

Sampled-Data Control Systems (036709)

lecture no. 7

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1/52

Outline

Integral action in state feedback

State-space design methods in discrete time: state observers

State-space design methods in discrete time: reduced-order observers

State-space design methods in discrete time: observer-based feedback

Generalized hold

2/52

State feedback with an integral action

Control law

$$u[t] = F_x x[t] + F_e \sum_{s=0}^t (r[s] - Cx[s])$$

or, equivalently,

$$U(z) = F_x X(z) + F_e \frac{z}{z-1} (R(z) - CX(z))$$

may be thought of as a state-feedback counterpart of PI.

Two questions that we address below:

- how to design gains $F_x \in \mathbb{R}^{1 \times n}$ and $F_e \in \mathbb{R}$?
- what closed-loop properties can be obtained by this configuration?

3/52

Design steps

If integral action required, standard trick is to

1. **augment** plant dynamics with those of $z/(z-1)$ (to ensure that the loop contains an integrator);
2. design state feedback for the augmented plant;
3. move integrator from plant to controller when implemented.

4/52

Step 1

Rewrite $u[t] = F_x x[t] + F_e \eta[t]$, where

$$\eta[t] := \sum_{s=0}^t (r[s] - Cx[s]) \iff \eta[t+1] = \eta[t] + r[t] - Cx[t].$$

The dynamics of η are fixed, so they can be

- added to the dynamics of the plant,
- which results in the new (augmented) “plant”

$$\begin{bmatrix} \eta[t+1] \\ x[t+1] \end{bmatrix} = \begin{bmatrix} 1 & -C \\ 0 & A \end{bmatrix} \begin{bmatrix} \eta[t] \\ x[t] \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r[t] + \begin{bmatrix} 0 \\ B \end{bmatrix} d[t].$$

Control law above is then just a static **state feedback** law,

$$u[t] = \begin{bmatrix} F_e & F_x \end{bmatrix} \begin{bmatrix} \eta[t] \\ x[t] \end{bmatrix}.$$

5/52

Step 2

Given

$$\tilde{A} := \begin{bmatrix} 1 & -C \\ 0 & A \end{bmatrix} \quad \text{and} \quad \tilde{B} := \begin{bmatrix} 0 \\ B \end{bmatrix},$$

design $\tilde{F} := \begin{bmatrix} F_e & F_x \end{bmatrix}$ assigning eigenvalues of $\tilde{A}_F = \tilde{A} + \tilde{B}\tilde{F}$ to the roots of

$$\tilde{\chi}_{cl}(z) = z^{n+1} + \tilde{\chi}_n z^n + \dots + \tilde{\chi}_1 z + \tilde{\chi}_0.$$

This is a standard design problem, solvable if $(\tilde{A}, \tilde{B})$ is reachable. By PBH,

$$\begin{bmatrix} \eta_1 & \eta'_2 \end{bmatrix} \begin{bmatrix} 1 - \lambda & -C & 0 \\ 0 & A - \lambda I & B \end{bmatrix} = \begin{bmatrix} \eta_1(1 - \lambda) & \eta'_2(A - \lambda I) - \eta_1 C & \eta'_2 B \end{bmatrix},$$

there are unreachable modes only if $P(1) = 0$, at $\lambda = 1$.

- If $\lambda \neq 1 \implies \eta_1 = 0 \implies \eta_2 = 0$, because (A, B) is reachable.
- If $\lambda = 1$, then $\eta'_2(A + BF - I) = \eta_1 C$ (for all F) and $\eta'_2 B = 0$, so that

$$\eta_1 C(I - A - BF)^{-1} B = 0 \iff \eta_1 T_{yr}(1) = 0 \iff \eta_1 P(1) = 0.$$

If $P(1) \neq 0 \implies \eta_1 = 0 \implies \eta_2 = 0$ by the reachability of (A, B) .

6/52

Step 3

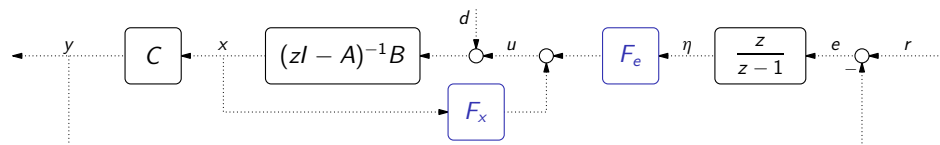
“Moving integrator to controller” means that we replace η with its value as a function of $r[t] - Cx[t]$:

$$\eta[t+1] = \eta[t] + r[t] - Cx[t] \iff \eta[t] = \sum_{s=0}^t (r[s] - Cx[s]).$$

Thus, we indeed end up with our PI:

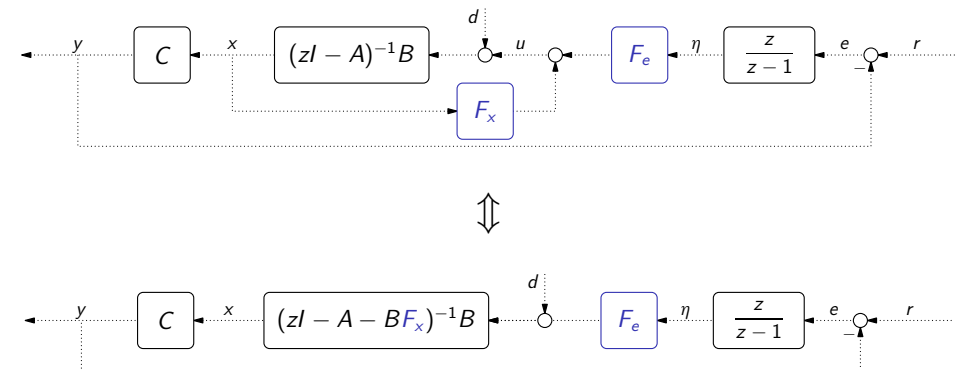
$$u[t] = F_x x[t] + F_e \sum_{s=0}^t (r[s] - Cx[s]),$$

which corresponds to



7/52

Closed-loop system



and we know that the **closed-loop system** **stable**. The integrator guarantees then that

- steady-state tracking error is zero for step r or d , exactly as with the classical PI controller.

8/52

Closed-loop dynamics

Combining

$$\begin{cases} \begin{bmatrix} \eta[t+1] \\ x[t+1] \end{bmatrix} = \begin{bmatrix} 1 & -C \\ 0 & A \end{bmatrix} \begin{bmatrix} \eta[t] \\ x[t] \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r[t] + \begin{bmatrix} 0 \\ B \end{bmatrix} d[t] \\ y[t] = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} \eta[t] \\ x[t] \end{bmatrix} \\ u[t] = \begin{bmatrix} F_e & F_x \end{bmatrix} \begin{bmatrix} \eta[t] \\ x[t] \end{bmatrix} \end{cases}$$

We end up with

$$T_{cl} : \begin{cases} \begin{bmatrix} \eta[t+1] \\ x[t+1] \end{bmatrix} = \begin{bmatrix} 1 & -C \\ BF_e & A + BF_x \end{bmatrix} \begin{bmatrix} \eta[t] \\ x[t] \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} r[t] \\ d[t] \end{bmatrix} \\ \begin{bmatrix} y[t] \\ u[t] \end{bmatrix} = \begin{bmatrix} 0 & C \\ F_e & F_x \end{bmatrix} \begin{bmatrix} \eta[t] \\ x[t] \end{bmatrix} \end{cases}$$

which has by now familiar inputs and outputs.

9/52

Closed-loop dynamics (contd)

The transfer functions of the closed-loop systems are

$$\begin{aligned} \begin{bmatrix} T_{yr}(z) & T_{yd}(z) \\ T_{ur}(z) & T_{ud}(z) \end{bmatrix} &= \begin{bmatrix} 0 & C \\ F_e & F_x \end{bmatrix} \left(zI - \begin{bmatrix} 1 & -C \\ BF_e & A + BF_x \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} 1 & -C & 1 & 0 \\ BF_e & A + BF_x & 0 & B \\ \hline 0 & C & 0 & 0 \\ F_e & F_x & 0 & 0 \end{array} \right]. \end{aligned}$$

10/52

Steady-state properties

If $r[t] = r_{ss}\mathbb{1}[1]$ and $d[t] = d_{ss}\mathbb{1}[1]$, in steady state we have

$$\begin{bmatrix} \eta_{ss} \\ x_{ss} \end{bmatrix} = \begin{bmatrix} 1 & -C \\ BF_e & A + BF_x \end{bmatrix} \begin{bmatrix} \eta_{ss} \\ x_{ss} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} r_{ss} \\ d_{ss} \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} 0 & C \\ -BF_e & I - A - BF_x \end{bmatrix} \begin{bmatrix} \eta_{ss} \\ x_{ss} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} r_{ss} \\ d_{ss} \end{bmatrix}$$

Therefore,

$$y_{ss} = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} \eta_{ss} \\ x_{ss} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} r_{ss} \\ d_{ss} \end{bmatrix} = r_{ss},$$

as expected. This is equivalent to

$$\begin{bmatrix} T_{yr}(1) & T_{yd}(1) \end{bmatrix} = \begin{bmatrix} 0 & C \end{bmatrix} \begin{bmatrix} 0 & C \\ -BF_e & I - A - BF_x \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

which can be proved also algebraically.

11/52

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Generalized hold

12/52

Observation problem

Given

$$\begin{cases} x[t+1] = Ax[t] + Bu[t] + Bd[t], & x[0] = x_0 \\ y[t] = Cx[t] + n[t] \end{cases}$$

with measured $u[t] \in \mathbb{R}^m$ and $y[t] \in \mathbb{R}^p$, reconstruct the unmeasured state $x[t] \in \mathbb{R}^n$. It has to be assumed that

- (C, A) is detectable.

It is also conventionally considered that

- either $x_0 = 0$ (normally, in optimal setups)
- or $d \equiv 0$ and $n \equiv 0$ (normally, in modal setups)

13/52

Observer idea

Underlying principle behind all observers is to

- reconstruct unmeasured x via imitating the dynamics of the system.

The naïve imitation,

$$\hat{x}[t+1] = A\hat{x}[t] + Bu[t], \quad \hat{x}[0] = \hat{x}_0$$

might not work, as the observation error, $\epsilon[t] := x[t] - \hat{x}[t]$, which satisfies

$$\epsilon[t+1] = A\epsilon[t] + Bd[t], \quad \epsilon[0] = \epsilon_0 := x_0 - \hat{x}_0$$

cannot be affected and might even be unbounded (if A is not Schur). But the measurement equation, $y[t] = Cx[t] + n[t]$, can provide valuable info,

- via the observation mismatch $y[t] - C\hat{x}[t]$ (innovations) signal, and should be used.

14/52

Observer

Construct the virtual plant, **observer**,

$$\hat{x}[t+1] = A\hat{x}[t] + Bu[t] - L(y[t] - C\hat{x}[t]), \quad \hat{x}[0] = \hat{x}_0$$

for some L . Denoting $A_L := A + LC$, the observer can also be written as

$$\hat{x}[t+1] = A_L\hat{x}[t] + Bu[t] - Ly[t], \quad \hat{x}[0] = \hat{x}_0.$$

This is an LSI system $\begin{bmatrix} K_{\hat{x}y} & K_{\hat{x}u} \end{bmatrix} : (y, u) \mapsto \hat{x}$, whose transfer functions

$$K_{\hat{x}y}(z) = -(zI - A_L)^{-1}L = \left[\begin{array}{c|c} A_L & -L \\ \hline I & 0 \end{array} \right]$$

and

$$K_{\hat{x}u}(z) = (zI - A_L)^{-1}B = \left[\begin{array}{c|c} A_L & B \\ \hline I & 0 \end{array} \right]$$

are stable.

15/52

Observer error

The observer error signal $\epsilon[t] := x[t] - \hat{x}[t]$ satisfies

$$\epsilon[t+1] = A_L\epsilon[t] + Bd[t] + Ln[t], \quad \epsilon[0] = \epsilon_0 := x_0 - \hat{x}_0$$

and we naturally choose L so that A_L is Schur.

- If $d \equiv 0$ and $n \equiv 0$, then

$$\epsilon[t+1] = A_L\epsilon[t], \quad \epsilon[0] = \epsilon_0 \implies \epsilon[t] = A_L^t \epsilon_0$$

and $\lim_{t \rightarrow \infty} (x[t] - \hat{x}[t]) = 0$ for all initial conditions.

- If $\epsilon_0 = 0$, then the error system $\begin{bmatrix} T_{\epsilon d} & T_{\epsilon n} \end{bmatrix} : (d, n) \mapsto \epsilon$ with

$$T_{\epsilon d}(z) = K_{\hat{x}u}(z) = \left[\begin{array}{c|c} A_L & B \\ \hline I & 0 \end{array} \right] \quad \text{and} \quad T_{\epsilon n}(z) = -K_{\hat{x}y}(z) = \left[\begin{array}{c|c} A_L & L \\ \hline I & 0 \end{array} \right]$$

is stable, so that ϵ is bounded for all bounded d and n .

16/52

State observer: design of L

A common approach is to

- assign poles of the error systems, which belong to $\text{spec}(A + LC)$.

Ackermann's formula (provided (C, A) is observable):

$$L = -\chi_{\text{obs}}(A)M_o^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where M_o is the observability matrix and

$$\chi_{\text{obs}}(z) = z^n + \hat{\chi}_{n-1}z^{n-1} + \dots + \hat{\chi}_1z + \hat{\chi}_0$$

is the required characteristic polynomial of $A_L = A + LC$. In the particular case of

- $\chi_{\text{obs}}(z) = z^n$ we have the dead-beat observer.

17/52

State observer: example

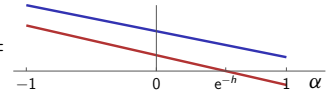
Return to the discretized DC motor, for which

$$\begin{cases} x[t+1] = \begin{bmatrix} 1 & 1 - e^{-h} \\ 0 & e^{-h} \end{bmatrix} x[t] + \begin{bmatrix} h - 1 + e^{-h} \\ 1 - e^{-h} \end{bmatrix} (u[t] + d[t]), \\ y[t] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[t] + n[t] \end{cases}$$

with $x_1[i] = y(ih)$ and $x_2[i] = \dot{y}(ih)$. As no sampling can be pathological in this case, the pair (C, A) is observable. Choose

$$\chi_{\text{obs}}(z) = z(z - \alpha)$$

for some $|\alpha| < 1$. By Ackermann, $L = -A(A - \alpha I)M_o^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, resulting in

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = - \begin{bmatrix} 1 + e^{-h} - \alpha \\ e^{-h}(e^{-h} - \alpha)/(1 - e^{-h}) \end{bmatrix} =$$


i.e. gains are decreasing functions of α .

18/52

Example: observer transfer functions

Thus,

$$K_{\hat{x}_1 y}(z) = \frac{(1 + e^{-h} - \alpha)z - e^{-h}}{z(z - \alpha)} \quad \text{and} \quad K_{\hat{x}_2 y}(z) = \frac{e^{-h}(\alpha - e^{-h})}{1 - e^{-h}} \frac{z - 1}{z(z - \alpha)}$$

and

$$K_{\hat{x}_1 u}(z) = \frac{(e^{-h} + h - 1)(z - z_z)}{z(z - \alpha)} \quad \text{and} \quad K_{\hat{x}_2 u}(z) = \frac{(1 - e^{-h})(z - z_\alpha)}{z(z - \alpha)}$$

where

$$z_z := -\frac{1 - e^{-h}(1 + h)}{e^{-h} + h - 1} \in (-1, 0)$$

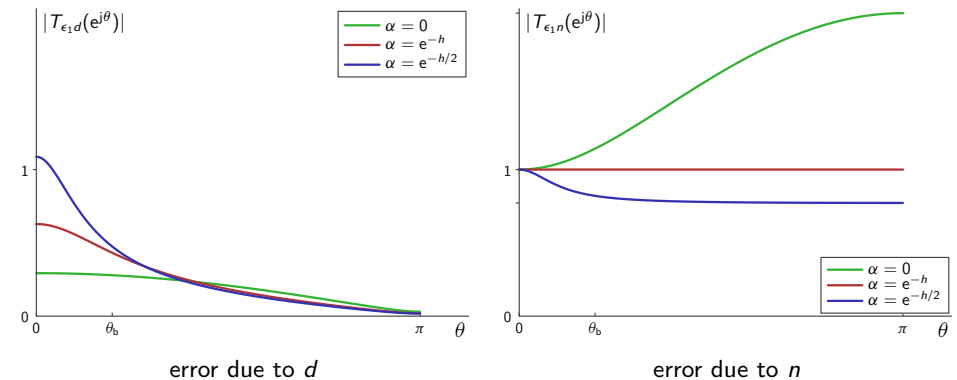
is the sampling zero of the plant and

$$z_\alpha := \frac{(1 - e^{-h}(h + 1))(\alpha - e^{-h})}{(1 - e^{-h})^2} \in (-1, 1)$$

($z_\alpha = z_z$ iff $\alpha = z_z$).

19/52

Example: error frequency responses, ϵ_1

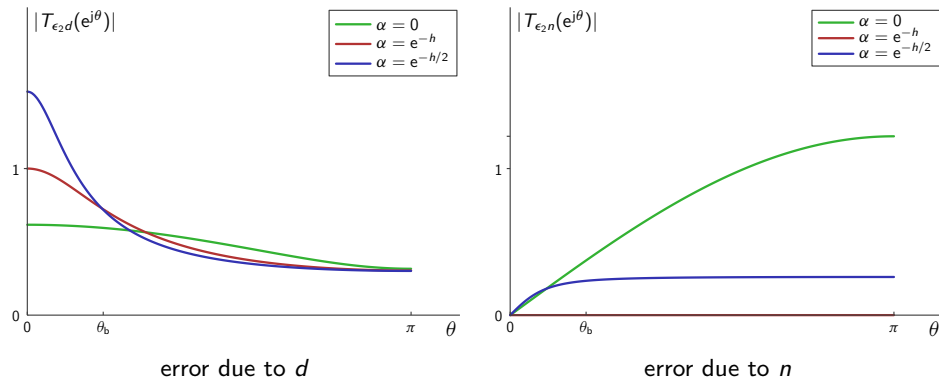


Sampling period $h = \pi/5 \approx 0.6283$, observer gains:

$$L \in \left\{ \begin{bmatrix} -1.533 \\ -0.61 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -0.803 \\ 0.225 \end{bmatrix} \right\}.$$

20/52

Example: error frequency responses, ϵ_2



Sampling period $h = \pi/5 \approx 0.6283$, observer gains:

$$L \in \left\{ \begin{bmatrix} -1.533 \\ -0.61 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -0.803 \\ 0.225 \end{bmatrix} \right\}.$$

21/52

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Generalized hold

22/52

Reduced-order observer

W.l.o.g. we may assume that $C = \begin{bmatrix} I_p & 0 \end{bmatrix}$. Rewrite then

$$\begin{cases} \begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u[t] + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d[t], \\ y[t] = x_1[t] + n[t] \end{cases}$$

If $n \equiv 0$, then $x_1 = y$ is measurable directly and need not be reconstructed. We may expect then to

- simplify the observer if it aims only at reconstructing x_2 .

This is plausible, as suggested by the partial (lower-dimensional) dynamics

$$x_2[t+1] = A_{22}x_2[t] + A_{21}y[t] + B_2u[t] + B_2d[t] - A_{21}n[t].$$

But

- how can we affect the convergence of an observer based on it?

23/52

Reduced-order observer (contd)

What we still have unexploited is the relation

$$y[t+1] = A_{11}y[t] + A_{12}x_2[t] + B_1u[t] + B_1d[t] + n[t+1] - A_{11}n[t].$$

It prompts the use of the correction term

$$L_2(y[t+1] - A_{11}y[t] - A_{12}\hat{x}_2[t] - B_1u[t]),$$

i.e. the observer

$$\begin{aligned} \hat{x}_2[t+1] &= A_{22}\hat{x}_2[t] + A_{21}y[t] + B_2u[t] \\ &\quad - L_2(y[t+1] - A_{11}y[t] - A_{12}\hat{x}_2[t] - B_1u[t]) \\ &= (A_{22} + L_2A_{12})\hat{x}_2[t] + (A_{21} + L_2A_{11})y[t] + (B_2 + L_2B_1)u[t] \\ &\quad - L_2y[t+1]. \end{aligned}$$

But

- how can we use the future $y[t+1]$?

24/52

Reduced-order observer (contd)

The trick is to define a new variable,

$$\xi[t] := \hat{x}_2[t] + L_2 y[t].$$

Because $\hat{x}_2[t] = \xi[t] - L_2 y[t]$, this variable satisfies

$$\begin{aligned} \xi[t+1] &= (A_{22} + L_2 A_{12})(\xi[t] - L_2 y[t]) + (A_{21} + L_2 A_{11})y[t] \\ &\quad + (B_2 + L_2 B_1)u[t]. \end{aligned}$$

Thus, we end up with the **reduced-order observer**

$$\begin{cases} \xi[t+1] = (A_{22} + L_2 A_{12})\xi[t] \\ \quad + (A_{21} + L_2 A_{11} - (A_{22} + L_2 A_{12})L_2)y[t] + (B_2 + L_2 B_1)u[t] \\ \hat{x}_2[t] = \xi[t] - L_2 y[t] \end{cases}$$

which is again a system $\begin{bmatrix} K_{\hat{x}_2 y} & K_{\hat{x}_2 u} \end{bmatrix} : (y, u) \mapsto \hat{x}_2$, of order $n - p$ now.

25/52

Reduced-order observer: transfer functions

Denote

$$A_{L2} := A_{22} + L_2 A_{12}, \quad A_{L1} := A_{21} + L_2 A_{11}, \quad \text{and} \quad B_{L2} := B_2 + L_2 B_1.$$

Direct substitutions yield then

$$K_{\hat{x}_2 y}(z) = -L_2 - (zI - A_{L2})^{-1}(A_{L2}L_2 - A_{L1}) = - \left[\begin{array}{c|c} \frac{A_{L2}}{I} & \frac{A_{L2}L_2 - A_{L1}}{L_2} \end{array} \right]$$

and

$$K_{\hat{x}_2 u}(z) = (zI - A_{L2})^{-1}B_{L2} = \left[\begin{array}{c|c} \frac{A_{L2}}{I} & \frac{B_{L2}}{0} \end{array} \right].$$

Note that $K_{\hat{x}_2 y}(z)$ is bi-proper.

26/52

Reduced-order observer: error system

Combining

$$x_2[t+1] = A_{22}x_2[t] + A_{21}y[t] + B_2u[t] + B_2d[t] - A_{21}n[t]$$

with

$$\begin{aligned} \hat{x}_2[t+1] &= A_{22}\hat{x}_2[t] + A_{21}y[t] + B_2u[t] \\ &\quad - L_2(A_{12}x_2[t] + B_1d[t] + n[t+1] - A_{11}n[t] - A_{12}\hat{x}_2[t]) \end{aligned}$$

(where $y[t+1] - A_{11}y[t] - B_1u[t] = A_{12}x_2 + B_1d[t] + n[t+1] - A_{11}n[t]$ is used), we end up with the observation error dynamics

$$\epsilon_2[t+1] = A_{L2}\epsilon_2[t] + B_{L2}d[t] + L_2n[t+1] - A_{L1}n[t], \quad \epsilon_2[0] = \epsilon_0$$

where $\epsilon_2 := x_2 - \hat{x}_2$. If $d \equiv 0$ and $n \equiv 0$, then $\epsilon_2[t] = A_{L2}^t \epsilon_0$ asymptotically vanishes if A_{L2} is Schur. As a matter of fact, $\epsilon_1[t] := x_1[t] - \underbrace{\hat{x}_1[t]}_{=y[t]} = -n[t]$.

27/52

Reduced-order observer: error system (contd)

Assuming $\epsilon_0 = 0$, the error system $\begin{bmatrix} T_{\epsilon_2 d} & T_{\epsilon_2 n} \end{bmatrix} : (d, n) \mapsto \epsilon_2$ in z -domain reads

$$E_2(z) = (zI - A_{L2})^{-1}(B_{L2}D(z) + (zL_2 - A_{L1})N(z)).$$

We thus end up with

$$T_{\epsilon_2 d} = K_{\hat{x}_2 u}(z) = \left[\begin{array}{c|c} \frac{A_{L2}}{I} & \frac{B_{L2}}{0} \end{array} \right]$$

and, taking into account that $(zI - A_{L2})^{-1}z = I + (zI - A_{L2})^{-1}A_{L2}$,

$$T_{\epsilon_2 n}(z) = -K_{\hat{x}_2 y}(z) = \left[\begin{array}{c|c} \frac{A_{L2}}{I} & \frac{A_{L2}L_2 - A_{L1}}{L_2} \end{array} \right].$$

Qualitatively, this is exactly what we had for the full-order observer.

28/52

Reduced-order observer: choice of L_2

Lemma

$\lambda \in \mathbb{C}$ is an unobservable mode of (C, A) iff it is an unobservable mode of (A_{12}, A_{22}) .

Proof: If λ is unobservable for (C, A) , then there is $\eta \neq 0$ such that

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \eta = 0 \iff \begin{bmatrix} A_{11} - \lambda I & A_{12} \\ A_{21} & A_{22} - \lambda I \\ I & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0.$$

Hence, $\eta_1 = 0$ and

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \eta = 0 \iff \begin{bmatrix} A_{22} - \lambda I \\ A_{12} \end{bmatrix} \eta_2 = 0.$$

This proves the result. $\square$

So if (C, A) is detectable, then there is L_2 such that $A_{22} + L_2 A_{12}$ is Schur.

29/52

Reduced-order observer: example

Return to the discretized DC motor, for which

$$\begin{cases} x[t+1] = \begin{bmatrix} 1 & 1 - e^{-h} \\ 0 & e^{-h} \end{bmatrix} x[t] + \begin{bmatrix} h - (1 - e^{-h}) \\ 1 - e^{-h} \end{bmatrix} (u[t] + d[t]), \\ y[t] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[t] + n[t] \end{cases}$$

with $x_1[i] = y(ih)$ (measurable) and $x_2[i] = \dot{y}(ih)$ (unmeasurable). Choose

$$\chi_{\text{obs}}(z) = z - \alpha, \quad |\alpha| < 1$$

for which

$$L_2 = \frac{\alpha - e^{-h}}{1 - e^{-h}} \implies A_{L2} = e^{-h} + \frac{\alpha - e^{-h}}{1 - e^{-h}}(1 - e^{-h}) = \alpha$$

and $A_{L1} = (\alpha - e^{-h})/(1 - e^{-h})$ and $B_{L2} = h + (1 - \alpha)(1 - h/(1 - e^{-h}))$. Note that $|L_2|$ is a decreasing function of α for $\alpha \leq e^{-h}$.

30/52

Example: observer transfer functions

Hence,

$$K_{\hat{x}_2 y}(z) = -\frac{\alpha - e^{-h}}{1 - e^{-h}} \frac{z - 1}{z - \alpha} = -T_{\epsilon_2 n}(z)$$

and

$$K_{\hat{x}_2 u}(z) = -\frac{\alpha - z_z}{1 - e^{-h}} \frac{1 - e^{-h} - h}{z - \alpha} = T_{\epsilon_2 d}(z),$$

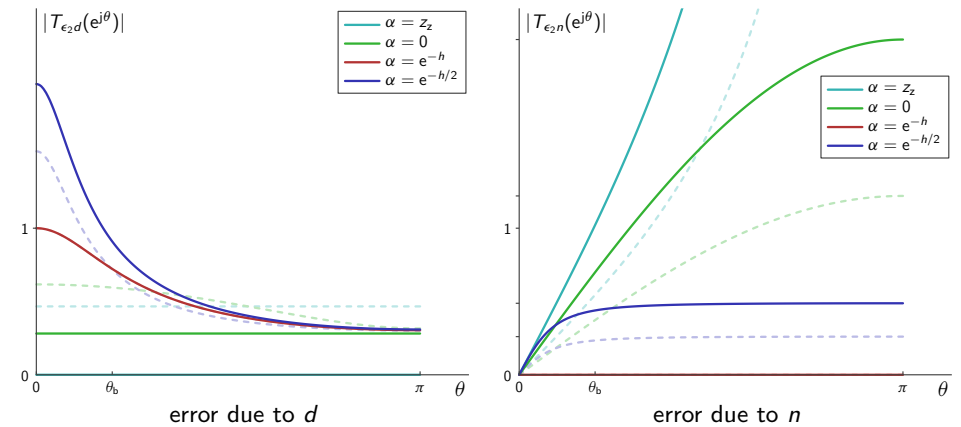
where

$$z_z := -\frac{1 - e^{-h}(1 + h)}{e^{-h} + h - 1} \in (-1, 0)$$

is the sampling zero of the plant.

31/52

Example: error frequency responses, ϵ_2



Sampling period $h = \pi/5 \approx 0.6283$, observer gains:

$$L_2 \in \{-2.883, -1.144, 0, 0.422\}.$$

32/52

Outline

Integral action in state feedback

State-space design methods in discrete time: state observers

State-space design methods in discrete time: reduced-order observers

State-space design methods in discrete time: observer-based feedback

Generalized hold

33/52

Observer-based feedback

Let

$$P : \begin{cases} x[t+1] = Ax[t] + Bu[t] + Bd[t] \\ y[t] = Cx[t] + n[t] \end{cases}$$

If we cannot measure its state x , we may try to use its observation instead:

$$u[t] = F\hat{x}[t], \quad \text{where } \hat{x}[t+1] = (A + LC)\hat{x}[t] + Bu[t] - Ly[t].$$

This control law, dubbed the **observer-based feedback**, can be presented as

$$K : \begin{cases} \hat{x}[t+1] = (A + BF + LC)\hat{x}[t] - Ly[t] \\ u[t] = F\hat{x}[t] \end{cases}$$

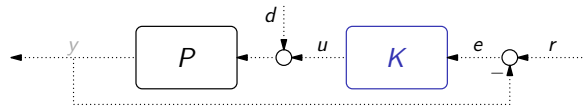
and has the transfer function (mind that the positive feedback is assumed)

$$K(z) = -F(zI - (A + BF + LC))^{-1}L = - \left[\begin{array}{c|c} A + BF + LC & L \\ \hline F & 0 \end{array} \right].$$

This is a *strictly causal* system, i.e. has at least one delay between y and u .

34/52

GoF for observer-based feedback



Let

$$P : \begin{cases} x[t+1] = Ax[t] + Bu_p[t] \\ y[t] = Cx[t] \end{cases} \quad \text{and} \quad K : \begin{cases} x_k[t+1] = A_k x_k[t] + B_k e[t] \\ u[t] = C_k x_k[t] + D_k e[t] \end{cases}$$

In Lect. 5, Slide 59 we saw that the relations for the GoF are

$$\begin{cases} \begin{bmatrix} x[t+1] \\ x_k[t+1] \end{bmatrix} = \begin{bmatrix} A - BD_k C & BC_k \\ -B_k C & A_k \end{bmatrix} \begin{bmatrix} x[t] \\ x_k[t] \end{bmatrix} + \begin{bmatrix} BD_k & B \\ B_k & 0 \end{bmatrix} \begin{bmatrix} r[t] \\ d[t] \end{bmatrix} \\ \begin{bmatrix} e[t] \\ u[t] \end{bmatrix} = \begin{bmatrix} -C & 0 \\ -D_k C & C_k \end{bmatrix} \begin{bmatrix} x[t] \\ x_k[t] \end{bmatrix} + \begin{bmatrix} I & 0 \\ D_k & 0 \end{bmatrix} \begin{bmatrix} r[t] \\ d[t] \end{bmatrix} \end{cases}$$

35/52

GoF for observer-based feedback (contd)

In our case $A_k = A + BF + LC$, $B_k = L$, $C_k = F$, and $D_k = 0$ (we switched to the negative feedback setting now). Hence,

$$\begin{cases} \begin{bmatrix} x[t+1] \\ \hat{x}[t+1] \end{bmatrix} = \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix} \begin{bmatrix} x[t] \\ \hat{x}[t] \end{bmatrix} + \begin{bmatrix} 0 & B \\ L & 0 \end{bmatrix} \begin{bmatrix} r[t] \\ d[t] \end{bmatrix} \\ \begin{bmatrix} e[t] \\ u[t] \end{bmatrix} = \begin{bmatrix} -C & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x[t] \\ \hat{x}[t] \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r[t] \\ d[t] \end{bmatrix} \end{cases}$$

This might appear messy, but it can be simplified by transforming the state

$$\begin{bmatrix} x[t] \\ \hat{x}[t] \end{bmatrix} \longrightarrow \begin{bmatrix} x[t] \\ \epsilon[t] \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x[t] \\ \hat{x}[t] \end{bmatrix}$$

In other words, use similarity transformation with

$$T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = T^{-1}.$$

36/52

Similarity transformation

We have:

$$\begin{aligned} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} &= \begin{bmatrix} A + BF & BF \\ 0 & A + LC \end{bmatrix}, \\ \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} 0 & B \\ L & 0 \end{bmatrix} &= \begin{bmatrix} 0 & B \\ -L & B \end{bmatrix}, \\ \begin{bmatrix} -C & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} &= \begin{bmatrix} -C & 0 \\ F & -F \end{bmatrix}. \end{aligned}$$

Hence,

$$\text{spec}(A_{cl}) = \text{spec}(A + BF) \cup \text{spec}(A + LC)$$

and the closed-loop system is

- asymptotically stable iff both the state feedback on which it is based is stabilizing and the observer used in it is stable.

37/52

GoF for observer-based feedback (contd)

The GoF, which are $(r, d) \mapsto (e, u)$,

$$\begin{cases} \begin{bmatrix} x[t+1] \\ \epsilon[t+1] \end{bmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x[t] \\ \epsilon[t] \end{bmatrix} + \begin{bmatrix} 0 & B \\ -L & B \end{bmatrix} \begin{bmatrix} r[t] \\ d[t] \end{bmatrix} \\ \begin{bmatrix} e[t] \\ u[t] \end{bmatrix} = \begin{bmatrix} -C & 0 \\ F & -F \end{bmatrix} \begin{bmatrix} x[t] \\ \epsilon[t] \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r[t] \\ d[t] \end{bmatrix} \end{cases}$$

and the corresponding closed-loop transfer functions are

$$\begin{aligned} \begin{bmatrix} S(z) & -T_d(z) \\ T_c(z) & -T(z) \end{bmatrix} &= \begin{bmatrix} I \\ K(z) \end{bmatrix} (I + P(z)K(z))^{-1} \begin{bmatrix} I & -P(z) \end{bmatrix} \\ &= \begin{bmatrix} A + BF & BF & 0 & B \\ 0 & A + LC & -L & B \\ -C & 0 & I & 0 \\ F & -F & 0 & 0 \end{bmatrix}. \end{aligned}$$

38/52

Outline

Integral action in state feedback

State-space design methods in discrete time: state observers

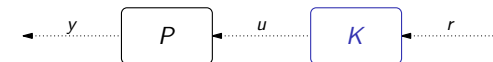
State-space design methods in discrete time: reduced-order observers

State-space design methods in discrete time: observer-based feedback

Generalized hold

39/52

Open-loop plant inversion



Ideal plant inversion,

$$K(z) = \frac{1}{P(z)},$$

yields $y = r$, but might require

- non-proper $K(z) \implies$ non-causal K (if $P(z)$ is strictly proper)
- unstable $K(z)$ (if $P(z)$ has zeros in $\mathbb{C} \setminus \mathbb{D}_1$)

Realistic plant inversion,

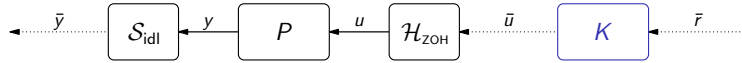
$$K(z) = \frac{T_r(z)}{P(z)},$$

yields $y = T_r r$ and results in an admissible $K(z)$ if

- the pole excess of $T_r(z)$ is at least that of $P(z)$,
- all unstable zeros of $P(z)$ are zeros of $T_r(z)$.

40/52

Open-loop plant inversion for discretized plants



The transfer function $\bar{P}_h(z)$ of the discretized plant $\bar{P}_h = S_{idl} P \mathcal{H}_{ZOH}$ is

- always strictly proper, so has a pole excess of at least 1,
- frequently nonminimum phase, because of sampling zeros.

Hence, admissible reference models T_r

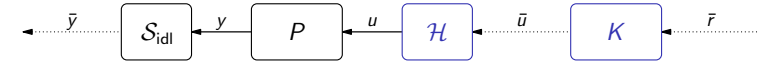
- must have at least one delay intuitive
- must contain unstable sampling zeros of $\bar{P}_h(z)$ not intuitive

But sampling zeros are functions of the hold, which suggests that

- control problems can be simplified by an appropriate choice of $\mathcal{H}$, which has a **control-centric** structure.

41/52

Generalized hold à la Kabamba



Let $P(s) = C(sI - A)^{-1}B$ with controllable (A, B) and consider the choice

$$\phi(t) = B'e^{A'(h-t)} \left(\int_0^h e^{As} B B' e^{A's} ds \right)^{-1} (\mathbb{1}(t) - \mathbb{1}(t-h))$$

Mind that $\phi(t) \in \mathbb{R}^{1 \times n}$, so $\bar{u}$ has a higher dimension than u . We know that

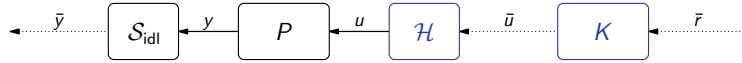
$$\begin{aligned} \bar{B} &= \int_0^h e^{A(h-t)} B \phi(t) dt = \int_0^h e^{A(h-t)} B B' e^{A'(h-t)} dt \left(\int_0^h e^{As} B B' e^{A's} ds \right)^{-1} \\ &= I_n \end{aligned}$$

so that $\bar{P}_h(z) = C(zI - \bar{A})^{-1}$, where $\bar{A} = e^{Ah}$, as always. Interestingly,

- $(\bar{A}, \bar{B}) = (\bar{A}, I)$ is reachable for all $h > 0$, even for pathological ones.

42/52

The miracle of generalized holds



It happens that $\bar{P}_h(z) = C(zI - \bar{A})^{-1}$ has no zeros at all. In particular, it is (almost, modulo one-step delay) right invertible by a stable system. Indeed, let

$$K(z) = z^{-1}(zI - \bar{A})C^\# = C^\# - \bar{A}C^\#z^{-1},$$

where $C^\# \in \mathbb{R}^n$ is a right inverse of $C \in \mathbb{R}^{1 \times n}$. This $K(z)$ is

- proper (hence, K is causal)
- stable (its only pole is at $z = 0$)

Moreover,

$$\bar{P}_h(z)K(z) = C(zI - \bar{A})^{-1} \cdot (zI - \bar{A})C^\#z^{-1} = z^{-1}.$$

In other words, we achieved

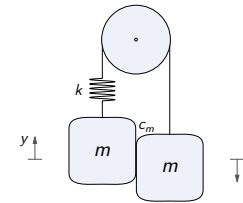
- $\bar{y}[i] = \bar{r}[i-1]$ regardless of properties of P !

43/52

Example

Consider the system $P : u \mapsto y$, which is at equilibrium with $y = u = 0$:

Its equation of motion (assuming zero damping):



$$\begin{aligned} m\ddot{y}(t) &= F_{\text{spring}}(t) + F_{\text{massfriction}}(t) \\ &= k(u(t) - y(t)) - c_m(\dot{u}(t) + \dot{y}(t)). \end{aligned}$$

We assume hereafter that $m = 1$, $k = 1$, $c_m = 2$.

Hence,

$$P(s) = \frac{-2s + 1}{s^2 + 2s + 1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & -2 & 0 \end{bmatrix}.$$

This is a nonminimum-phase transfer function, i.e. it is not stably invertible.

Its discretization under $\mathcal{H}_{ZOH}$,

$$\bar{P}_h(z) = \frac{(1 - e^{-h} - 3e^{-h}h)z - e^{-h}(1 - e^{-h} - 3h)}{(z - e^{-h})^2}$$

is also nonminimum phase for all $h \in (0, 2.83844638)$.

44/52

Example: hold and discretized model

The hold function for $t \in [0, h]$ is

$$\phi(t) = \frac{4e^{t-h}}{(1-e^{-2h})^2 - (2e^{-h}h)^2} \left(\begin{bmatrix} h-t & 1-h+t \end{bmatrix} - e^{-2h} \begin{bmatrix} h-t+2th & 1+h+t+2th \end{bmatrix} \right)$$

and the discretized plant is

$$\bar{P}_h(z) = \left[\begin{array}{cc|cc} e^{-h}(1+h) & e^{-h}h & 1 & 0 \\ -e^{-h}h & e^{-h}(1-h) & 0 & 1 \\ \hline 1 & 2 & 0 & 0 \end{array} \right].$$

45/52

Example: controller

Then the controller inverting $z\bar{P}_h(z)$ is (here $C^\# = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, non-unique)

$$K(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -h-1 \\ h \end{bmatrix} e^{-h} z^{-1}.$$

This controller generates the discrete control signal

$$\bar{u}[i] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{r}[i] + \begin{bmatrix} -h-1 \\ h \end{bmatrix} e^{-h} \bar{r}[i-1].$$

The discrete state is $\bar{x}[i] = C^\# \bar{r}[i-1]$, resulting in the sampled output

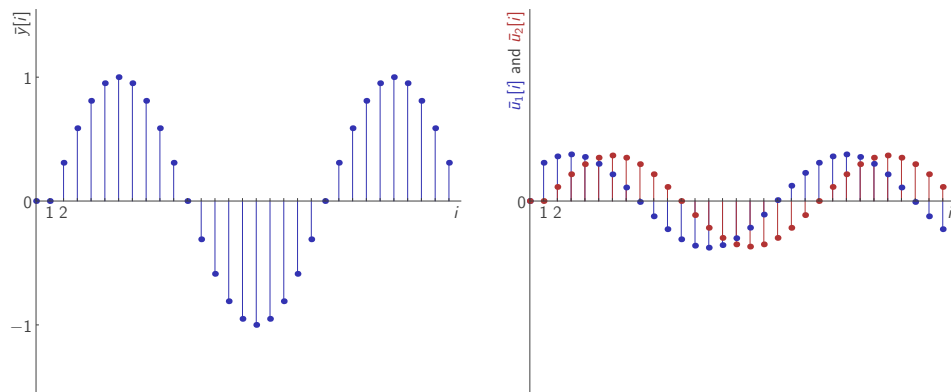
$$\bar{y}[i] = C \bar{x}[i] = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} \bar{r}[i-1] \\ 0 \end{bmatrix} = \bar{r}[i-1]$$

for all $\bar{r}$, as expected.

46/52

Example: simulations

With $r(t) = \sin(0.1\pi t)\mathbb{1}(t)$ and $h = 1$:



This is what we expect, $\bar{y}[i] = \bar{r}[i-1] \iff y(ih) = r((i-1)h)$.

47/52

Example: analog behavior

The analog control signal in this case

$$u(ih + \tau) = \phi(\tau) \bar{u}[i] = \frac{4e^{\tau-h}}{(1-e^{-2h})^2 - (2e^{-h}h)^2} (\phi_0(\tau) \bar{r}[i] + \phi_1(\tau) \bar{r}[i-1])$$

for

$$\begin{bmatrix} \phi_0(\tau) \\ \phi_1(\tau) \end{bmatrix} := \begin{bmatrix} h-\tau \\ e^{-h}\tau \end{bmatrix} (1-e^{-2h}) - \begin{bmatrix} e^{-h}\tau \\ h-\tau \end{bmatrix} 2e^{-h}h.$$

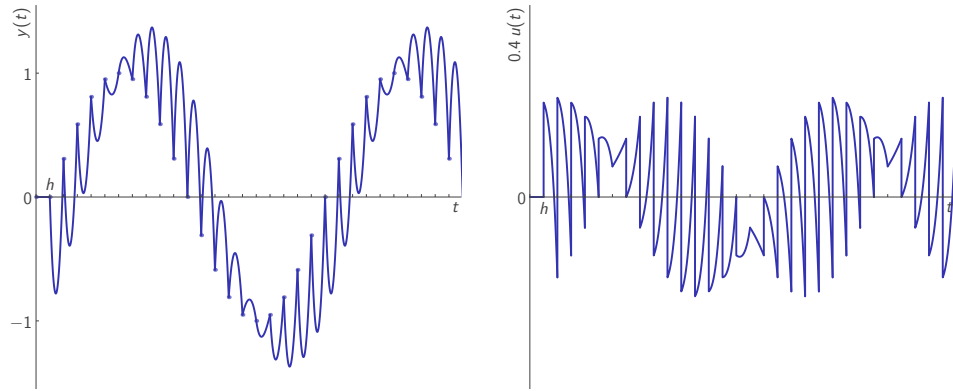
The analog output can be computed according to

$$y(ih + \tau) = Ce^{A\tau} \bar{x}[i] + C \int_0^\tau e^{A(\tau-\sigma)} Bu(ih + \sigma) d\sigma.$$

48/52

Example: simulations (contd)

With $r(t) = \sin(0.1\pi t)\mathbb{1}(t)$ and $h = 1$:



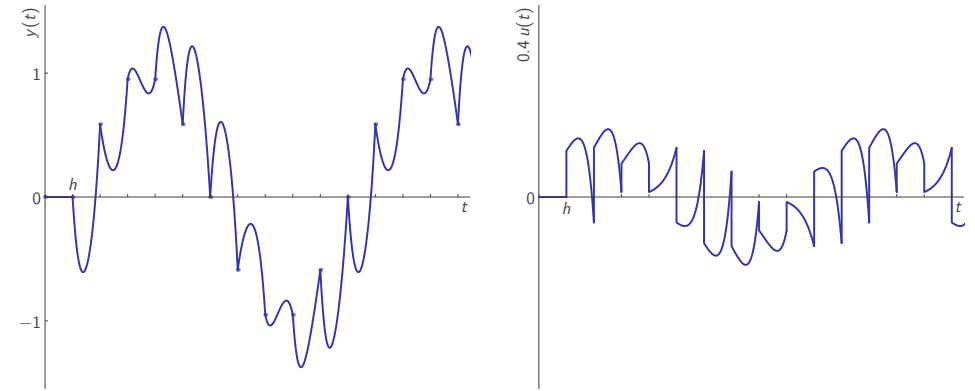
Perhaps, not what we expected, as

- analog output $y(t)$ is poor in intersample
- control signal $u(t)$ is “nervous”

49/52

Example: simulations (contd)

With $r(t) = \sin(0.1\pi t)\mathbb{1}(t)$ and $h = 2$:



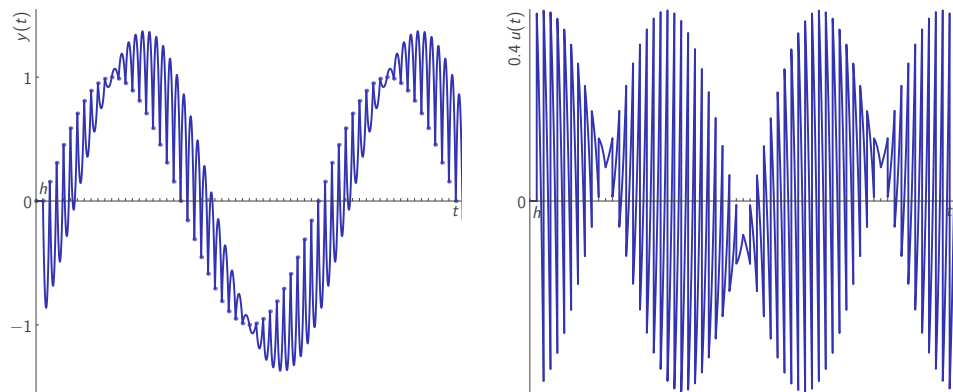
Perhaps, not what we expected, as

- analog output $y(t)$ is poor in intersample
- control signal $u(t)$ is “nervous”

50/52

Example: simulations (contd)

With $r(t) = \sin(0.1\pi t)\mathbb{1}(t)$ and $h = 0.5$:

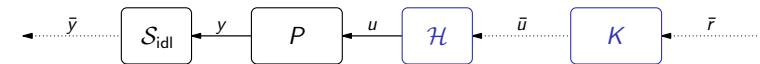


Perhaps, not what we expected, as

- analog output $y(t)$ is poor in intersample
- control signal $u(t)$ is “nervous”

51/52

The miracle of generalized holds (contd)



Control-oriented design of $\mathcal{H}$

- may simplify discrete analysis and controller design
- might produce poor analog response and unacceptable control signals well, there's no such thing as a free lunch...

A more control-oriented moral:

- endeavors to overcome limitations of analog control are costly and normally futile.

52/52