

Sampled-Data Control Systems (036709)

lecture no. 4

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT



Outline

Computer-controlled systems

The world according to $\bar{C}$

Discretization of LTI systems

Classical methods for discrete systems (mostly stability)

Outline

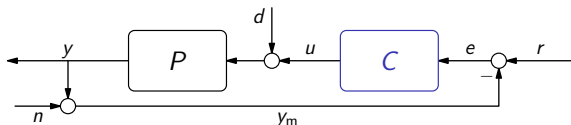
Computer-controlled systems

The world according to $\bar{C}$

Discretization of LTI systems

Classical methods for discrete systems (mostly stability)

Analog unity-feedback control

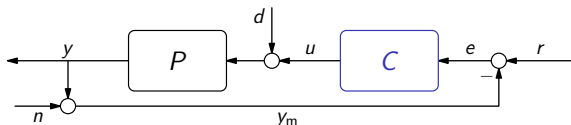


- tracking: $y \approx r$ or $|S(j\omega)| \ll 1$ in the spectrum of r
- disturbance attenuation: $|T_d(j\omega)| \ll 1$ in the spectrum of d
- noise sensitivity: $|T(j\omega)| \ll 1$ in the spectrum of n
- control effort: $|T_c(j\omega)| \not\gg 1$ (provided data is normalized)
- robustness: $|T(j\omega)| < 1/\ell(\omega)$ or $|T_c(j\omega)| < 1/(\ell(\omega)|P(j\omega)|)$

Normally, r and d are “slow,” whereas n and modeling uncertainty are “fast.” Therefore, we typically aim at

- choosing a “right” crossover ω_c , separating low and high frequencies
- increasing $|C(j\omega)|$ at low frequencies
- decreasing $|C(j\omega)|$ at high frequencies

Analog unity-feedback control

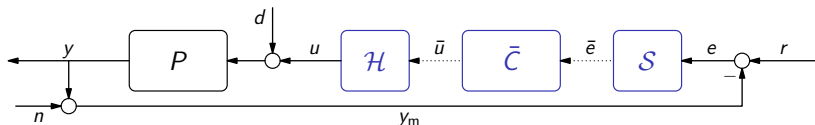


- tracking: $y \approx r$ or $|S(j\omega)| \ll 1$ in the spectrum of r
- disturbance attenuation: $|T_d(j\omega)| \ll 1$ in the spectrum of d
- noise sensitivity: $|T(j\omega)| \ll 1$ in the spectrum of n
- control effort: $|T_c(j\omega)| \not\gg 1$ (provided data is normalized)
- robustness: $|T(j\omega)| < 1/\ell(\omega)$ or $|T_c(j\omega)| < 1/(\ell(\omega)|P(j\omega)|)$

Normally, r and relevant d are “slow,” whereas n and modeling uncertainty are “fast.” Therefore, we typically aim at

- choosing a “right” crossover ω_c , separating low and high frequencies
- increasing $|C(j\omega)|$ at low frequencies
- decreasing $|C(j\omega)|$ at high frequencies

Sampled-data unity-feedback control



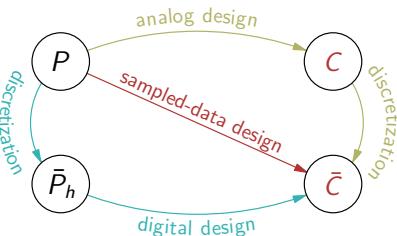
- tracking: $y \approx r$
- disturbance attenuation
- noise sensitivity
- control effort
- robustness

with the same separation of properties of r & d and n & uncertainty. So,

- how to design $\bar{C}$ (and possibly $\mathcal{S}$ and $\mathcal{H}$) to attain these goals?

Approaches to computer control design

There are essentially 3 paths from P to $\bar{C}$:



1. Discrete-time design

(discretize the problem first, then do your favorite discrete design)

2. Digital redesign of analog controllers

(do your favorite analog design first, then discretize the resulting controller)

3. Direct digital (sampled-data) design

(design $\bar{C}$, and possibly $\mathcal{S}$ and $\mathcal{H}$, directly for analog plant and specs)

Outline

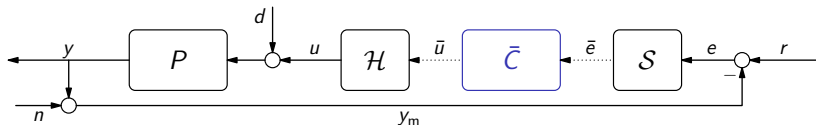
Computer-controlled systems

The world according to $\bar{C}$

Discretization of LTI systems

Classical methods for discrete systems (mostly stability)

What does $\bar{C}$ see?



Input:

$$\bar{e} = S r - S n - S P d - S P \mathcal{H} \bar{u},$$

where $\bar{u}$ is its output (cf. the analog $e = r - n - P d - P u$). **Observations:**

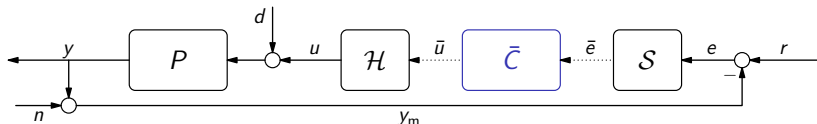
- the discrete $\bar{P}_h := S P \mathcal{H} : \bar{u} \mapsto \bar{e}$ is the plant from the viewpoint of $\bar{C}$,
- sampled reference signal $\bar{r} := S r$ replaces r ,
- sampled noise signal $\bar{n} := S n$ replaces n ,
- $S P d$ doesn't fit, unless we assume that $d \approx \mathcal{H} \bar{d}$ for some $\bar{d}$

In other words,

$$\bar{e} \approx \bar{r} - \bar{n} - S P \mathcal{H} \bar{d} - S P \mathcal{H} \bar{u} = \bar{r} - \bar{n} - \bar{P}_h \bar{d} - \bar{P}_h \bar{u}$$

(if d can be viewed as having the same structure as u).

What does $\bar{C}$ see?



Input:

$$\bar{e} = S r - S n - S P d - S P \mathcal{H} \bar{u},$$

where $\bar{u}$ is its output (cf. the analog $e = r - n - P d - P u$). Observations:

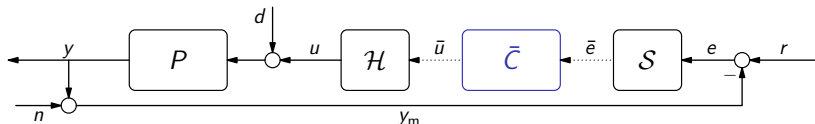
- the discrete $\bar{P}_h := S P \mathcal{H} : \bar{u} \mapsto \bar{e}$ is the plant from the viewpoint of $\bar{C}$,
- sampled reference signal $\bar{r} := S r$ replaces r ,
- sampled noise signal $\bar{n} := S n$ replaces n ,
- $S P d$ doesn't fit, unless we **assume** that $d \approx \mathcal{H} \bar{d}$ for some $\bar{d}$

In other words,

$$\bar{e} \approx \bar{r} - \bar{n} - S P \mathcal{H} \bar{d} - S P \mathcal{H} \bar{u} = \bar{r} - \bar{n} - \bar{P}_h \bar{d} - \bar{P}_h \bar{u}$$

(if d can be viewed as having the same structure as u)

What does $\bar{C}$ see?



Input:

$$\bar{e} = Sr - Sn - SPd - SP\mathcal{H}\bar{u},$$

where $\bar{u}$ is its output (cf. the analog $e = r - n - Pd - Pu$). Observations:

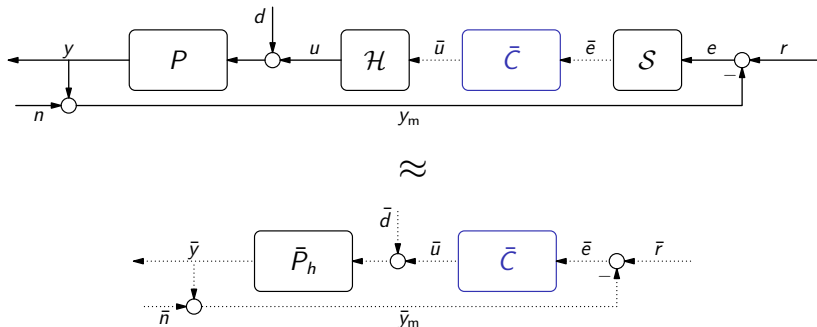
- the discrete $\bar{P}_h := SP\mathcal{H} : \bar{u} \mapsto \bar{e}$ is the plant from the viewpoint of $\bar{C}$,
- sampled reference signal $\bar{r} := Sr$ replaces r ,
- sampled noise signal $\bar{n} := Sn$ replaces n ,
- SPd doesn't fit, unless we **assume** that $d \approx \mathcal{H}\bar{d}$ for some $\bar{d}$

In other words,

$$\bar{e} \approx \bar{r} - \bar{n} - SP\mathcal{H}\bar{d} - SP\mathcal{H}\bar{u} = \bar{r} - \bar{n} - \bar{P}_h\bar{d} - \bar{P}_h\bar{u}$$

(if d can be viewed as having the same structure as u).

What does $\bar{C}$ see? (contd)



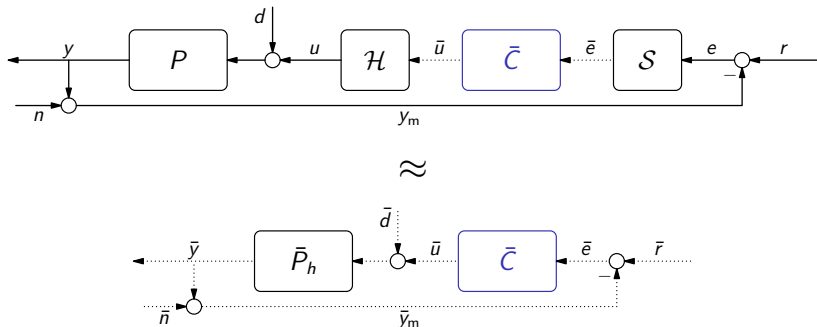
Thus, $\bar{C}$ lives in a pure discrete (stroboscopic) world. But

- does this stroboscopic world reflect the reality?

Specifically,

- can we still rely on spectra separation between $\bar{r}$ and $\bar{n}$?
- does low-pass P yield low-pass $\bar{P}_h$ (and so relevant $\bar{d}$ is "slow")?
- is $\bar{C}$, designed for its own reality, any good for the original problem?

What does $\bar{\mathcal{C}}$ see? (contd)



Thus, $\bar{\mathcal{C}}$ lives in a pure discrete (stroboscopic) world. But

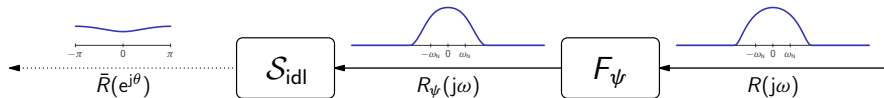
- does this stroboscopic world reflect the reality?

Specifically,

- can we still rely on spectra separation between $\bar{r}$ and $\bar{n}$?
- does low-pass P yield low-pass $\bar{P}_h$ (and so relevant $\bar{d}$ is “slow”)?
- is $\bar{\mathcal{C}}$, designed for its own reality, any good for the original problem?

Does “slow” r imply “slow” $\bar{r}$?

Not necessarily, if sampling is slow (i.e. ω_N is not large enough vis-à-vis the bandwidth of r):



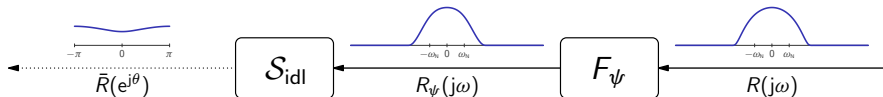
Yes, if sampling is fast (i.e. ω_N is large vis-à-vis the bandwidth of r):

Thus

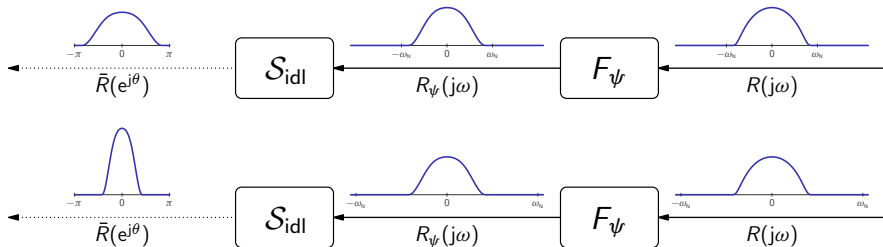
– “fast enough” sampling resolves this issue.

Does “slow” r imply “slow” $\bar{r}$?

Not necessarily, if sampling is slow (i.e. ω_N is not large enough vis-à-vis the bandwidth of r):



Yes, if sampling is fast (i.e. ω_N is large vis-à-vis the bandwidth of r):

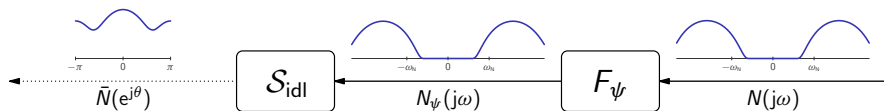


Thus

- “fast enough” sampling resolves this issue.

Does “fast” n imply “fast” $\bar{n}$?

No, if fast ingredients of n , aliasing as slow ingredients of $\bar{n}$, are substantial:



Example

Let $y(t) = \text{const}$ be measured via a noisy sensor with $n(t) = \sin(2\omega_N t + \alpha)$:

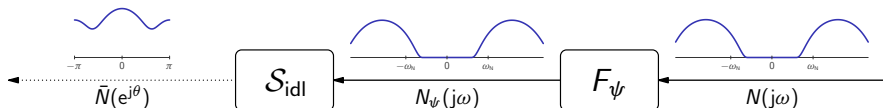
Their spectra are well separated, but the sampled measured signal

$$\bar{y}[i] = y(ih) + \sin \alpha$$

is offset, with no way to tell the contribution of y from that of n .

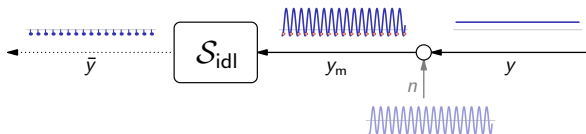
Does “fast” n imply “fast” $\bar{n}$?

No, if fast ingredients of n , aliasing as slow ingredients of $\bar{n}$, are substantial:



Example

Let $y(t) = \text{const}$ be measured via a noisy sensor with $n(t) = \sin(2\omega_N t + \alpha)$:



Their spectra are well separated, but the sampled measured signal

$$\bar{y}[i] = y(ih) + \sin \alpha$$

is offset, with no way to tell the contribution of y from that of n .



Remedy: antialiasing filtering

In general, the only way to prevent this is to filter out ultra- ω_N parts of the spectrum of n **before** the sampling operation. In other words, we need

- A/D converters with low-pass F_ψ ,

whose bandwidth doesn't exceed the Nyquist frequency. Such F_ψ is dubbed an **antialiasing filter**. Antialiasing is a good practice, not only in control.

Example

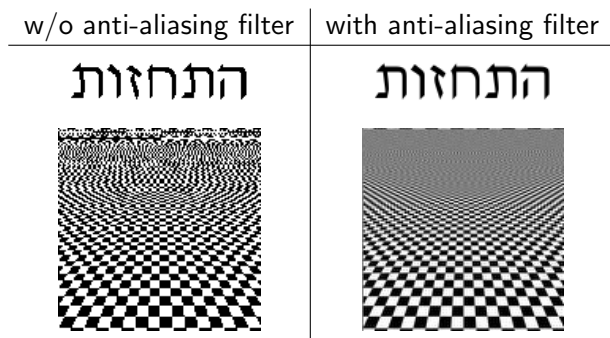
Remedy: antialiasing filtering

In general, the only way to prevent this is to filter out ultra- ω_N parts of the spectrum of n **before** the sampling operation. In other words, we need

- A/D converters with low-pass F_ψ ,

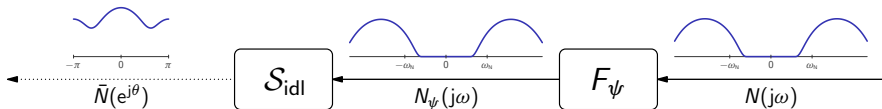
whose bandwidth doesn't exceed the Nyquist frequency. Such F_ψ is dubbed an **antialiasing filter**. Antialiasing is a good practice, not only in control.

Example

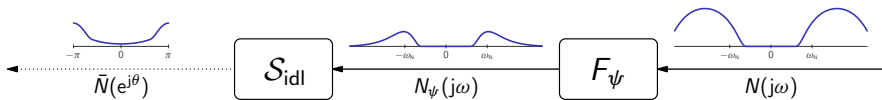


Does “fast” n imply “fast” $\bar{n}$? (contd)

No, if fast ingredients of n , aliasing as slow ingredients of $\bar{n}$, are substantial:



Yes, if ultra-Nyquist parts of n are filtered out prior sampling (better F_ψ):



Thus

- appropriate antialiasing resolves this issue
(but should be cautious, antialiasing filters introduce undesired **phase lags**).
- As a matter of fact,
- antialiasing is simplified / its costs are alleviated under fast sampling.

Can “slow” $\bar{P}_h \bar{d}$ explain “slow” Pd ?

Outline

Computer-controlled systems

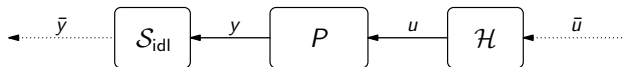
The world according to $\bar{C}$

Discretization of LTI systems

Classical methods for discrete systems (mostly stability)

Discretization problem

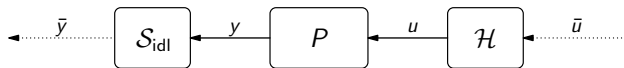
Find the discrete $\bar{P}_h : \bar{u} \mapsto \bar{y}$ for



Assumptions:

- sampling is periodic, with a sampling period $h > 0$
- sampler is $\mathcal{S}_{\text{idl}}$
(antialiasing low-pass F_ψ can always be absorbed into P)
- P is causal and LTI, with strictly proper $P(s)$
(no loss of generality, as it always contains an antialiasing filter)
- hold $\mathcal{H}$ is causal and LSI

Linearity



If $\bar{u} = \alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2$, then

1. $u = \alpha_1 \mathcal{H} \bar{u}_1 + \alpha_2 \mathcal{H} \bar{u}_2$

2. $y = \alpha_1 P \mathcal{H} \bar{u}_1 + \alpha_2 P \mathcal{H} \bar{u}_2$

3. $\bar{y} = \alpha_1 \mathcal{S}_{\text{idl}} P \mathcal{H} \bar{u}_1 + \alpha_2 \mathcal{S}_{\text{idl}} P \mathcal{H} \bar{u}_2$

because $\mathcal{H}$ is linear

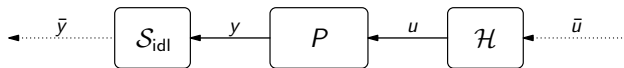
because P is linear

because $\mathcal{S}_{\text{idl}}$ is linear

Therefore, $\bar{P}_h(\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2) = \alpha_1 \bar{P}_h \bar{u}_1 + \alpha_2 \bar{P}_h \bar{u}_2$, i.e.

- the discretized $\bar{P}_h = \mathcal{S}_{\text{idl}} P \mathcal{H}$ is always **linear**.

Shift invariance



If $\bar{u}$ is shifted by one step, then

1. u is shifted by h
2. y is shifted by h
3. $\bar{y}$ is shifted by one step

because $\mathcal{H}$ is shift invariant
because P is time invariant
because $\mathcal{S}_{\text{idl}}$ is shift invariant

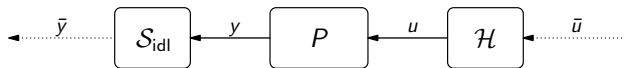
Therefore,

- the discretized $\bar{P}_h = \mathcal{S}_{\text{idl}} P \mathcal{H}$ is always **shift invariant**,

with all consequences of that, like

- impulse response / convolution
- frequency- and z-domain analysis
- ...

Impulse response



If $\bar{u}[j] = \delta[j]$, then

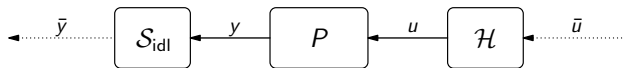
$$\begin{aligned}
 \bar{y}[i] &= y(ih) = \int_{\mathbb{R}} p(ih - t) u(t) dt = \int_{\mathbb{R}} p(ih - t) \sum_{j \in \mathbb{Z}} \phi(t - jh) \bar{u}[j] dt \\
 &= \int_{\mathbb{R}} p(ih - t) \phi(t) dt.
 \end{aligned}$$

Thus, the impulse response of $\bar{P}_h$ is

$$\bar{p}_h[i] = \int_{\mathbb{R}} p(ih - t) \phi(t) dt.$$

Note that $\bar{p}_h[i] = p_\phi(ih)$, where $p_\phi(t)$ is the impulse response of $P_\phi := P\Phi$, as one would expect.

Impulse response



If $\bar{u}[j] = \delta[j]$, then

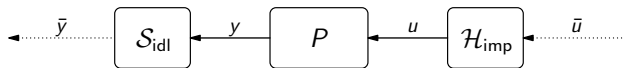
$$\begin{aligned}\bar{y}[i] &= y(ih) = \int_{\mathbb{R}} p(ih - t) u(t) dt = \int_{\mathbb{R}} p(ih - t) \sum_{j \in \mathbb{Z}} \phi(t - jh) \bar{u}[j] dt \\ &= \int_{\mathbb{R}} p(ih - t) \phi(t) dt.\end{aligned}$$

Thus, the impulse response of $\bar{P}_h$ is

$$\bar{p}_h[i] = \int_{\mathbb{R}} p(ih - t) \phi(t) dt.$$

Note that $\bar{p}_h[i] = p_\phi(ih)$, where $p_\phi(t)$ is the impulse response of $P_\phi := PF_\phi$, as one would expect (why?).

Impulse response with $\mathcal{H}_{\text{imp}}$



In this case

$$\phi(t) = \delta(t),$$

so that

$$\bar{p}_h[i] = \bar{p}[i] = p(ih).$$

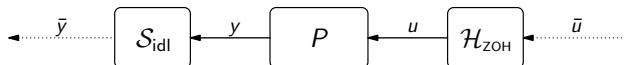
Example

If $P(s) = 1/(s - a)$ for some $a \in \mathbb{C}$, then $p(t) = e^{at}\mathbb{1}(t)$ and

$$\bar{p}_h[i] = e^{ahi}\mathbb{1}[i] = (e^{ah})^i\mathbb{1}[i] =$$



Impulse response with $\mathcal{H}_{\text{ZOH}}$



In this case

$$\phi(t) = \mathbb{1}(t) - \mathbb{1}(t - h),$$

so that

$$\bar{p}_h[i] = \int_0^h p(ih - t) dt = \int_{(i-1)h}^{ih} p(t) dt.$$

Since $p(t) = 0$ for all $t < 0$, $\bar{p}_h[i] = 0$ for all $i \leq 0$, i.e. $\bar{P}_h$ is **strictly causal**.

Example

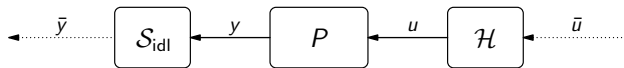
If $P(s) = 1/(s - a)$, then $p(t) = e^{at}\mathbb{1}(t)$ and

$$\bar{p}_h[i] = \int_0^h e^{a(ih-t)} \mathbb{1}(ih-t) dt = \frac{e^{ah} - 1}{a} (e^{ah})^{i-1} \mathbb{1}[i-1] =$$

and $\bar{p}_h[i] = h\mathbb{1}[i-1]$ for $a = 0$.



Transfer function and frequency response



Naturally,

$$\bar{P}_h(z) = \mathfrak{Z}\{\bar{p}_h\} = \sum_{i \in \mathbb{Z}} \bar{p}_h[i] z^{-i} = \sum_{i \in \mathbb{Z}} p_\phi(ih) z^{-i},$$

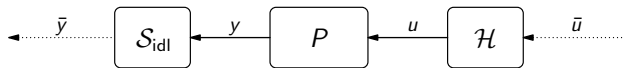
which is known as the **pulse-transfer function** of P . More informative is its frequency response, which at every $\theta \in [-\pi, \pi]$ is

$$\bar{P}_h(e^{j\theta}) = \sum_{i \in \mathbb{Z}} p_\phi(ih) e^{-j\theta i} = \frac{1}{h} \sum_{i \in \mathbb{Z}} P_\phi(j\omega_i) = \frac{1}{h} \sum_{i \in \mathbb{Z}} P(j\omega_i) F_\phi(j\omega_i),$$

where $\omega_i = (\theta + 2\pi i)/h$ are frequencies, aliasing as θ . If $|P(j\omega)F_\phi(j\omega)|$ is negligibly small at ultra- ω_N frequencies, then

$$\bar{P}_h(e^{j\theta}) \approx \frac{1}{h} P(j\theta/h) F_\phi(j\theta/h).$$

Transfer function and frequency response



Naturally,

$$\bar{P}_h(z) = \mathfrak{Z}\{\bar{p}_h\} = \sum_{i \in \mathbb{Z}} \bar{p}_h[i] z^{-i} = \sum_{i \in \mathbb{Z}} p_\phi(ih) z^{-i},$$

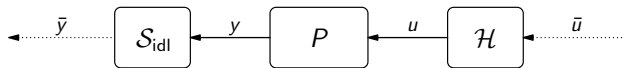
which is known as the **pulse-transfer function** of P . More informative is its frequency response, which at every $\theta \in [-\pi, \pi]$ is

$$\bar{P}_h(e^{j\theta}) = \sum_{i \in \mathbb{Z}} p_\phi(ih) e^{-j\theta i} = \frac{1}{h} \sum_{i \in \mathbb{Z}} P_\phi(j\omega_i) = \frac{1}{h} \sum_{i \in \mathbb{Z}} P(j\omega_i) F_\phi(j\omega_i).$$

where $\omega_i = (\theta + 2\pi i)/h$ are frequencies, aliasing as θ . If $|P(j\omega)F_\phi(j\omega)|$ is negligibly small at ultra- ω_ω frequencies, then

$$\bar{P}_h(e^{j\theta}) \approx \frac{1}{h} P(j\theta/h) F_\phi(j\theta/h).$$

Transfer function and frequency response



Naturally,

$$\bar{P}_h(z) = \mathfrak{Z}\{\bar{p}_h\} = \sum_{i \in \mathbb{Z}} \bar{p}_h[i] z^{-i} = \sum_{i \in \mathbb{Z}} p_\phi(ih) z^{-i},$$

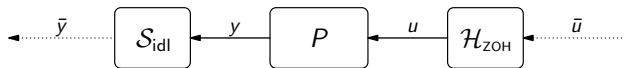
which is known as the **pulse-transfer function** of P . More informative is its frequency response, which at every $\theta \in [-\pi, \pi]$ is

$$\bar{P}_h(e^{j\theta}) = \sum_{i \in \mathbb{Z}} p_\phi(ih) e^{-j\theta i} = \frac{1}{h} \sum_{i \in \mathbb{Z}} P_\phi(j\omega_i) = \frac{1}{h} \sum_{i \in \mathbb{Z}} P(j\omega_i) F_\phi(j\omega_i).$$

where $\omega_i = (\theta + 2\pi i)/h$ are frequencies, aliasing as θ . If $|P(j\omega)F_\phi(j\omega)|$ is negligibly small at ultra- ω_N frequencies, then

$$\bar{P}_h(e^{j\theta}) \approx \frac{1}{h} P(j\theta/h) F_\phi(j\theta/h).$$

Transfer function with $\mathcal{H}_{\text{ZOH}}$

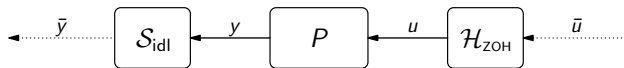


In this case

$$\begin{aligned}
 \bar{P}_h(z) &= \sum_{i \in \mathbb{Z}} \int_0^h p(ih - t) dt z^{-i} = \sum_{i \in \mathbb{Z}} \int_{(i-1)h}^{ih} p(t) dt z^{-i} \\
 &= \sum_{i \in \mathbb{Z}} \left(\int_{-\infty}^{ih} p(t) dt - \int_{-\infty}^{(i-1)h} p(t) dt \right) z^{-i} \\
 &= \sum_{i \in \mathbb{Z}} \int_{-\infty}^{ih} p(t) dt z^{-i} - \sum_{i \in \mathbb{Z}} \int_{-\infty}^{ih} p(t) dt z^{-i-1} \\
 &= (1 - z^{-1}) \sum_{i \in \mathbb{Z}} \underbrace{\int_{-\infty}^{ih} p(t) dt}_{p_{\text{step}}(ih)} z^{-i}
 \end{aligned}$$

where $p_{\text{step}}(t)$ is the step response of P .

Transfer function with $\mathcal{H}_{\text{ZOH}}$

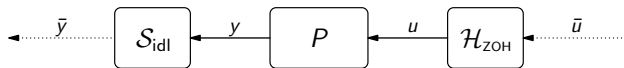


In this case

$$\begin{aligned}
 \bar{P}_h(z) &= \sum_{i \in \mathbb{Z}} \int_0^h p(ih - t) dt z^{-i} = \sum_{i \in \mathbb{Z}} \int_{(i-1)h}^{ih} p(t) dt z^{-i} \\
 &= \sum_{i \in \mathbb{Z}} \left(\int_{-\infty}^{ih} p(t) dt - \int_{-\infty}^{(i-1)h} p(t) dt \right) z^{-i} \\
 &= \sum_{i \in \mathbb{Z}} \int_{-\infty}^{ih} p(t) dt z^{-i} - \sum_{i \in \mathbb{Z}} \int_{-\infty}^{ih} p(t) dt z^{-i-1} \\
 &= (1 - z^{-1}) \sum_{i \in \mathbb{Z}} \underbrace{\int_{-\infty}^{ih} p(t) dt}_{p_{\text{step}}(ih)} z^{-i} = (1 - z^{-1}) \mathfrak{Z}\{S_{\text{idl}} p_{\text{step}}\},
 \end{aligned}$$

where $p_{\text{step}}(t)$ is the step response of P .

Frequency response with $\mathcal{H}_{\text{ZOH}}$



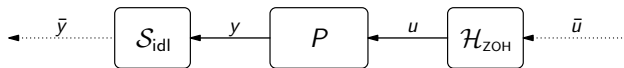
In this case $F_\phi(s) = (1 - e^{-sh})/s$, so that (again, $\omega_i = (\theta + 2\pi i)/h$)

$$\begin{aligned}\bar{P}_h(e^{j\theta}) &= \frac{1}{h} \sum_{i \in \mathbb{Z}} P(j\omega_i) \frac{1 - e^{-j\omega_i h}}{j\omega_i} = (1 - e^{-j\theta}) \sum_{i \in \mathbb{Z}} \frac{P(j\omega_i)}{j\omega_i h} \\ &= \frac{1 - e^{-j\theta}}{j\theta} P(j\theta/h) + (1 - e^{-j\theta}) \sum_{i \in \mathbb{Z} \setminus \{0\}} \frac{P(j\omega_i)}{j\omega_i h}.\end{aligned}$$

If $P(j2\omega_N/h)$ are finite for all $i \in \mathbb{N}$, then

– $\bar{P}_h(1) = P(0)$, i.e. discretization normally preserves the static gain.

Frequency response with $\mathcal{H}_{\text{ZOH}}$



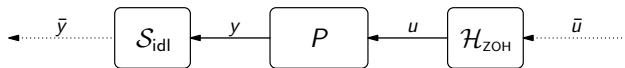
In this case $F_\phi(s) = (1 - e^{-sh})/s$, so that (again, $\omega_i = (\theta + 2\pi i)/h$)

$$\begin{aligned}
 \bar{P}_h(e^{j\theta}) &= \frac{1}{h} \sum_{i \in \mathbb{Z}} P(j\omega_i) \frac{1 - e^{-j\omega_i h}}{j\omega_i} = (1 - e^{-j\theta}) \sum_{i \in \mathbb{Z}} \frac{P(j\omega_i)}{j\omega_i h} \\
 &= \frac{1 - e^{-j\theta}}{j\theta} P(j\theta/h) + (1 - e^{-j\theta}) \sum_{i \in \mathbb{Z} \setminus \{0\}} \frac{P(j\omega_i)}{j\omega_i h}.
 \end{aligned}$$

If $P(j2\omega_N i)$ are finite for all $i \in \mathbb{N}$, then

- $\bar{P}_h(1) = P(0)$, i.e. discretization normally preserves the static gain.

Frequency response with $\mathcal{H}_{\text{ZOH}}$: no aliasing case

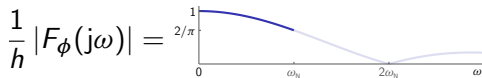


If aliasing in $P_\phi(j\omega)$ is negligible, then for all $\omega \in [-\omega_N, \omega_N]$

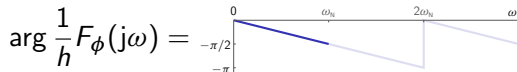
$$\bar{P}_h(e^{j\omega h}) \approx \frac{1}{h} P_\phi(j\omega) = P(j\omega) \frac{1 - e^{-j\omega h}}{j\omega h} = P(j\omega) \text{sinc}(\omega h/2) e^{-j\omega h/2},$$

where $\text{sinc}(t) := \sin(t)/t$. Thus, discretization (actually, the hold)

- decreases gain

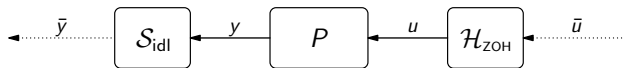


- adds a phase lag



Remark: If $\mathcal{H} = \mathcal{H}_{\text{imp}}$, then $F_\psi = 1$ and no distortion is introduced by the hold. Strangely enough, this implies that $\mathcal{H}_{\text{imp}}$ is preferable over $\mathcal{H}_{\text{ZOH}}$ from the feedback control point of view. But requirements to the decay of $|P(j\omega)|$ become harsher to avoid aliasing effects.

Discretization of finite-dimensional systems



Assume that P is finite dimensional and such that its transfer function

$$P(s) = \sum_{i=1}^n \frac{b_i}{s - a_i}, \quad \text{for } a_i \in \mathbb{C} \text{ such that } a_i \neq a_j \text{ and } b_i \in \mathbb{C} \setminus \{0\}$$

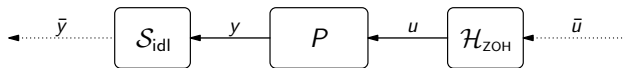
(i.e. no multiple poles). Its impulse response $p(t) = \sum_{i=1}^n b_i e^{a_i t} \mathbb{1}(t)$ and, by the linearity of the discretization operation,

$$\bar{p}_h[t] = \sum_{i=1}^n \bar{b}_i \bar{a}_i^{t-1} \mathbb{1}[t-1], \quad \text{where } \bar{a}_i := e^{a_i h} \text{ and } \bar{b}_i := \frac{e^{a_i h} - 1}{a_i} b_i.$$

Hence,

$$\bar{P}_h(z) = \sum_{i=1}^n \frac{\bar{b}_i}{z - \bar{a}_i}$$

Discretization of finite-dimensional systems



Assume that P is finite dimensional and such that its transfer function

$$P(s) = \sum_{i=1}^n \frac{b_i}{s - a_i}, \quad \text{for } a_i \in \mathbb{C} \text{ such that } a_i \neq a_j \text{ and } b_i \in \mathbb{C} \setminus \{0\}$$

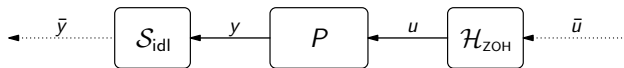
(i.e. no multiple poles). Its impulse response $p(t) = \sum_{i=1}^n b_i e^{a_i t} \mathbb{1}(t)$ and, by the linearity of the discretization operation,

$$\bar{p}_h[t] = \sum_{i=1}^n \bar{b}_i \bar{a}_i^{t-1} \mathbb{1}[t-1], \quad \text{where } \bar{a}_i := e^{a_i h} \text{ and } \bar{b}_i := \frac{e^{a_i h} - 1}{a_i} b_i.$$

Hence,

$$\bar{P}_h(z) = \sum_{i=1}^n \frac{\bar{b}_i}{z - \bar{a}_i}.$$

Discretization of finite-dimensional systems (contd)

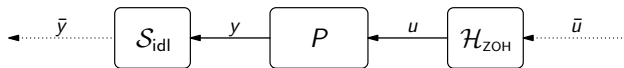


The pulse-transfer function

$$\bar{P}_h(z) = \sum_{i=1}^n \frac{\bar{b}_i}{z - \bar{a}_i}$$

- is always rational, of order at most n
(i.e. $\bar{P}_h$ remains finite dimensional)
- is always strictly proper, as $\lim_{|z| \rightarrow \infty} \bar{P}_h(z) = 0$
(agrees with the fact that $\bar{P}$ is always strictly causal)
- can have its poles only in the set $\{e^{a_i h}\}_{i=1}^n$
 - $a_i \in \mathbb{C} \setminus \bar{\mathbb{C}}_0 \Rightarrow \bar{a}_i \in \bar{\mathbb{D}}_1$ (unstable remains stable)
 - $a_i \in j\mathbb{R} \Rightarrow \bar{a}_i \in \mathbb{T}$ (marginally stable remains on boundary)
 - $a_i \in \bar{\mathbb{C}}_0 \Rightarrow \bar{a}_i \in \mathbb{C} \setminus \bar{\mathbb{D}}_1$ (stable remains unstable)

Discretization of finite-dimensional systems (contd)



The pulse-transfer function

$$\bar{P}_h(z) = \sum_{i=1}^n \frac{\bar{b}_i}{z - \bar{a}_i}$$

- is always rational, of order at most n
(i.e. $\bar{P}_h$ remains finite dimensional)
- is always strictly proper, as $\lim_{|z| \rightarrow \infty} \bar{P}_h(z) = 0$
(agrees with the fact that $\bar{P}$ is always strictly causal)

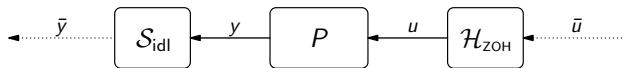
— can have its poles only in the set $\{e^{s_i h}\}_{i=1}^n$

$$- a_i \in \mathbb{C} \setminus \mathbb{C}_0 \Rightarrow \bar{a}_i \in \mathbb{D}_1$$

$$- a_i \in j\mathbb{R} \Rightarrow \bar{a}_i \in \mathbb{T}$$

$$- a_i \in \mathbb{C}_0 \Rightarrow \bar{a}_i \in \mathbb{C} \setminus \bar{\mathbb{D}}_1$$

Discretization of finite-dimensional systems (contd)



The pulse-transfer function

$$\bar{P}_h(z) = \sum_{i=1}^n \frac{\bar{b}_i}{z - \bar{a}_i}$$

- is always rational, of order at most n
(i.e. $\bar{P}_h$ remains finite dimensional)
- is always strictly proper, as $\lim_{|z| \rightarrow \infty} \bar{P}_h(z) = 0$
(agrees with the fact that $\bar{P}$ is always strictly causal)
- can have its poles only in the set $\{e^{a_i h}\}_{i=1}^n$
 - $a_i \in \mathbb{C} \setminus \bar{\mathbb{C}}_0 \implies \bar{a}_i \in \mathbb{D}_1$
 - $a_i \in j\mathbb{R} \implies \bar{a}_i \in \mathbb{T}$
 - $a_i \in \mathbb{C}_0 \implies \bar{a}_i \in \mathbb{C} \setminus \bar{\mathbb{D}}_1$

stable remains stable

stability boundary remains on boundary

unstable remains unstable

Discretization: example 1

Let

$$P(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

For this choice

$$\bar{a}_1 = 1, \quad \bar{b}_1 = h, \quad \bar{a}_2 = e^{-h}, \quad \text{and} \quad \bar{b}_2 = e^{-h} - 1.$$

Hence,

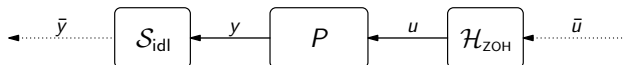
$$\bar{P}_h(z) = \frac{h}{z-1} + \frac{e^{-h}-1}{z-e^{-h}} = \frac{(h+e^{-h}-1)z + 1 - (1+h)e^{-h}}{(z-1)(z-e^{-h})}.$$

This $\bar{P}_h(z)$ has

- a pole at $z = 1$ (integrator) and a pole at $z = e^{-h} \in (0, 1)$,
- a zero at $z = \frac{1 - (1+h)e^{-h}}{1 - e^{-h} - h} \in (-1, 0)$.

The same number of poles as $P(s)$, different number of zeros.

Sampling zeros



Zeros of $\bar{P}_h(z)$ are mess. We only know that

- the number of finite zeros of $\bar{P}_h(z)$ is $n - 1$ for (almost) all h

if $P(s)$ has m finite zeros $z_1, \dots, z_m$, then as $h \downarrow 0$

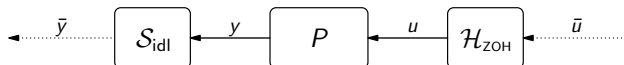
- m zeros of $\bar{P}_h(z)$ approach $e^{s_i h}$,

the remaining $n - m - 1$ zeros, aka **sampling zeros**, approach the roots of Euler–Frobenius polynomials $Q_{n-m-1}(z)$, independent of $P(s)$:

$n - m$	$Q_{n-m-1}(z)$
2	$z + 1$
3	$z^2 + 4z + 1$
4	$z^3 + 11z^2 + 11z + 1$
5	$z^4 + 26z^3 + 66z^2 + 26z + 1$

As $Q_k(z) = z^k Q_k(1/z)$ and $Q_k(0) \neq 0$, $Q_k(z_0) = 0 \iff Q_k(1/z_0) = 0$.
Therefore, $Q_k(z)$ has root(s) in $\mathbb{C} \setminus \bar{\mathbb{D}}_1$ for all $k \geq 2$.

Sampling zeros



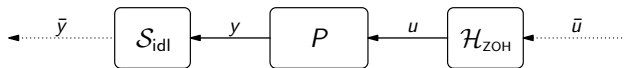
Zeros of $\bar{P}_h(z)$ are mess. We only know that

- the number of finite zeros of $\bar{P}_h(z)$ is $n - 1$ for (almost) all h
- if $P(s)$ has m finite zeros ($m < n$), z_i , then as $h \downarrow 0$
 - m zeros of $\bar{P}_h(z)$ approach $e^{z_i h}$,
 - the remaining $n - m - 1$ zeros, aka **sampling zeros**, approach the roots of Euler–Frobenius polynomials $Q_{n-m-1}(z)$, independent of $P(s)$:

$n - m$	$Q_{n-m-1}(z)$
2	$z + 1$
3	$z^2 + 4z + 1$
4	$z^3 + 11z^2 + 11z + 1$
5	$z^4 + 26z^3 + 66z^2 + 26z + 1$

As $Q_k(z) = z^k Q_k(1/z)$ and $Q_k(0) \neq 0$, $Q_k(z_0) = 0 \iff Q_k(1/z_0) = 0$.
 Therefore, $Q_k(z)$ has root(s) in $\mathbb{C} \setminus \bar{\mathbb{D}}_1$ for all $k \geq 2$.

Sampling zeros: implication



Let $P(s)$ has n poles and $m < n$ (finite) zeros.

- If $n - m > 2$, then $\bar{P}_h(z)$ is **nonminimum phase** for sufficiently **small h** , regardless zeros of $P(s)$.

Example

If $P(s) = 2/[s(s+1)(s+2)]$, then

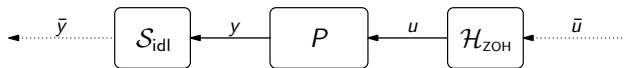
$$\bar{P}_h(z) = \frac{2h - 3 + 4e^{-h} - e^{-2h}}{2} \frac{(z - z_{h,1})(z - z_{h,2})}{(z - 1)(z - e^{-h})(z - e^{-2h})},$$

where

$$\begin{bmatrix} z_{h,1} \\ z_{h,2} \end{bmatrix} =$$



Sampling zeros: implication



Let $P(s)$ has n poles and $m < n$ (finite) zeros.

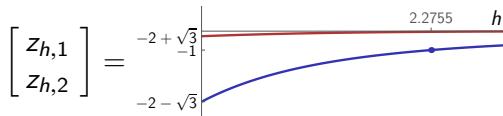
— If $n - m > 2$, then $\bar{P}_h(z)$ is **nonminimum phase** for sufficiently **small h** , regardless zeros of $P(s)$.

Example

If $P(s) = 2/[s(s+1)(s+2)]$, then

$$\bar{P}_h(z) = \frac{2h - 3 + 4e^{-h} - e^{-2h}}{2} \frac{(z - z_{h,1})(z - z_{h,2})}{(z - 1)(z - e^{-h})(z - e^{-2h})},$$

where



Discretization: example 2

Let

$$P(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{j\omega_n/2}{s + j\omega_n} - \frac{j\omega_n/2}{s - j\omega_n}$$

(so $p(t) = \omega_n \sin(\omega_n t)$). For this choice

$$\bar{a}_1 = e^{-j\omega_n h}, \quad \bar{b}_1 = \frac{1 - e^{-j\omega_n h}}{2}, \quad \bar{a}_2 = e^{j\omega_n h}, \quad \text{and} \quad \bar{b}_2 = \frac{1 - e^{j\omega_n h}}{2}.$$

Hence,

$$\bar{P}_h(z) = \frac{1}{2} \left(\frac{1 - e^{-j\omega_n h}}{z - e^{-j\omega_n h}} + \frac{1 - e^{j\omega_n h}}{z - e^{j\omega_n h}} \right) = \frac{(1 - \cos(\omega_n h))(z + 1)}{z^2 - 2\cos(\omega_n h)z + 1}.$$

If

- $\cos(\omega_n h) \neq \pm 1$, then $\bar{P}_h(z)$ has two poles at $e^{\pm j\omega_n h}$ and a zero at -1 ,
- $\cos(\omega_n h) = 1$, then $\bar{P}_h(z) = 0$,
- $\cos(\omega_n h) = -1$, then $\bar{P}_h(z) = 2/(z + 1)$.

Even the order is not always preserved.

When order drops?

Consider

$$\bar{P}_h(z) = \sum_{i=1}^n \frac{\bar{b}_i}{z - \bar{a}_i} \quad \text{where } \bar{a}_i := e^{a_i h} \text{ and } \bar{b}_i := \frac{e^{a_i h} - 1}{a_i} b_i.$$

Two pathological cases, where the order of $\bar{P}_h(z)$ is smaller than n :

1. $\bar{a}_i = \bar{a}_j$, although $a_i \neq a_j$, which is equivalent to

$$e^{a_i h} = e^{a_j h} \iff a_i h = a_j h + j2\pi k \text{ for some } k \in \mathbb{Z} \setminus \{0\}$$

or $a_i - a_j = j2\omega_N k$.

2. $\bar{b}_i = 0$, although $b_i \neq 0$, which is equivalent to

$$(e^{a_i h} = 1) \wedge (a_i \neq 0) \iff a_i h = j2\pi k \text{ for some } k \in \mathbb{Z} \setminus \{0\}$$

or $a_i = j2\omega_N k$. But if the latter condition holds, then $\exists j \neq i$ such that $a_j = -j2\omega_N k$. Hence, $a_i - a_j = j2\omega_N (2k)$ and this case is covered by 1.

When order drops?

Consider

$$\bar{P}_h(z) = \sum_{i=1}^n \frac{\bar{b}_i}{z - \bar{a}_i} \quad \text{where } \bar{a}_i := e^{a_i h} \text{ and } \bar{b}_i := \frac{e^{a_i h} - 1}{a_i} b_i.$$

Two pathological cases, where the order of $\bar{P}_h(z)$ is smaller than n :

1. $\bar{a}_i = \bar{a}_j$, although $a_i \neq a_j$, which is equivalent to

$$e^{a_i h} = e^{a_j h} \iff a_i h = a_j h + j2\pi k \text{ for some } k \in \mathbb{Z} \setminus \{0\}$$

or $a_i - a_j = j2\omega_N k$.

2. $\bar{b}_i = 0$, although $b_i \neq 0$, which is equivalent to

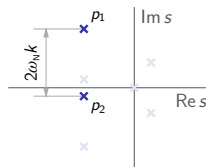
$$(e^{a_i h} = 1) \wedge (a_i \neq 0) \iff a_i h = j2\pi k \text{ for some } k \in \mathbb{Z} \setminus \{0\}$$

or $a_i = j2\omega_N k$. But if the latter condition holds, then $\exists j \neq i$ such that $a_j = -j2\omega_N k$. Hence, $a_i - a_j = j2\omega_N(2k)$ and this case is covered by 1.

Pathological sampling

We say that sampling is **pathological** with respect to P if there are at least 2 poles of $P(s)$, say p_1 and p_2 , such that

$$p_1 - p_2 = j \frac{2\pi}{h} k$$

 $\iff$ 

for some $k \in \mathbb{Z} \setminus \{0\}$. If sampling is pathological, then

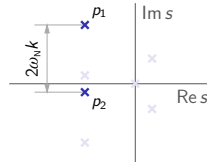
— some parts of dynamics of P are not visible by the discrete controller.
But these parts don't disappear, they are just in the blind spot of $\tilde{C}$, which cannot counteract anything caused by them (e.g. instability or oscillations).

As the minimum distance between poles for h being pathological is $2\omega_n$,

— “sufficiently fast” sampling is never pathological.

Pathological sampling

We say that sampling is **pathological** with respect to P if there are at least 2 poles of $P(s)$, say p_1 and p_2 , such that

$$p_1 - p_2 = j \frac{2\pi}{h} k \iff$$


The diagram shows the complex s-plane with a horizontal real axis (Re s) and a vertical imaginary axis (Im s). Two poles, p_1 and p_2 , are marked with blue 'x' symbols. p_1 is located at a positive imaginary value, and p_2 is located at a negative imaginary value. A vertical double-headed arrow between the horizontal lines passing through p_1 and p_2 is labeled $2\omega_n k$. Several other poles are shown as light blue 'x' marks: one on the positive imaginary axis, one on the negative imaginary axis, and two in the right half-plane (positive real part).

for some $k \in \mathbb{Z} \setminus \{0\}$. If sampling is pathological, then

- some parts of dynamics of P are not visible by the *discrete* controller.

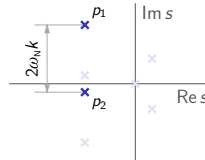
But these parts don't disappear, they are just in the blind spot of $\bar{C}$, which cannot counteract anything caused by them (e.g. instability or oscillations).

As the minimum distance between poles for $\bar{h}$ being pathological is $2\omega_n$,

- “sufficiently fast” sampling is never pathological.

Pathological sampling

We say that sampling is **pathological** with respect to P if there are at least 2 poles of $P(s)$, say p_1 and p_2 , such that

$$p_1 - p_2 = j \frac{2\pi}{h} k \iff$$


The diagram shows the complex s-plane with a horizontal real axis (Re s) and a vertical imaginary axis (Im s). Two poles, p1 and p2, are marked with blue 'x' symbols. p1 is located at a positive imaginary value, and p2 is located at a negative imaginary value. A vertical double-headed arrow between the horizontal lines passing through p1 and p2 is labeled 2\omega_N k. Several other poles are shown as light blue 'x' symbols, including one at the origin and others in the right half-plane.

for some $k \in \mathbb{Z} \setminus \{0\}$. If sampling is pathological, then

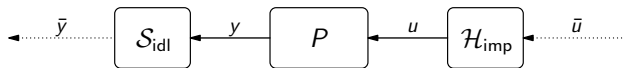
- some parts of dynamics of P are not visible by the *discrete* controller.

But these parts don't disappear, they are just in the blind spot of $\bar{C}$, which cannot counteract anything caused by them (e.g. instability or oscillations).

As the minimum distance between poles for h being pathological is $2\omega_N$,

- “sufficiently fast” sampling is never pathological.

Discretization of finite-dimensional systems with $\mathcal{H}_{\text{imp}}$



Assume that P is finite dimensional and such that its transfer function

$$P(s) = \sum_{i=1}^n \frac{b_i}{s - a_i}, \quad \text{for } a_i \in \mathbb{C} \text{ such that } a_i \neq a_j \text{ and } b_i \in \mathbb{C} \setminus \{0\}$$

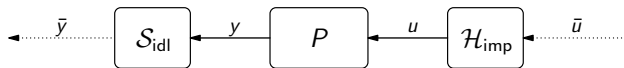
(i.e. no multiple poles). Its impulse response $p(t) = \sum_{i=1}^n b_i e^{a_i t} \mathbb{1}(t)$ and, by the linearity of the discretization operation,

$$\bar{p}_h[t] = \sum_{i=1}^n b_i \bar{a}_i^t \mathbb{1}[t], \quad \text{where } \bar{a}_i := e^{a_i h}.$$

Hence,

$$\bar{P}_h(z) = \sum_{i=1}^n \frac{b_i z}{z - \bar{a}_i}.$$

Pathological sampling with $\mathcal{H}_{\text{imp}}$



In

$$\bar{P}_h(z) = \sum_{i=1}^n \frac{b_i z}{z - \bar{a}_i}, \quad \text{where } \bar{a}_i := e^{a_i h},$$

gains can never be zero, but $\bar{a}_i = \bar{a}_j$ can still happen. This suggests that

- the pathological sampling phenomenon is fundamental

and not caused by specific A/D or D/A conversion mechanisms, at least as far as all involved signals are scalar.

Outline

Computer-controlled systems

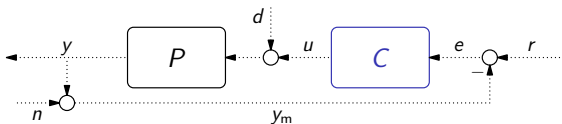
The world according to $\tilde{C}$

Discretization of LTI systems

Classical methods for discrete systems (mostly stability)

Discrete unity feedback

We may now drop all signs of discretization and consider a discrete system,

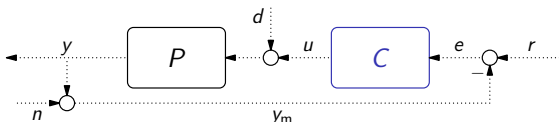


for a given

$$P(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \frac{N_P(z)}{D_P(z)}$$

with $b_m \neq 0$ and $m \leq n$ (typically, $m = n - 1$).

Internal stability



The closed-loop system is said to be

- **internally stable** if all Gang of Four transfer functions

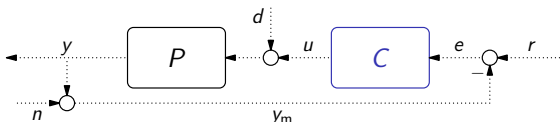
$$\begin{bmatrix} S(z) & T_d(z) \\ T_c(z) & T(z) \end{bmatrix} := \frac{1}{1 + P(z)C(z)} \begin{bmatrix} 1 \\ C(z) \end{bmatrix} \begin{bmatrix} 1 & P(z) \end{bmatrix}$$

are stable,

in whatever sense. Because we study *causal finite-dimensional LSI* systems, every stability notion boils down to the requirement that the corresponding transfer function is **proper** and has **no poles in $\mathbb{C} \setminus \mathbb{D}_1$** .

Internal stability is the formalism helping to avoid unstable cancellations.

Internal stability



The closed-loop system is said to be

- **internally stable** if all Gang of Four transfer functions

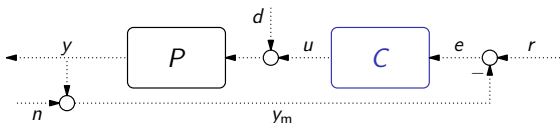
$$\begin{bmatrix} S(z) & T_d(z) \\ T_c(z) & T(z) \end{bmatrix} := \frac{1}{1 + P(z)C(z)} \begin{bmatrix} 1 \\ C(z) \end{bmatrix} \begin{bmatrix} 1 & P(z) \end{bmatrix}$$

are stable,

in whatever sense. Because we study *causal finite-dimensional LSI* systems, every stability notion boils down to the requirement that the corresponding transfer function is **proper** and has **no poles in $\mathbb{C} \setminus \mathbb{D}_1$** .

Internal stability is the formalism helping to **avoid unstable cancellations**.

Characteristic polynomial



If $C(z) = N_C(z)/D_C(z)$ is proper, then the closed-loop system is internally stable iff its **characteristic polynomial**

$$\chi_{cl}(z) = N_P(z)N_C(z) + D_P(z)D_C(z)$$

has **all roots in $\mathbb{D}_1$** (such polynomials are known as **Schur**).

Jury stability criterion

Let $\chi_{cl}(z) = \chi_n z^n + \chi_{n-1} z^{n-1} + \dots + \chi_1 z + \chi_0$. Construct the Jury table:

	z^n	z^{n-1}		z^2	z	1
1	χ_n	χ_{n-1}	$\cdots$	χ_2	χ_1	χ_0
2	χ_0	χ_1	$\cdots$	χ_{n-2}	χ_{n-1}	χ_n
3	$\chi_{1,n-1}$	$\chi_{1,n-2}$	$\cdots$	$\chi_{1,1}$	$\chi_{1,0}$	
4	$\chi_{1,0}$	$\chi_{1,1}$	$\cdots$	$\chi_{1,n-2}$	$\chi_{1,n-1}$	
	$\vdots$					
$2n+1$	$\chi_{n,0}$					

where for each $i \in \mathbb{Z}_{1..n}$

$2i+1$ th row is the $(2i-1)$ th row minus $(2i)$ th row multiplied by $\frac{\chi_{i,0}}{\chi_{i,n+1-i}}$,
 $2i+2$ th row is the $(2i+1)$ th row in reverse order.

Jury criterion:

- if $\chi_n > 0$, then $\chi_{cl}(z)$ is Schur iff $\chi_{i,n+1-i}$ are positive for all $i \in \mathbb{Z}_{1..n}$
- if $\chi_{i,n+1-i} \neq 0$ for all i , then # of $\chi_{i,n+1-i} < 0$ equals # of roots in $\mathbb{C} \setminus \mathbb{D}_1$

Jury stability criterion

Let $\chi_{cl}(z) = \chi_n z^n + \chi_{n-1} z^{n-1} + \dots + \chi_1 z + \chi_0$. Construct the Jury table:

	z^n	z^{n-1}		z^2	z	1
1	χ_n	χ_{n-1}	$\cdots$	χ_2	χ_1	χ_0
2	χ_0	χ_1	$\cdots$	χ_{n-2}	χ_{n-1}	χ_n
3	$\chi_{1,n-1}$	$\chi_{1,n-2}$	$\cdots$	$\chi_{1,1}$	$\chi_{1,0}$	
4	$\chi_{1,0}$	$\chi_{1,1}$	$\cdots$	$\chi_{1,n-2}$	$\chi_{1,n-1}$	
	$\vdots$					
$2n+1$	$\chi_{n,0}$					

where for each $i \in \mathbb{Z}_{1..n}$

$2i+1$ th row is the $(2i-1)$ th row minus $(2i)$ th row multiplied by $\frac{\chi_{i,0}}{\chi_{i,n+1-i}}$,
 $2i+2$ th row is the $(2i+1)$ th row in reverse order.

Jury criterion:

- if $\chi_n > 0$, then $\chi_{cl}(z)$ is Schur iff $\chi_{i,n-i}$ are positive for all $i \in \mathbb{Z}_{1..n}$
- if $\chi_{i,n-i} \neq 0$ for all i , then $\#$ of $\chi_{i,n-i} < 0$ equals $\#$ of roots in $\mathbb{C} \setminus \mathbb{D}_1$

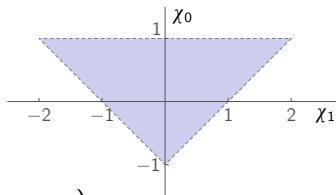
Jury stability criterion for 2-order polynomials

If $n = 2$ and, wlog, $\chi_2 = 1$, then the Jury table is

	z^2	z	1
1	1	χ_1	χ_0
2	χ_0	χ_1	1
3	$1 - \chi_0^2$	$\chi_1(1 - \chi_0)$	
4	$\chi_1(1 - \chi_0)$	$(1 + \chi_0)(1 - \chi_0)$	
5	$1 - \chi_0^2 - \chi_1^2(1 - \chi_0)/(1 + \chi_0)$		

and $\chi_{cl}(z) = z^2 + \chi_1 z + \chi_0$ is Schur iff

$$\begin{cases} |\chi_0| < 1 \\ |\chi_1| < 1 + \chi_0 \end{cases} \iff$$



(more complex than in the continuous-time case).

Jury criterion: example

Consider

$$P(z) = \frac{(h + e^{-h} - 1)z + 1 - (1 + h)e^{-h}}{(z - 1)(z - e^{-h})},$$

which is the discretization of $P(s) = 1/[s(s + 1)]$, with $C(z) = k$. Then

$$\begin{aligned}\chi_{cl}(z) &= (z - 1)(z - e^{-h}) + k(h + e^{-h} - 1)z + k(1 - (1 + h)e^{-h}) \\ &= z^2 + (k(h + e^{-h} - 1) - 1 - e^{-h})z + e^{-h} + k(1 - e^{-h} - he^{-h})\end{aligned}$$

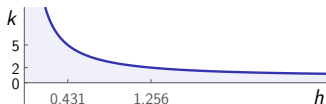
Two conditions:

$$|\chi_0| < 1 \quad -\frac{1+e^{-h}}{1-(h+1)e^{-h}} < k < \frac{1-e^{-h}}{1-(h+1)e^{-h}}$$

$$|\chi_1| < 1 + \chi_0 \quad 0 < k < \frac{2(1+e^{-h})}{h-2+(h+2)e^{-h}}$$

Hence,

$$0 < k < \frac{1 - e^{-h}}{1 - (h + 1)e^{-h}} :$$



Root locus

The technique is exactly as in the continuous-time case. Start with writing

$$\chi_{\text{cl}}(z) = 0 \quad \Longleftrightarrow \quad -\frac{1}{k} = G_k(z),$$

where k is a parameter to change, in $(0, \infty)$, and $G_k(z)$ is a proper transfer function. This representation is termed the **root-locus form**. All rules, which we know from the continuous-time analysis, apply then literally.

What changes is the meaning of the results, because

- stability / performance areas become different.

For example, no asymptote remains in the stability area ($\mathbb{D}_1$), which implies that we can afford

- no high-gain feedback in discrete setting if $P(z)$ is strictly proper, which is normally the case.

Root locus

The technique is exactly as in the continuous-time case. Start with writing

$$\chi_{\text{cl}}(z) = 0 \quad \Longleftrightarrow \quad -\frac{1}{k} = G_k(z),$$

where k is a parameter to change, in $(0, \infty)$, and $G_k(z)$ is a proper transfer function. This representation is termed the **root-locus form**. All rules, which we know from the continuous-time analysis, apply then literally.

What changes is the meaning of the results, because

- stability / performance areas become different.

For example, no asymptote remains in the stability area ($\mathbb{D}_1$), which implies that we can afford

- **no high-gain feedback** in discrete setting if $P(z)$ is strictly proper, which is normally the case.

Root locus: example

Consider again

$$P(z) = \frac{\overbrace{(h + e^{-h} - 1)z}^{>0} + \overbrace{1 - (1 + h)e^{-h}}^{>0}}{(z - 1)(z - e^{-h})},$$

which is the discretization of $P(s) = 1/[s(s + 1)]$, and the “P” $C(z) = k$.

Root locus: example

Consider again

$$P(z) = \frac{\overbrace{(h + e^{-h} - 1)z + 1}^{>0} - \overbrace{(1 + h)e^{-h}}^{>0}}{(z - 1)(z - e^{-h})},$$

which is the discretization of $P(s) = 1/[s(s + 1)]$, and the “P” $C(z) = k$.

– start: $z = 1$ and $z = e^{-h}$ (poles of $G_k(z) = P(z)$);

– end: $z = -\frac{1 - (1 + h)e^{-h}}{h + e^{-h} - 1} \in (-1, 0)$ and $z \rightarrow -\infty + j0$, as the pole excess is 1 (one asymptote, with the angle 180°);

– real axis: between the poles and to the left of the zero

– breakaway / break-in: by $dP(z)/dz = 0$ for real z ,

$$z_{1,2} = e^{-h} + \frac{(1 - e^{-h})\sqrt{1 - e^{-h}}}{\sqrt{1 - e^{-h}} \pm \sqrt{h}} = 1 \mp \frac{(1 - e^{-h})\sqrt{h}}{\sqrt{1 - e^{-h}} \pm \sqrt{h}}$$

with $e^{-h} < z_1 < 1$ (breakaway) and z_2 to the left of the zero (break-in) and $z_2 \leq -1$ if $0 < h < 3.720754$ and $-1 < z_1 < 0$ if $h \geq 3.720754$.

Root locus: example

Consider again

$$P(z) = \frac{\overbrace{(h + e^{-h} - 1)z}^{>0} + \overbrace{1 - (1 + h)e^{-h}}^{>0}}{(z - 1)(z - e^{-h})},$$

which is the discretization of $P(s) = 1/[s(s + 1)]$, and the “P” $C(z) = k$.

- start: $z = 1$ and $z = e^{-h}$ (poles of $G_k(z) = P(z)$);
- end: $z = -\frac{1 - (1 + h)e^{-h}}{h + e^{-h} - 1} \in (-1, 0)$ and $z \rightarrow -\infty + j0$, as the pole excess is 1 (one asymptote, with the angle 180°);

— real axis: between the poles and to the left of the zero

— breakaway / break-in: by $dP(z)/dz = 0$ for real z ,

$$z_{1,2} = e^{-h} + \frac{(1 - e^{-h})\sqrt{1 - e^{-h}}}{\sqrt{1 - e^{-h}} \pm \sqrt{h}} = 1 \mp \frac{(1 - e^{-h})\sqrt{h}}{\sqrt{1 - e^{-h}} \pm \sqrt{h}}$$

with $e^{-h} < z_1 < 1$ (breakaway) and z_2 to the left of the zero (break-in)
and $z_2 \leq -1$ if $0 < h < 3.720754$ and $-1 < z_1 < 0$ if $h \geq 3.720754$.

Root locus: example

Consider again

$$P(z) = \frac{\overbrace{(h + e^{-h} - 1)z}^{>0} + \overbrace{1 - (1 + h)e^{-h}}^{>0}}{(z - 1)(z - e^{-h})},$$

which is the discretization of $P(s) = 1/[s(s + 1)]$, and the “P” $C(z) = k$.

- start: $z = 1$ and $z = e^{-h}$ (poles of $G_k(z) = P(z)$);
- end: $z = -\frac{1 - (1 + h)e^{-h}}{h + e^{-h} - 1} \in (-1, 0)$ and $z \rightarrow -\infty + j0$, as the pole excess is 1 (one asymptote, with the angle 180°);
- real axis: between the poles and to the left of the zero

— breakaway / break-in: by $dP(z)/dz = 0$ for real z ,

$$z_{1,2} = e^{-h} + \frac{(1 - e^{-h})\sqrt{1 - e^{-h}}}{\sqrt{1 - e^{-h}} \pm \sqrt{h}} = 1 \mp \frac{(1 - e^{-h})\sqrt{h}}{\sqrt{1 - e^{-h}} \pm \sqrt{h}}$$

with $e^{-h} < z_1 < 1$ (breakaway) and z_2 to the left of the zero (break-in)
and $z_2 \leq -1$ if $0 < h < 3.720754$ and $-1 < z_1 < 0$ if $h \geq 3.720754$.

Root locus: example

Consider again

$$P(z) = \frac{\overbrace{(h + e^{-h} - 1)z}^{>0} + \overbrace{1 - (1 + h)e^{-h}}^{>0}}{(z - 1)(z - e^{-h})},$$

which is the discretization of $P(s) = 1/[s(s + 1)]$, and the “P” $C(z) = k$.

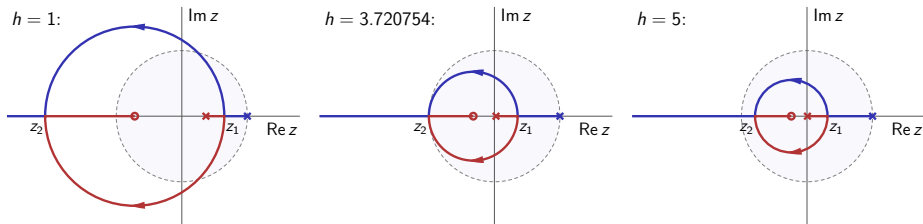
- start: $z = 1$ and $z = e^{-h}$ (poles of $G_k(z) = P(z)$);
- end: $z = -\frac{1 - (1 + h)e^{-h}}{h + e^{-h} - 1} \in (-1, 0)$ and $z \rightarrow -\infty + j0$, as the pole excess is 1 (one asymptote, with the angle 180°);
- real axis: between the poles and to the left of the zero
- breakaway / break-in: by $dP(z)/dz = 0$ for real z ,

$$z_{1,2} = e^{-h} + \frac{(1 - e^{-h})\sqrt{1 - e^{-h}}}{\sqrt{1 - e^{-h}} \pm \sqrt{h}} = 1 \mp \frac{(1 - e^{-h})\sqrt{h}}{\sqrt{1 - e^{-h}} \pm \sqrt{h}}$$

with $e^{-h} < z_1 < 1$ (breakaway) and z_2 to the left of the zero (break-in) and $z_2 \leq -1$ if $0 < h < 3.720754$ and $-1 < z_1 < 0$ if $h \geq 3.720754$.

Root locus: example (contd)

For various sampling periods,

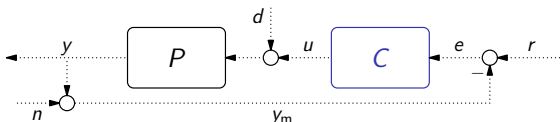


In all cases the system is stable only if k is sufficiently small. In fact, for

$$0 < k < \frac{1 - e^{-h}}{1 - (h + 1)e^{-h}},$$

which we know from the analysis of the characteristic polynomial.

Nyquist criterion



The same logic, as in the continuous-time case. The return difference

$$M(z) = 1 + L(z) = 1 + P(z)C(z) = \frac{\chi_{cl}(z)}{\chi_{ol}(z)}$$

still has open-loop poles as its poles and closed-loop poles as its zeros. The line of reasonings is then

1. define simple closed contour Γ_z containing all $\mathbb{C} \setminus \bar{\mathbb{D}}_1$;
2. determine the mapping Γ_L of Γ_z by the loop gain $L(z) = M(z) - 1$;
3. count the number ν of *clockwise* encirclings of $(-1, 0)$ by Γ_L .

By the argument principle, $\nu = \#_{\text{clsd-loop unstable poles}} - \#_{\text{opn-loop unstable poles}}$

Nyquist contour

The contour encircling the unstable region $\mathbb{C} \setminus \bar{\mathbb{D}}_1$ is cumbersome. A simple workaround is to redefine $z \rightarrow 1/\lambda$. The unstable region in terms of λ is $\mathbb{D}_1$ and the contour around it is the unit circle, $\Gamma_\lambda = \mathbb{T}$. Some observations:

- the (clockwise) Γ_λ is mapped by $L(\lambda)$ as the frequency response $L(e^{j\theta})$ under increasing θ (the frequency for λ is $-\theta$);
- if $L(\lambda)$, equivalently $L(z)$, has poles at $\mathbb{T}$, the contour is altered as

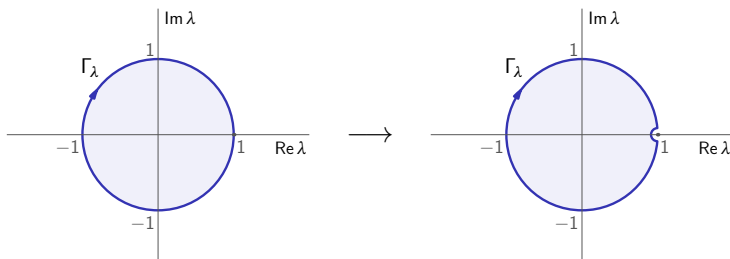


with the same completion rules as in the continuous-time case.

Nyquist contour

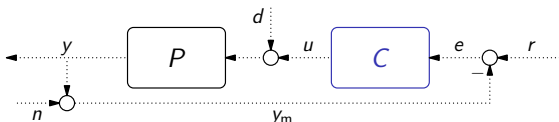
The contour encircling the unstable region $\mathbb{C} \setminus \bar{\mathbb{D}}_1$ is cumbersome. A simple workaround is to redefine $z \rightarrow 1/\lambda$. The unstable region in terms of λ is $\mathbb{D}_1$ and the contour around it is the unit circle, $\Gamma_\lambda = \mathbb{T}$. Some observations:

- the (clockwise) Γ_λ is mapped by $L(\lambda)$ as the frequency response $L(e^{j\theta})$ under increasing θ (the frequency for λ is $-\theta$);
- if $L(\lambda)$, equivalently $L(z)$, has poles at $\mathbb{T}$, the contour is altered as



with the same completion rules as in the continuous-time case.

Steady-state performance



Nothing changes vis-à-vis the continuous-time case, except replacing $s = 0$ with $z = 1$. For example, if $d[t] = \mathbb{1}[t]$, then by the Final Value Theorem

$$y_{ss} := \lim_{t \rightarrow \infty} y[t] = \lim_{z \rightarrow 1} (z - 1) T_d(z) D(z) = \lim_{z \rightarrow 1} (z - 1) T_d(z) \frac{z}{z - 1} = T_d(1),$$

which is the static gain of (stable) T_d . Moreover,

$$y_{ss} = 0 \quad \Longleftrightarrow \quad (P(1) = 0) \vee (C(1) = \infty),$$

where the latter condition requires an integral action in C .

Transient performance and poles

Messier, e.g. discrete 1-order systems can exhibit oscillations and the role of zeros is not clear. So normally understood via discretized models.

Because $1 = \mathcal{H}_{\text{ZOH}} \bar{1}$, we have $S_{\text{id}} G \bar{1} = \bar{G}_h \bar{1}$, i.e. the step response of the discrete $\bar{G}_h$ is the sampled version of that of G . If $G(s) = \omega_n^2 / (s^2 + 2\zeta\omega_n s + \omega_n^2)$ for $\zeta \in [0, 1]$, then $\bar{G}_h(z)$ has its poles at $z = e^{-\zeta\omega_n h} e^{\pm j\sqrt{1-\zeta^2}\omega_n h}$. Constant ζ and $\omega_n h$ contours are

and

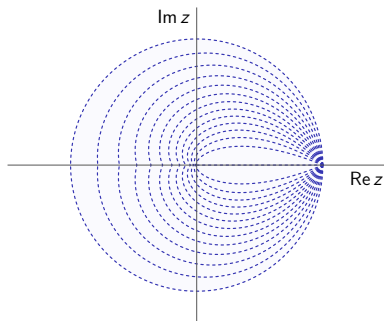
Transient performance and poles

Messier, e.g. discrete 1-order systems can exhibit oscillations and the role of zeros is not clear. So normally understood via discretized models.

Because $\mathbb{1} = \mathcal{H}_{\text{ZOH}} \bar{\mathbb{1}}$, we have $\mathcal{S}_{\text{idl}} G \mathbb{1} = \bar{G}_h \bar{\mathbb{1}}$, i.e. the

- step response of the discrete $\bar{G}_h$ is the sampled version of that of G .

If $G(s) = \omega_n^2 / (s^2 + 2\zeta\omega_n s + \omega_n^2)$ for $\zeta \in [0, 1]$, then $\bar{G}_h(z)$ has its poles at $z = e^{-\zeta\omega_n h} e^{\pm j\sqrt{1-\zeta^2}\omega_n h}$. Constant ζ and $\omega_n h$ contours are



and

