{\text{rem}}$.

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Tustin vs. matched

These two are perhaps predominant. Comparisons are hard, as both are ad hoc, with somewhat vague justifications. Still, I would almost always go for *Tustin with pre-warping*. Reasons:

- more transparent frequency-response mapping
matched discretization provides no clear relation between of $C(j\omega)$ and $\bar{C}(e^{j\theta})$
- gain reduction at high frequencies if C is low pass
 $\bar{C}(-1) = 0$ whenever $C(s)$ is strictly proper, not the case for the matched approach
- no artificial delays
the number of delays after matched discretization equals the pole excess of $C(s)$

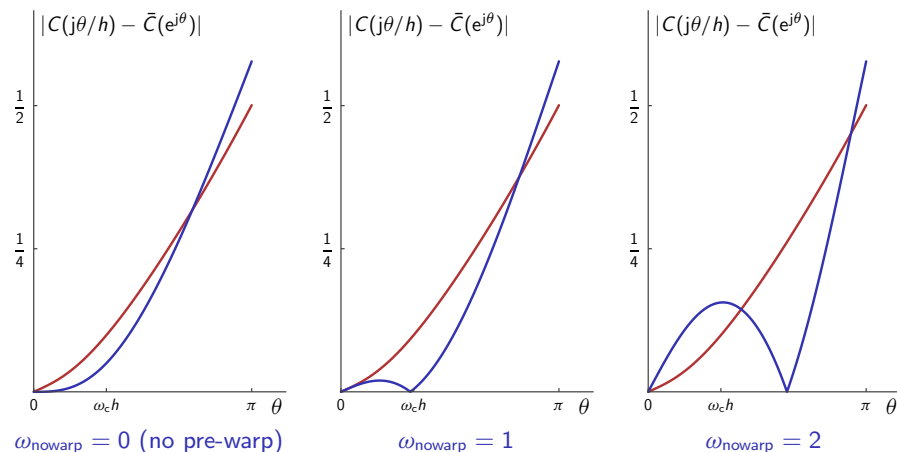
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Tustin vs. matched: lead

Consider the lead

$$C(s) = \frac{\sqrt{\alpha}s + \omega_c}{s + \sqrt{\alpha}\omega_c}.$$

With $\alpha = 3$ (30° phase lead at $\omega = \omega_c$), $\omega_c = \omega_N/3$, and $h = 1$.



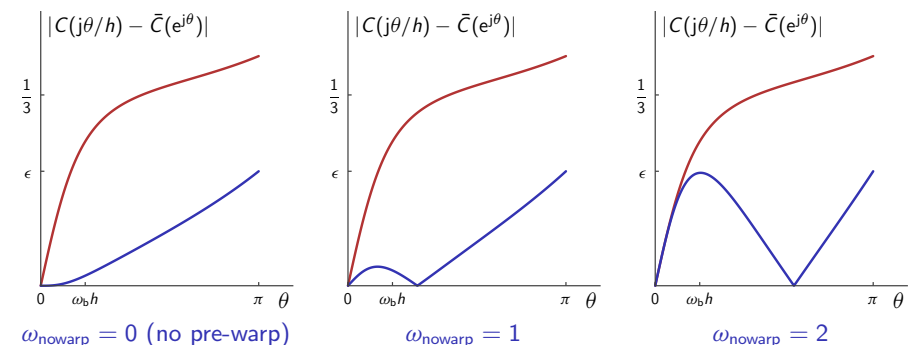
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Tustin vs. matched: low-pass filter

Consider

$$C(s) = \frac{1}{\sqrt{\epsilon^{-2} - 1} s / \omega_N + 1},$$

which is Butterworth with the stopband edge at $\omega = \omega_N$. With $\epsilon = 0.2$ and $h = 1$ we have:



Note that $\gamma = 2/h$ yields the best approximation at $\omega = 0$.

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Outline

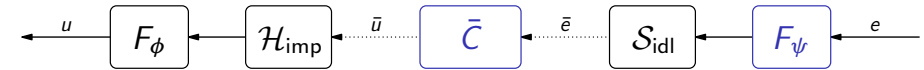
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Analog low-pass elements



There are 2 low-pass elements in the sampled-data controller:

- the anti-aliasing filter, F_ψ
- the “analog part” of the ZOH, F_ϕ

They are always present and thus

- F_ψ and⁶ F_ϕ/h would better be absorbed into the plant P to design C .

While F_ψ is normally finite dimensional, F_ϕ is not. This is not an obstacle in loop-shaping, but might require approximations, like

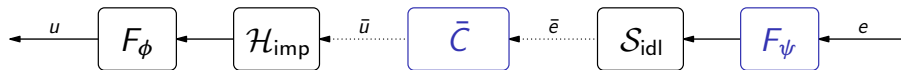
$$\frac{F_\phi(s)}{h} \approx e^{-sh/2}, \quad \frac{F_\phi(s)}{h} \approx \frac{e^{-sh/(3+\sqrt{3})}}{sh/\sqrt{12+1}}, \quad \text{or} \quad \frac{F_\phi(s)}{h} \approx \frac{12}{h^2s^2 + 6hs + 12}$$

in more analytic design methods.

⁶Because S_{idl} includes the factor $1/h$ in the frequency domain.

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High-frequency properties of $\bar{C}$



It should be remembered that

- F_ψ is an analog part of the controller, whereas $\bar{C}$ is its digital part.

This means that they do not necessarily compensate each other, even if the analog controller C compensates parts of F_ψ . For example, a non-decaying high-frequency gain of $C(j\omega)$ can counteract the roll-off of $F_\psi(j\omega)$. But a non-decaying high-frequency gain of $\bar{C}(e^{j\theta})$ has a limited effect on ultra- ω_N properties of $F_\psi(j\omega)$. As a result,

- we may afford being more aggressive with the design of C .

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(Re)design example

Let

$$P(s) = \frac{1}{s(s+1)}$$

and consider 3 analog H_∞ loop shaping designs tuned for $h \in \{0.2, 0.3, 0.4\}$, containing an integral action and aiming at the crossover $\omega_c \approx 1.6$ rad/sec.

For each intended h , design steps are

1. select $F_\psi(s)$ as the 1-order Butterworth with $\epsilon = 0.1$
2. select $F_\phi(s) = 12/(h^2s^2 + 6hs + 12)$
3. construct the augmented plant $P_{aug} := F_\psi P F_\phi$ (it depends on h)
4. design a controller $C = \bar{C}_{PI} \bar{C}_{rem}$ for P_{aug}
5. approximate C by
 - 5.1 matched
 - 5.2 Tustin
 - 5.3 Tustin for $\bar{C}_{PI}$ and Tustin with pre-warping at $\omega_{nowarp} = 0.8\omega_c$ for $\bar{C}_{rem}$
6. check the stability of $\bar{C}$ for the stroboscopic plant $S_{idl} F_\psi P \mathcal{H}_{ZOH}$

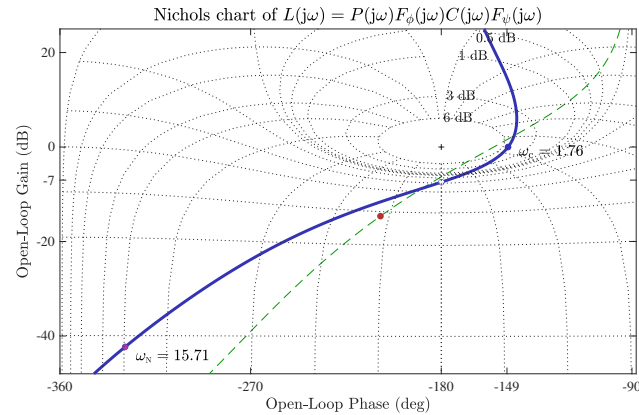
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(Re)design example: $h = 0.2$

In this case

$$C(s) = \frac{s + 0.4334}{s} \frac{29.697(s^2 + 2.921s + 2.318)(s^2 + 30s + 300)}{(s^2 + 6.887s + 29.41)(s^2 + 30.46s + 306.4)}$$

yields



Maximum stabilizing

1. $h_{\max,m} = 0.5716$
2. $h_{\max,T} = 0.7060$
3. $h_{\max,Tpw} = 0.8958$

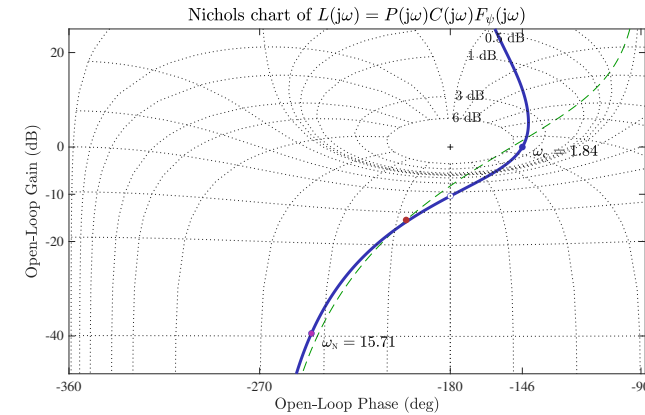
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(Re)design example: $h = 0.2$ and “forgotten” F_ϕ

In this case

$$C(s) = \frac{s + 0.4392}{s} \frac{26.434(s^2 + 2.953s + 2.392)}{s^2 + 7.325s + 24.89}$$

yields



Maximum stabilizing

1. $h_{\max,m} = 0.4864$
2. $h_{\max,T} = 0.6221$
3. $h_{\max,Tpw} = 0.705$

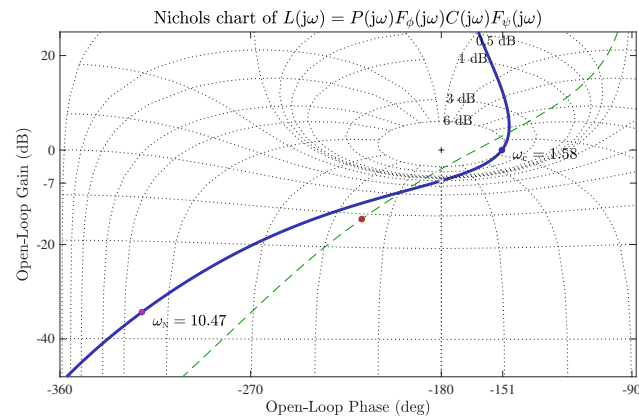
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(Re)design example: $h = 0.3$

In this case

$$C(s) = \frac{s + 0.4358}{s} \frac{32.936(s^2 + 2.336s + 1.555)(s^2 + 20s + 133.3)}{(s^2 + 5.645s + 23.6)(s^2 + 20.8s + 140.9)}$$

yields



Maximum stabilizing

1. $h_{\max,m} = 0.6379$
2. $h_{\max,T} = 0.7477$
3. $h_{\max,Tpw} = 0.9758$

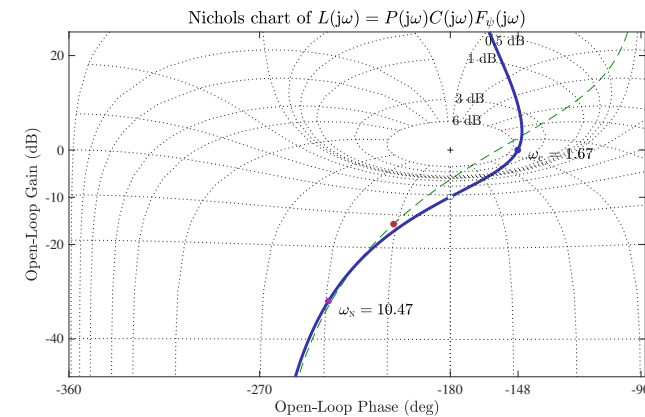
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(Re)design example: $h = 0.3$ and “forgotten” F_ϕ

In this case

$$C(s) = \frac{s + 0.4438}{s} \frac{28.17(s^2 + 2.374s + 1.627)}{s^2 + 6.418s + 19.43}$$

yields



Maximum stabilizing

1. $h_{\max,m} = 0.506$
2. $h_{\max,T} = 0.6406$
3. $h_{\max,Tpw} = 0.7198$

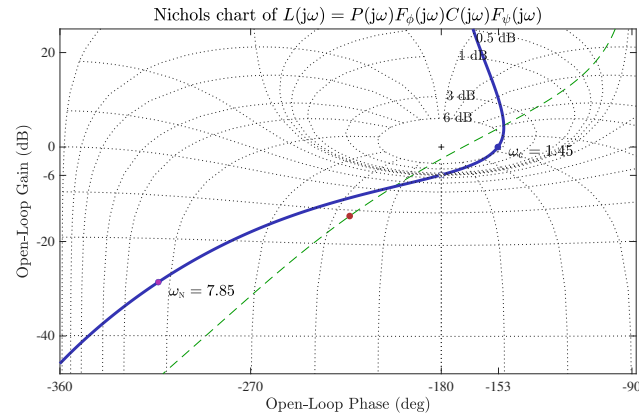
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(Re)design example: $h = 0.4$

In this case

$$C(s) = \frac{s + 0.4436}{s} \frac{35.485(s^2 + 2.02s + 1.165)(s^2 + 15s + 75)}{(s^2 + 4.737s + 19.81)(s^2 + 16.15s + 83.51)}$$

yields



Maximum stabilizing

1. $h_{\max, m} = 0.7104$
2. $h_{\max, T} = 0.7902$
3. $h_{\max, Tpw} = 1.0296$

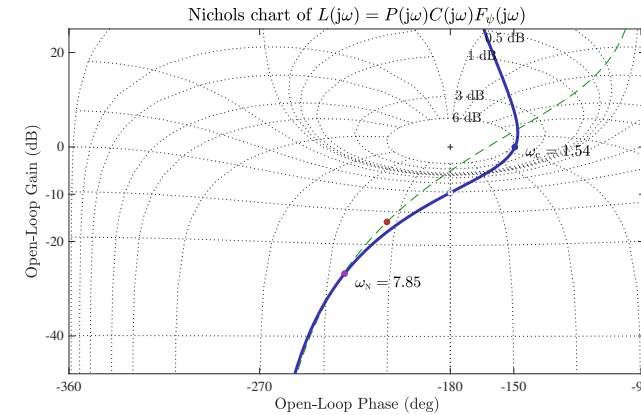
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(Re)design example: $h = 0.4$ and “forgotten” F_ϕ

In this case

$$C(s) = \frac{s + 0.4528}{s} \frac{29.315(s^2 + 2.063s + 1.236)}{s^2 + 5.862s + 16.31}$$

yields



Maximum stabilizing

1. $h_{\max, m} = 0.5310$
2. $h_{\max, T} = 0.6702$
3. $h_{\max, Tpw} = 0.7439$

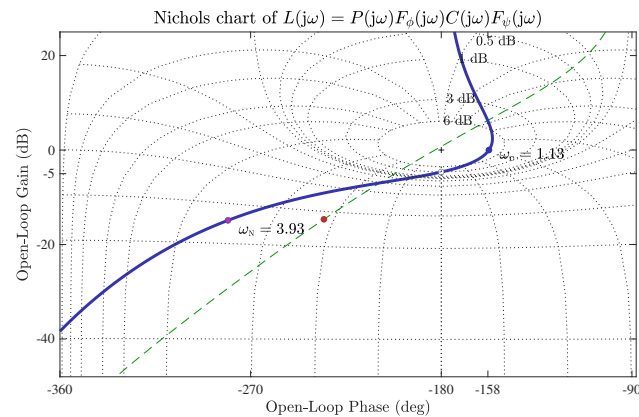
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(Re)design example: $h = 0.8$ (limit)

In this case

$$C(s) = \frac{s + 0.8896}{s} \frac{42.716(s^2 + 1.066s + 0.308)(s^2 + 7.5s + 18.75)}{(s^2 + 9.553s + 28.03)(s^2 + 2.698s + 11.34)}$$

yields



Maximum stabilizing

1. $h_{\max, m} = 0.9674$
2. $h_{\max, T} = 0.8$
3. $h_{\max, Tpw} = 0.9477$

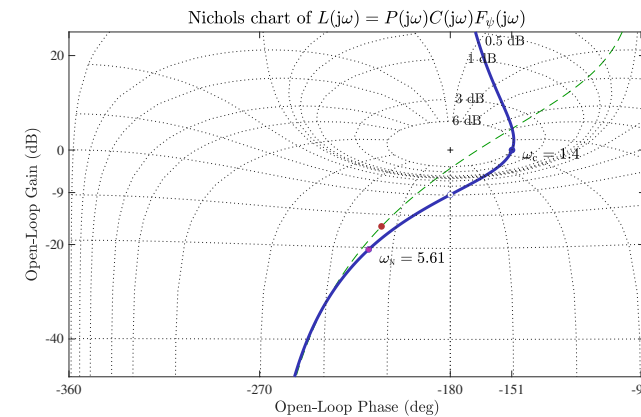
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(Re)design example: $h = 0.56$ (limit)

In this case

$$C(s) = \frac{s + 0.4793}{s} \frac{30.528(s^2 + 1.764s + 0.8763)}{s^2 + 5.281s + 13.27}$$

yields



Maximum stabilizing

1. $h_{\max, m} = 0.56$
2. $h_{\max, T} = 0.7118$
3. $h_{\max, Tpw} = 0.7877$

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