

Sampled-Data Control Systems (036709)

lecture no. 6

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Outline

Discretization in state space

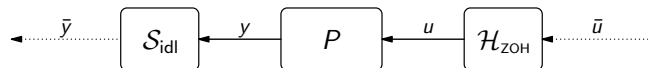
Stability

State-space design methods in discrete time: state feedback

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Discretization problem, again

Find the discrete $\bar{P}_h : \bar{u} \mapsto \bar{y}$ for



Assume now that $P(s) = C(sI - A)^{-1}B$ and aim at deriving a state-space realization of $\bar{P}_h$ (we already know that $\bar{P}_h$ is strictly causal, LSI, and finite dimensional).

The impulse response of P is $p(t) = Ce^{At}B\mathbb{1}(t)$ and that of $\bar{P}_h$ for $i \in \mathbb{N}$ is

$$\begin{aligned}\bar{p}_h[i] &= \int_0^h p(ih - t)dt = \int_0^h Ce^{A(ih-t)}Bdt = Ce^{A(i-1)h} \int_0^h e^{A(h-t)}dtB \\ &= \underbrace{C(e^{Ah})}_{\bar{A}} \underbrace{\int_0^h e^{At}dt}_{\bar{B}} B.\end{aligned}$$

This is the impulse response of a system, whose realization is $(\bar{A}, \bar{B}, C, 0)$.

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Van Loan's formula

Consider the matrix

$$M := \int_0^h e^{Es} G e^{Fs} ds.$$

It is readily seen that

$$M e^{-Fh} = \int_0^h e^{E(h-s)} G e^{F(h-s)} ds e^{-Fh} = \int_0^h e^{E(h-s)} G e^{-Fs} ds.$$

This is the response $y(h)$ of $(sI - E)^{-1}G$ to the input $u(t) = e^{-Ft}\mathbb{1}(t)$. But $u(t)$ is the impulse response of $(sI - (-F))^{-1}$. Hence,

— $M e^{-Fh}$ is the impulse response, at $t = h$, of the cascade

$$\left[\begin{array}{c|c} E & G \\ \hline I & 0 \end{array} \right] \left[\begin{array}{c|c} -F & I \\ \hline I & 0 \end{array} \right] = \left[\begin{array}{cc|c} E & G & 0 \\ 0 & -F & I \\ \hline I & 0 & 0 \end{array} \right].$$

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Van Loan's formula (contd)

As the impulse response of $C(sI - A)^{-1}B$ at $t = h$ is $Ce^{Ah}B$,

$$Me^{-Fh} = \begin{bmatrix} I & 0 \end{bmatrix} \exp \left(\begin{bmatrix} E & G \\ 0 & -F \end{bmatrix} h \right) \begin{bmatrix} 0 \\ I \end{bmatrix},$$

i.e. it is the (1,2) sub-block of the matrix exponential. But because

$$\exp \left(\begin{bmatrix} A_{11} & \times \\ 0 & A_{22} \end{bmatrix} t \right) = \begin{bmatrix} e^{A_{11}t} & \times \\ 0 & e^{A_{22}t} \end{bmatrix},$$

e^{-Fh} is the (2,2) sub-block of the same matrix exponential and we have

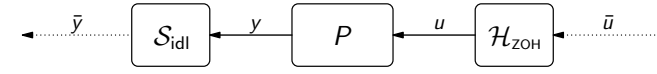
$$\int_0^h e^{Es} G e^{Fs} ds = \Gamma_{12} \Gamma_{22}^{-1}, \quad \text{where} \quad \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{bmatrix} := \exp \left(\begin{bmatrix} E & G \\ 0 & -F \end{bmatrix} h \right).$$

Equivalently,

$$\exp \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} t \right) = \begin{bmatrix} e^{A_{11}t} & \int_0^t e^{A_{11}s} A_{12} e^{A_{22}(t-s)} ds \\ 0 & e^{A_{22}t} \end{bmatrix}.$$

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Discretization problem in state space



Thus,

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] \rightarrow \bar{P}_h(z) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array} \right],$$

where, using Van Loan's formula,

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} = \begin{bmatrix} e^{Ah} & \int_0^h e^{At} dt B \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \exp \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} h \right).$$

The default response of MATLAB's c2d command yields this discretization.

Note that

$$\lambda \in \text{spec}(A) \iff e^{\lambda h} \in \text{spec}(\bar{A}),$$

which is expected, in light connections between poles of $P(s)$ and $\bar{P}_h(z)$.

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Reachability under discretization

It should be clear, from the definition, that

- an uncontrollable (A, B) cannot result in a reachable $(\bar{A}, \bar{B})$

(control signal is limited to be piecewise-constant). A stronger statement:

Proposition

If λ is an uncontrollable mode of (A, B) , then $e^{\lambda h}$ is an unreachable mode of $(\bar{A}, \bar{B})$.

Proof: If λ is an uncontrollable mode of (A, B) , then there is $\eta \neq 0$ such that $\eta' A = \lambda \eta'$ and $\eta' B = 0$. Because

$$\eta' \bar{A} = \eta' e^{Ah} = e^{\lambda h} \eta' \quad \text{and} \quad \eta' \bar{B} = \eta' \int_0^h e^{At} dt B = \int_0^h e^{\lambda t} dt \eta' B = 0,$$

we have

$$\eta' [\bar{A} - e^{\lambda h} I \quad \bar{B}] = 0$$

and this mode is unreachable. $\square$

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Reachability under discretization (contd)

Less obvious is that

- controllable (A, B) might not result in reachable $(\bar{A}, \bar{B})$.

Example

If

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} 0 & \omega_n & 0 \\ -\omega_n & 0 & \omega_n \end{bmatrix}, \quad \text{spec}(A) = \{\pm j\omega_n\}$$

which is controllable iff $\omega_n \neq 0$, then

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} = \begin{bmatrix} \cos(\omega_n h) & \sin(\omega_n h) & 1 - \cos(\omega_n h) \\ -\sin(\omega_n h) & \cos(\omega_n h) & \sin(\omega_n h) \end{bmatrix}.$$

Hence,

$$\bar{M}_c = \begin{bmatrix} 1 - \cos(\omega_n h) & -\cos(2\omega_n h) + \cos(\omega_n h) \\ \sin(\omega_n h) & \sin(2\omega_n h) - \sin(\omega_n h) \end{bmatrix}$$

and $\det \bar{M}_c = 2 \sin(\omega_n h)(\cos(\omega_n h) - 1) = 0 \iff \omega_n h = \pi k, k \in \mathbb{Z}_+.$ ∇

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Useful function

Define

$$\alpha_\lambda(x) := e^{\lambda h} \int_0^h e^{(x-\lambda)t} dt = \frac{e^{xh} - e^{\lambda h}}{x - \lambda}, \quad x \in \mathbb{C}.$$

Because $\alpha_\lambda(\lambda) = e^{\lambda h} h \neq 0$,

$$\alpha_\lambda(x) = 0 \iff x = \lambda + j\frac{2\pi}{h}k, \quad k \in \mathbb{Z} \setminus \{0\}$$

which are all points where the numerator is zero, but the denominator isn't. Hence,

$$\det \alpha_\lambda(A) \neq 0 \iff \lambda + j\frac{2\pi}{h}k \notin \text{spec}(A), \quad k \in \mathbb{Z} \setminus \{0\}.$$

Facts:

- $e^{Ah} - e^{\lambda h}I = \alpha_\lambda(A)(A - \lambda I) = (A - \lambda I)\alpha_\lambda(A) \implies \bar{A} = I + \alpha_0(A)A$
- $\int_0^h e^{At} dt = \alpha_0(A) \implies \bar{B} = \alpha_0(A)B$

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Reachability under discretization (contd)

Theorem

If h is non-pathological with respect to $\text{spec}(A)$, then $(\bar{A}, \bar{B})$ is reachable iff (A, B) is controllable.

Proof: If h is non-pathological, then $\lambda \in \text{spec}(A) \implies \lambda + j\frac{2\pi}{h}k \notin \text{spec}(A)$ for all $k \in \mathbb{Z} \setminus \{0\}$. Hence, $\alpha_\lambda(A)$ is nonsingular for all $\lambda \in \mathbb{C}$ and

$$\bar{A} - e^{\lambda h}I = (A - \lambda I)\alpha_\lambda(A) = \alpha_0(A)(A - \lambda I)[\alpha_0(A)]^{-1}\alpha_\lambda(A).$$

This yields

$$\begin{bmatrix} \bar{A} - e^{\lambda h}I & \bar{B} \end{bmatrix} = \alpha_0(A) \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} [\alpha_0(A)]^{-1}\alpha_\lambda(A) & 0 \\ 0 & I \end{bmatrix}$$

and, consequently, proves the statement (as the rank is not affected by pre- and post-multiplications by nonsingular matrices). $\square$

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Stabilizability under discretization

Corollary

If h is non-pathological with respect to $\text{spec}(A) \cap \bar{\mathbb{C}}_0$, then $(\bar{A}, \bar{B})$ is stabilizable iff (A, B) is stabilizable.

Proof (outline): $\det \alpha_\lambda(A) \neq 0$ for all $\lambda \in \bar{\mathbb{C}}_0$ and λ is an unstable mode of A iff $e^{\lambda h}$ is an unstable mode of $\bar{A}$ (cf. Lect. 4, Slide 26). $\square$

Remark: Actually, stronger results can be proved.

1. If λ is a controllable mode of (A, B) and $\lambda + j\frac{2\pi}{h}k \notin \text{spec}(A)$, $\forall k \in \mathbb{Z} \setminus \{0\}$, then $e^{\lambda h}$ is a reachable mode of $(\bar{A}, \bar{B})$.
2. Let $B \in \mathbb{R}^{n \times 1}$. If λ is a controllable mode of (A, B) , then $e^{\lambda h}$ is a reachable mode of $(\bar{A}, \bar{B})$ iff $\lambda + j\frac{2\pi}{h}k \notin \text{spec}(A)$, $\forall k \in \mathbb{Z} \setminus \{0\}$ (might not be true in the MIMO case).

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Observability under discretization

Theorem

If λ is an observable mode of (C, A) and $\lambda + j\frac{2\pi}{h}k \notin \text{spec}(A)$, $\forall k \in \mathbb{Z} \setminus \{0\}$, then $e^{\lambda h}$ is an observable mode of $(C, \bar{A})$.

Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue of A . The result follows by

$$\begin{bmatrix} \bar{A} - e^{\lambda h}I \\ C \end{bmatrix} = \begin{bmatrix} \alpha_\lambda(A)(A - \lambda I) \\ C \end{bmatrix} = \begin{bmatrix} \alpha_\lambda(A) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

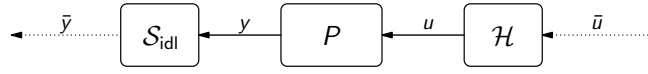
and the fact that $\alpha_\lambda(A)$ is nonsingular. $\square$

Corollary

If h is non-pathological with respect to $\text{spec}(A) \cap \bar{\mathbb{C}}_0$, then $(C, \bar{A})$ is detectable iff (C, A) is detectable.

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Discretization problem, yet again



Now assume that $\mathcal{H}$ is a static hold with the hold function $\phi(t)$ (by Lect. 3, Slide 31, it has support in $[0, h]$). The impulse response of $\bar{P}_h = S_{idl} P \mathcal{H}$ in this case for $i \in \mathbb{N}$ is

$$\begin{aligned}\bar{p}_h[i] &= \int_{\mathbb{R}} p(ih - t) \phi(t) dt = \int_0^h p(ih - t) \phi(t) dt = \int_0^h C e^{A(ih-t)} B \phi(t) dt \\ &= \underbrace{C(e^{Ah})^{i-1}}_{\bar{A}} \underbrace{\int_0^h e^{A(h-t)} B \phi(t) dt}_{\bar{B}}.\end{aligned}$$

This is the impulse response of a system, whose realization is $(\bar{A}, \bar{B}, C, 0)$.

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Alternative derivation

If P has the state equation $\dot{x}(t) = Ax(t) + Bu(t)$, then for all $\tau \in [0, h]$

$$\begin{aligned}x(ih + \tau) &= e^{A\tau} x(ih) + \int_0^\tau e^{A(\tau-\sigma)} B u(ih + \sigma) d\sigma \\ &= e^{A\tau} x(ih) + \int_0^\tau e^{A(\tau-\sigma)} B \phi(\sigma) d\sigma \bar{u}[i]\end{aligned}$$

(mind that $u = \mathcal{H}\bar{u} \implies u(ih + \sigma) = \phi(\sigma)\bar{u}[i]$). Because $y(ih) = Cx(ih)$, we only need the sampled state $\bar{x} = S_{idl}x$ and end up with the realization

$$\bar{P}_h : \begin{cases} \bar{x}[i+1] = e^{Ah} \bar{x}[i] + \int_0^h e^{A(h-t)} B \phi(t) dt \bar{u}[i] = \bar{A} \bar{x}[i] + \bar{B} \bar{u}[i] \\ \bar{y}[i] = C \bar{x}[i] \end{cases}$$

Note that

- the state of $\bar{P}_h$ is the sampled state of P , i.e. $\bar{x}[i] = x(ih)$.

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Outline

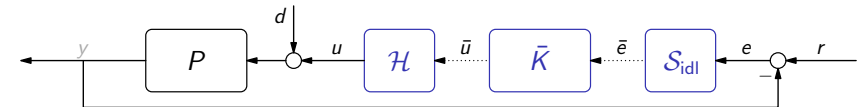
Discretization in state space

Stability

State-space design methods in discrete time: state feedback

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Lyapunov stability of sampled-data system



Assume that

- P is LTI, with $P(s) = C(sI - A)^{-1}B$ and state $x(t) \in \mathbb{R}^n$,
- $\bar{K}$ is LSI, with $\bar{K}(z) = \bar{D}_k + \bar{C}_k(zI - \bar{A}_k)^{-1}\bar{B}_k$ and state $\bar{x}_k[i] \in \mathbb{R}^{n_k}$,
- $\mathcal{H}$ is static, whose hold function $\phi(t)$ is bounded with support in $[0, h]$.

Definition

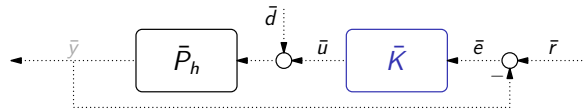
The closed-loop system is said to be asymptotically stable if the trajectories $x(t)$ and $\bar{x}_k[i]$ are bounded and

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \bar{x}_k[i] = 0$$

for all $x(0) \in \mathbb{R}^n$ and $\bar{x}_k[0] \in \mathbb{R}^{n_k}$ under zero exogenous signals r and d .

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Closed-loop discretized realization



Already familiar realization,

$$\begin{cases} \begin{bmatrix} \bar{x}[i+1] \\ \bar{x}_k[i+1] \end{bmatrix} = \begin{bmatrix} \bar{A} - \bar{B}\bar{D}_k C & \bar{B}\bar{C}_k \\ -\bar{B}_k C & \bar{A}_k \end{bmatrix} \begin{bmatrix} \bar{x}[i] \\ \bar{x}_k[i] \end{bmatrix} + \begin{bmatrix} \bar{B}\bar{D}_k & \bar{B} \\ \bar{B}_k & 0 \end{bmatrix} \begin{bmatrix} \bar{r}[i] \\ \bar{d}[i] \end{bmatrix} \\ \begin{bmatrix} \bar{e}[i] \\ \bar{u}[i] \end{bmatrix} = \begin{bmatrix} -C & 0 \\ -\bar{D}_k C & \bar{C}_k \end{bmatrix} \begin{bmatrix} \bar{x}[i] \\ \bar{x}_k[i] \end{bmatrix} + \begin{bmatrix} I & 0 \\ \bar{D}_k & 0 \end{bmatrix} \begin{bmatrix} \bar{r}[i] \\ \bar{d}[i] \end{bmatrix} \end{cases}$$

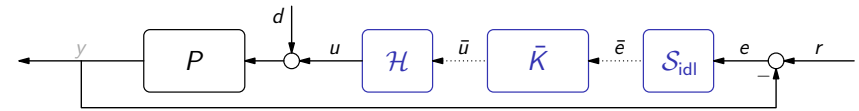
Define

$$\bar{A}_{cl} := \begin{bmatrix} \bar{A} - \bar{B}\bar{D}_k C & \bar{B}\bar{C}_k \\ -\bar{B}_k C & \bar{A}_k \end{bmatrix},$$

whose Schur property is equivalent to the asymptotic stability of the system (requires the stabilizability and detectability of $(C, \bar{A}, \bar{B})$ and $(\bar{C}_k, \bar{A}_k, \bar{B}_k)$).

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Lyapunov stability of sampled-data system (contd)



Theorem

The system is asymptotically stable iff $\bar{A}_{cl}$ Schur.

Proof:

only if: If $\bar{A}_{cl}$ is not Schur, then $(\bar{x}[i], \bar{x}_k[i])$ is not bounded. Hence, neither is $(x(t), \bar{x}_k[i])$ and $\rho(\bar{A}_{cl}) < 1$ is necessary.

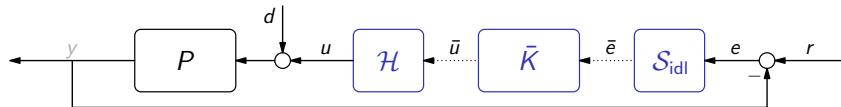
if: If $\bar{A}_{cl}$ is Schur, then the discretized system is stable, with bounded $\bar{x}[i]$ and $\bar{x}_k[i]$, vanishing as $i \rightarrow \infty$. Then $\bar{u}[i] = \bar{C}_k \bar{x}_k[i] - \bar{D}_k C \bar{x}[i]$ is also bounded and vanishes as $i \rightarrow \infty$. Taking into account that

$$x(ik + \tau) = e^{A\tau} \bar{x}[i] + \int_0^\tau e^{A(\tau-\sigma)} B \phi(\sigma) d\sigma \bar{u}[i],$$

$\|x(ik + \tau)\| \leq e^{\|A\|\tau} \|\bar{x}[i]\| + \tau e^{\|A\|\tau} \|B\| \max_{\sigma \in [0, \tau]} \|\phi(\sigma)\| \|\bar{u}[i]\|$, so $x(ik + \tau)$ is bounded and vanishes as $i \rightarrow \infty$ for all bounded τ . $\square$

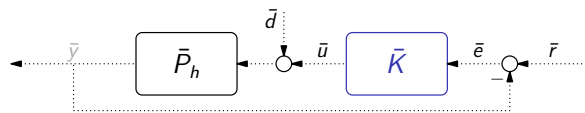
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Internal i/o stability of sampled-data system



Theorem

If the sampling period h is non-pathological with respect to unstable poles of $P(s)$, then $\bar{K}$ stabilizes the system iff it stabilizes the discretized system



Implication: if sampling is non-pathological with respect to unstable poles of the analog plant, then

- the closed-loop stability of the discretized system is equivalent to that of the original system.

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Outline

Discretization in state space

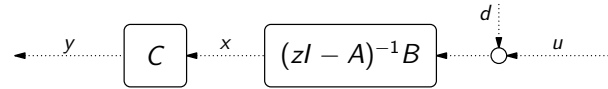
Stability

State-space design methods in discrete time: state feedback

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Discrete system with measured state

Let a discrete plant P be given by its state-space realization $(A, B, C, 0)$. It can be viewed as the cascade of $P_x : u + d \mapsto x$ and (static) $C : x \mapsto y$:



For the time being, *assume* that

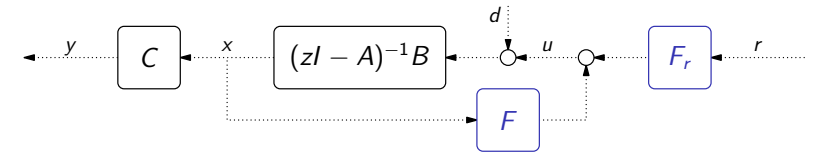
- the state $x[t]$ can be directly **measured**,

which might not be practical, but is quite useful conceptually. Normally, we may still be interested to control y or some its functions (like $\dot{y}$). But

- measurement of x is beneficial to control y .

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State feedback



The static control law

$$u[t] = Fx[t] + F_r r[t]$$

is known as (static) **state feedback**. The resulting closed-loop system is

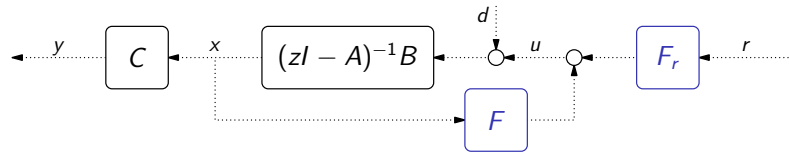
$$\begin{cases} x[t+1] = (A + BF)x[t] + BF_r r[t] + Bd[t] \\ y[t] = Cx[t] \end{cases}$$

(obtain by substituting the control law to $x[t+1] = Ax[t] + Bu[t] + Bd[t]$). It should be clear that

- closed-loop system can be stabilized iff (A, B) is stabilizable.

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State feedback: closed-loop transfer functions



In the z -domain, denoting $A_F := A + BF$,

$$X(z) = (zI - A_F)^{-1}(BF_r R(z) + BD(z)) \quad \text{and} \quad Y(z) = CX(z).$$

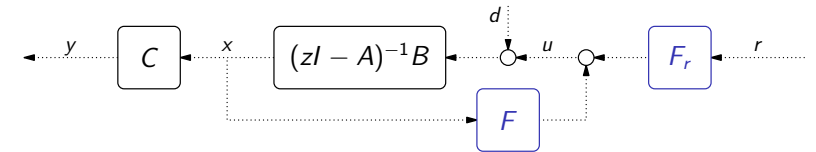
Hence, closed-loop system $T_{cl} : (r, d) \mapsto (y, u)$ has the transfer function

$$T_{cl}(z) = \begin{bmatrix} 0 & 0 \\ F_r & 0 \end{bmatrix} + \begin{bmatrix} C \\ F \end{bmatrix} (zI - A_F)^{-1} \begin{bmatrix} BF_r & B \end{bmatrix} = \left[\begin{array}{c|cc} A_F & BF_r & B \\ \hline C & 0 & 0 \\ F & F_r & 0 \end{array} \right]$$

(they are not complete analogs of GoF).

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State feedback: zeros of $T_{yr}(z)$



Because $T_{yr}(z) = P(z)T_{ur}(z)$,

- zeros of $T_{yr}(z)$ are those of $P(z)$ or $T_{ur}(z)$.

But

$$T_{ur}(z) = \left[\begin{array}{c|c} A_F & B \\ \hline F & 1 \end{array} \right] F_r = \left[\begin{array}{c|c} A & B \\ \hline -F & 1 \end{array} \right]^{-1} F_r.$$

Hence, zeros of $T_{ur}(z)$ are among poles of $P(z)$, which are different from zeros of $P(z)$. Therefore,

- zeros of $T_{yr}(z)$ coincide with zeros of $P(z)$.

In other words, **state feedback does not shift zeros**, which is important.

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State feedback: static gain of $T_{yr}(z)$

Consider the transfer function of the closed-loop system $T_{yr} : r \mapsto y$,

$$T_{yr}(z) = C(zI - A_F)^{-1}BF_r = \left[\frac{A_F}{C} \middle| \frac{BF_r}{0} \right].$$

Its static gain $T_{yr}(1) = C(I - A_F)^{-1}BF_r$, well defined whenever $\rho(A_F) < 1$. If $r[t] = \mathbb{1}[t]$, then the steady-state response of the tracking error $e = r - y$

$$\lim_{t \rightarrow \infty} e[t] = \lim_{t \rightarrow \infty} r[t] - \lim_{t \rightarrow \infty} y[t] = 1 - T_{yr}(1) = 1 - C(I - A_F)^{-1}BF_r.$$

To render it zero, we need

$$F_r = \frac{1}{C(I - A_F)^{-1}B}.$$

This choice is possible iff $P(1) \neq 0$ (as zeros of $P(z)$ are those of $T_{yr}(z)$).

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State feedback: static gain of $T_{yd}(z)$

Consider the transfer function of the closed-loop system $T_{yd} : d \mapsto y$,

$$T_{yd}(z) = C(zI - A_F)^{-1}B = \left[\frac{A_F}{C} \middle| \frac{B}{0} \right].$$

Its static gain $T_{yd}(1) = C(I - A_F)^{-1}B$, well defined whenever $\rho(A_F) < 1$. If $d[t] = \mathbb{1}[t]$, then the steady-state response,

$$\lim_{t \rightarrow \infty} y[t] = T_{yd}(1) = C(I - A_F)^{-1}B.$$

It is zero iff $T_{yd}(z)$ has a zero at $z = 1$. But the latter condition holds iff

- $P(z)$ has a zero at $z = 1$,

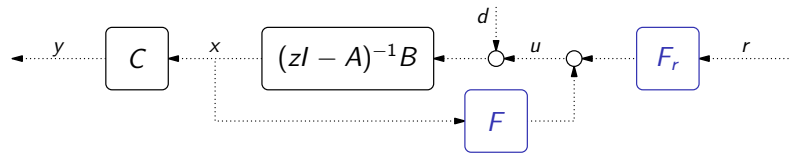
which does not depend on the design. In other words,

- state feedback cannot completely reject constant load disturbance.

It can reduce the effect of d by increasing the gain F though.

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State feedback: design of F



A common approach is to

- assign closed-loop poles, which belong to $\text{spec}(A + BF)$.

This method is

- easy to implement,
- transparent, with the classical modal behavior insight,
- clean, as it does not alter zeros.

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Pole assignment: companion form

First, assume that (A, B) is in the companion form:

$$A = A_{cf} := \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \quad \text{and} \quad B = B_{cf} := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

If $F = [f_0 \ f_1 \ \dots \ f_{n-1}]$, then

$$A_F = A + BF = A_{cf} + B_{cf}F = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ f_0 - a_0 & f_1 - a_1 & \dots & f_{n-1} - a_{n-1} \end{bmatrix}.$$

This is still in the companion form, hence

$$\chi_{cl}(z) = z^n + (a_{n-1} - f_{n-1})z^{n-1} + \dots + (a_1 - f_1)z + (a_0 - f_0).$$

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Pole assignment: companion form (contd)

The choice

$$F = F_{cf} := \begin{bmatrix} a_0 - \chi_0 & a_1 - \chi_1 & \dots & a_{n-1} - \chi_{n-1} \end{bmatrix}$$

renders

$$\chi_{cl}(z) = z^n + \chi_{n-1}z^{n-1} + \dots + \chi_1z + \chi_0$$

for any choice of χ_i , $i = 0, \dots, n-1$. Design has then two simple steps:

1. choose a desired n -degree closed-loop characteristic polynomial $\chi_{cl}(z)$,
2. choose F as **above** based on coefficients of $\chi_{ol}(z) - \chi_{cl}(z)$.

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Pole assignment: arbitrary realization

Conceptually, all we need to do is to

- transform (A, B) into the companion form by similarity transformation.

Indeed, for any controllable (A, B) there is (we'll show this by construction) nonsingular T such that

$$A_{cf} = TAT^{-1} \quad \text{and} \quad B_{cf} = TB.$$

Hence, $A = T^{-1}A_{cf}T$, $B = T^{-1}B_{cf}$, and

$$F = F_{cf}T$$

does the job, as

$$A + BF = T^{-1}A_{cf}T + T^{-1}B_{cf}F_{cf}T = T^{-1}(A_{cf} + B_{cf}F_{cf})T.$$

Elegant algorithm to construct required $F_{cf}T$ w/o an explicit calculation of T is offered by **Ackermann's formula**.

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Ackermann's formula: preliminaries

1. By Cayley-Hamilton Theorem, $\chi_{ol}(A_{cf}) = 0$, so that

$$\begin{aligned} \chi_{cl}(A_{cf}) &= A_{cf}^n + \chi_{n-1}A_{cf}^{n-1} + \dots + \chi_1A_{cf} + \chi_0I \\ &= -a_{n-1}A_{cf}^{n-1} - \dots - a_1A_{cf} - a_0I \\ &\quad + \chi_{n-1}A_{cf}^{n-1} + \dots + \chi_1A_{cf} + \chi_0I \\ &= (\chi_{n-1} - a_{n-1})A_{cf}^{n-1} + \dots + (\chi_1 - a_1)A_{cf} + (\chi_0 - a_0)I. \end{aligned}$$

2. If e_i is the i th standard basis in $\mathbb{R}^n$, then $\forall i = 1, \dots, n-1$,

$$e'_i A_{cf} = e'_{i+1} \quad \text{or, equivalently,} \quad e'_1 A_{cf}^i = e'_{i+1}.$$

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Ackermann's formula: preliminaries (contd)

3. If $A_{cf} = TAT^{-1}$ and $B_{cf} = TB^{-1}$, then

$$M_{c,cf} = TM_c \iff T = M_{c,cf}M_c^{-1}.$$

4. Combining 1 and 2:

$$\begin{aligned} F_{cf} &= \begin{bmatrix} a_0 - \chi_0 & a_1 - \chi_1 & \dots & a_{n-1} - \chi_{n-1} \end{bmatrix} \\ &= (a_0 - \chi_0)e'_1 + (a_1 - \chi_1)e'_2 + \dots + (a_{n-1} - \chi_{n-1})e'_n \\ &= (a_0 - \chi_0)e'_1 + (a_1 - \chi_1)e'_1 A_{cl} + \dots + (a_{n-1} - \chi_{n-1})e'_1 A_{cl}^{n-1} \\ &= e'_1((a_0 - \chi_0)I + (a_1 - \chi_1)A_{cl} + \dots + (a_{n-1} - \chi_{n-1})A_{cl}^{n-1}) \\ &= -e'_1 \chi_{cl}(A_{cf}) \end{aligned}$$

5. By 2 (and the fact that $B_{cf} = e_n$):

$$\begin{aligned} e'_1 M_{c,cf} &= e'_1 \begin{bmatrix} B_{cf} & A_{cf}B_{cf} & \dots & A_{cf}^{n-1}B_{cf} \end{bmatrix} \\ &= \begin{bmatrix} e'_1 e_n & e'_2 e_n & \dots & e'_n e_n \end{bmatrix} = e'_n \end{aligned}$$

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Ackermann's formula: derivation

We have:

$$\begin{aligned}
 F_{cf} &= -e'_1 \chi_{cl}(A_{cf}) && \text{by item 4} \\
 &= -e'_1 \chi_{cl}(TAT^{-1}) = -e'_1 T \chi_{cl}(A) T^{-1} \\
 &= -e'_1 M_{c,cf} M_c^{-1} \chi_{cl}(A) T^{-1} && \text{by item 3} \\
 &= -e'_n M_c^{-1} \chi_{cl}(A) T^{-1} && \text{by item 5}
 \end{aligned}$$

Now it is time to return to the original coordinates:

$$\begin{aligned}
 F &= F_{cf} T = -e'_n M_c^{-1} \chi_{cl}(A) T^{-1} T \\
 &= -e'_n M_c^{-1} \chi_{cl}(A),
 \end{aligned}$$

voilà!

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Ackermann's formula

The feedback gain assigning the closed-loop poles to the roots of $\chi_{cl}(z)$ is

$$F = - \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} M_c^{-1} \chi_{cl}(A),$$

where

$$M_c = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

and

$$\chi_{cl}(A) = A^n + \chi_{n-1}A^{n-1} + \dots + \chi_1A + \chi_0I.$$

This gain F is called **Ackermann's formula** and indeed depends only on the original (controllable) realization.

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Dead-beat state feedback

The particular choice $\chi_{cl}(z) = z^n$ results in

$$F = - \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} M_c^{-1} A^n,$$

known as the **dead-beat** gain. If A_F has this characteristic polynomial, then

$$A_F^i = 0, \quad \forall i \geq n.$$

Hence, the closed-loop impulse response

$$t_{cl}[t] = \begin{bmatrix} C \\ F \end{bmatrix} A_F^{t-1} \begin{bmatrix} BF_r & B \end{bmatrix} = 0, \quad \forall t \geq n+1,$$

meaning that transients under dead-beat feedback end in finite time. This property is attractive, but

- frequently entails large control signals and poor robustness.

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Example 1: DC motor with rigid mechanical load

Consider an analog plant having the transfer function

$$P(s) = \frac{1}{s(s+1)},$$

which can be thought of as a voltage controlled DC motor with normalized time constant. Bring in its companion form

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

in which case the state comprises $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$.

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Example 1: discretization

We have:

$$[\bar{A} \mid \bar{B}] = \begin{bmatrix} 1 & 1 - e^{-h} & h - (1 - e^{-h}) \\ 0 & e^{-h} & 1 - e^{-h} \end{bmatrix}$$

(not in companion form anymore, but $\bar{x}_1[i] = y(ih)$ and $\bar{x}_2[i] = \dot{y}(ih)$) and

$$\bar{P}_h(z) = \frac{(h + e^{-h} - 1)z + 1 - e^{-h}(1 + h)}{(z - 1)(z - e^{-h})}.$$

Its sampling zero is at (cf. Lect 4, Slide 27)

$$z_z = \frac{1 - e^{-h}(1 + h)}{1 - e^{-h} - h} = \begin{matrix} 0 & 1 & 3 & h \\ -0.391 & & & \\ -0.718 & & & \\ -1 & & & \end{matrix}$$

Note that

- as both poles of the system are real, no sampling is pathological in this case, so the realization $(\bar{A}, \bar{B}, C, 0)$ is both reachable and observable.

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Example 1: design

The state feedback in this case,

$$\bar{u}[i] = F\bar{x}[i] + F_r\bar{r}[i] = F_1y(ih) + F_2\dot{y}(ih) + F_rr(ih),$$

can be viewed as a PD control law. Choose

$$\chi_{cl}(z) = z(z - \alpha)$$

for some $|\alpha| < 1$. By Ackermann, $F = -[0 \ 1] \bar{M}_c^{-1} \bar{A}(\bar{A} - \alpha I)$, yielding

$$F_1 = -\underbrace{\frac{1 - \alpha}{(1 - e^{-h})h}}_{k_p > 0} \quad \text{and} \quad F_2 = F_1 \underbrace{\left(\frac{1 - e^{-h}(1 + h)}{1 - e^{-h}} + \frac{e^{-h}h}{1 - \alpha} \right)}_{\tau_d > 0}.$$

The absolute values of both these gains are decreasing functions of both α and h .

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Example 1: closed-loop transfer functions

Command response:

$$T_{yr}(z) = \frac{(e^{-h} + h - 1)(z - z_z)}{z(z - \alpha)} F_r \quad \text{and} \quad T_{ur}(z) = \frac{(z - 1)(z - e^{-h})}{z(z - \alpha)} F_r$$

and

$$T_{yr}(1) = -\frac{F_r}{F_1} \implies F_r = -F_1 = \frac{1 - \alpha}{(1 - e^{-h})h}.$$

Thus, we end up with the sampled-data version of the classical PD control:

$$\bar{u}[i] = k_p(e(ih) - \tau_d \dot{y}(ih)), \quad \text{where } e(t) := r(t) - y(t).$$

Disturbance response:

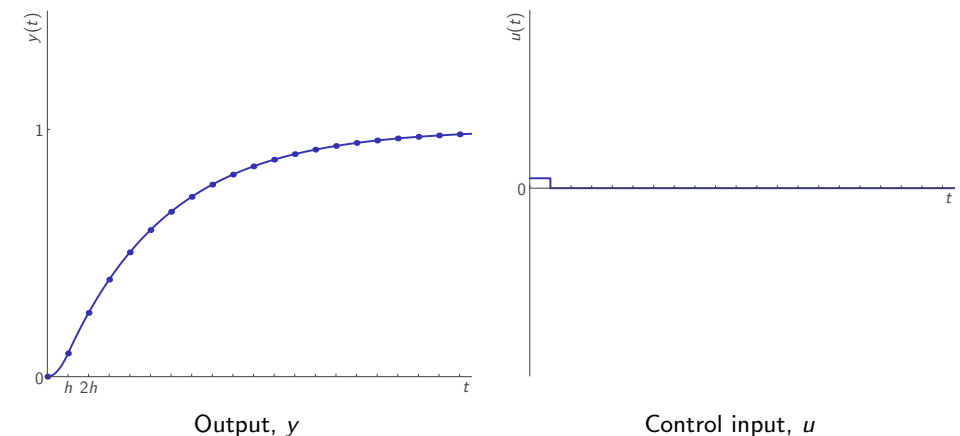
$$T_{yd}(z) = \frac{(e^{-h} + h - 1)(z - z_z)}{z(z - \alpha)} \quad \text{and} \quad T_{ud}(z) = -\frac{(1 - \alpha + e^{-h})z - e^{-h}}{z(z - \alpha)}.$$

Because $|T_{yd}(1)| = 1/|F_1| = 1/k_p$, the steady-state response to a step $d(t)$ decreases as the P gain increases.

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Example 1: command response simulations

Let $\alpha = e^{-h} \approx 0.819$ (keeps the open-loop stable pole untouched). Then:

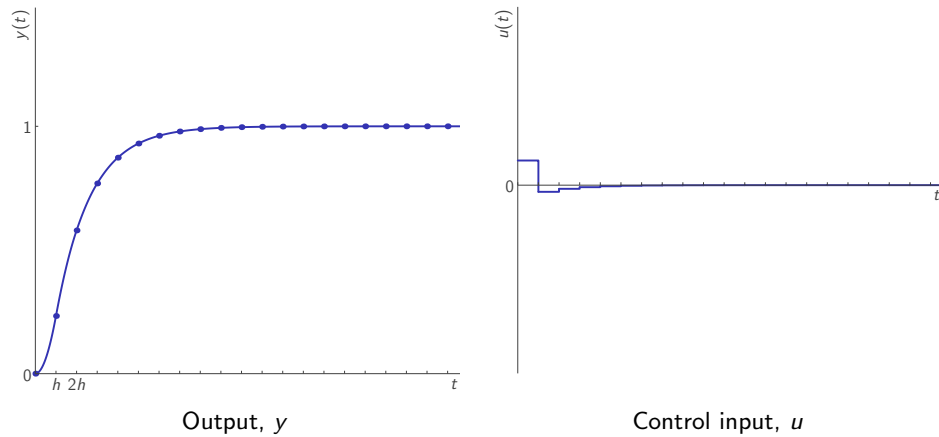


with $k_p = 5$ & $\tau_d = 1$.

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Example 1: command response simulations (contd)

Let $\alpha = e^{-3h} \approx 0.549$ (accelerates closed-loop response). Then:

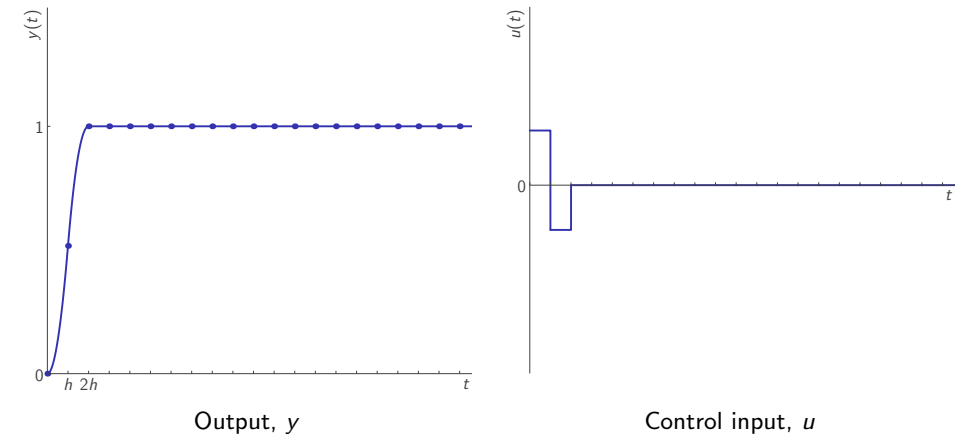


with $k_p = 12.4453$ & $\tau_d = 0.459591$.

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Example 1: command response simulations (contd)

Let $\alpha = 0$ (dead-beat strategy). Then:

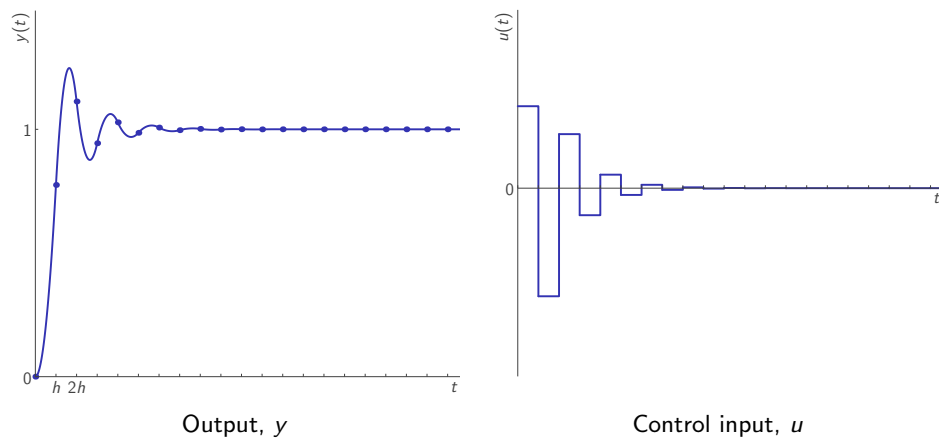


with $k_p = 27.5833$ & $\tau_d = 0.260415$.

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Example 1: command response simulations (contd)

Let $\alpha = -0.5$ (increases the gain). Then:

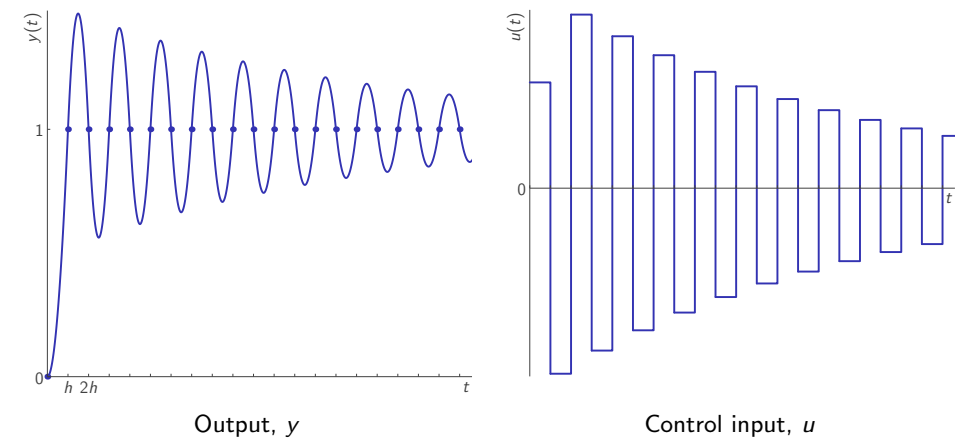


with $k_p = 41.3749$ & $\tau_d = 0.205833$.

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Example 1: command response simulations (contd)

Let $\alpha = z_s \approx -0.936$ (canceling lightly-damped zero, visible in T_{ur}). Then:



with $k_p = 53.3881$ & $\tau_d = 0.18127$. We see that the analog output exhibits
— oscillations, not visible at the sampling instances (**hidden oscillations**).

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