

Sampled-Data Control Systems (036709)

lecture no. 3

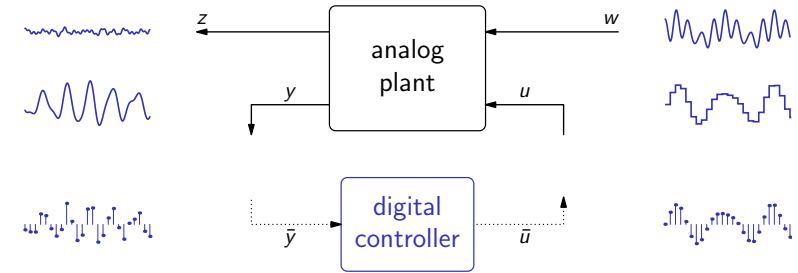
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Sampled-data control systems

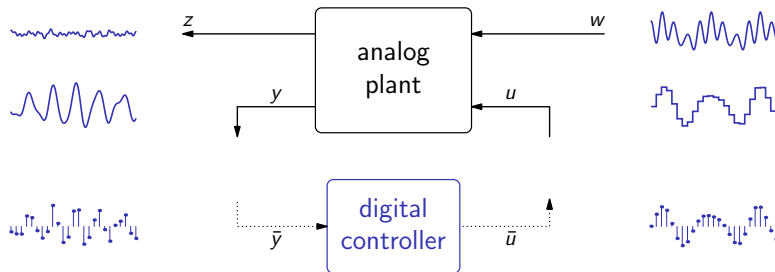


Provided **analog** and **digital** components are understood, the next question is

- how do they talk to each other?

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Synchronizing continuous and discrete times



Times of the plant, $t \in \mathbb{R}$, and the controller, $i \in \mathbb{Z}$, are linked via the

- sequence of **sampling instances**, $\{t_i\}_{i \in \mathbb{Z}}$ (should be increasing), which establishes a one-to-one correspondence between them. The intervals (t_i, t_{i+1}) are dubbed **intersample** time. We say that
 - sampling is uniform (periodic) if $t_i = ih$ for a *sampling period* $h > 0$
 - sampling is aperiodic if *sampling intervals* $h_i := t_{i+1} - t_i$ vary with i
 - sampling is Zeno if there are infinitely many t_i 's in some finite $\mathbb{I} \subset \mathbb{R}$

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Outline

A/D conversion in time domain

D/A conversion in time domain

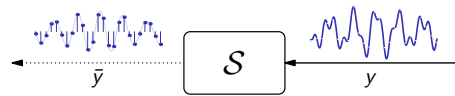
A/D conversion in frequency domain

D/A conversion in frequency domain

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Sampling

Analog-to-digital (A/D) converters, aka **samplers**, are devices transforming continuous-time signals into discrete sequences:



Conceptually, samplers are systems, $\mathcal{S} : y \mapsto \bar{y}$, just with

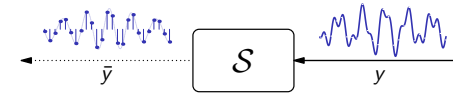
- different time variables for input and output signals.

Such systems are known as hybrid.

Remark: Dimensions of $y(t)$ and $\bar{y}[i]$ need not coincide in general.

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Sampling: basic properties

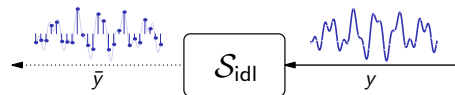


$\mathcal{S} : y \mapsto \bar{y}$ is said to be

- **linear** if $\mathcal{S}(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1(\mathcal{S}y_1) + \alpha_2(\mathcal{S}y_2)$ (superposition)
- **shift invariant** if
 1. sampling is uniform (say, with a sampling period $h > 0$)
 2. h -shift of y results in the unit shift of $\bar{y}$
- **causal** if $\bar{y}[i]$ may only depend on $y(t)$ for $t \leq t_i$
- **L_p -stable** if
 1. $y \in L_p \implies \bar{y} \in \ell_p$
 2. $\|\bar{y}\|_p \leq \gamma \|y\|_p$ for all $y \in L_p$ and some $\gamma \geq 0$ independent of y .
- **static** (or memoryless) if $\bar{y}[i]$ may only depend on $y(t)$ for $t \in [t_{i-1}, t_i]$

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The ideal sampler



Acts as

$$\bar{y}[i] = y(t_i)$$

and requires the *continuity* of $y(t)$ at sampling instances. $\mathcal{S}_{\text{idl}}$ is

- linear
- shift invariant if $t_i = ih$
- causal
- L_∞ -stable
- L_2 -**unstable**, strangely enough
- static

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The ideal sampler: L_2 stability

Bump function:

$$f_{\text{bump}}(t) = \begin{cases} e^{t^2/(t^2-1)} & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases} = \text{graph of a bump function on } [-1, 1]$$

(smooth). Let now

$$y(t) = \sum_{i \in \mathbb{N}} f_{\text{bump}}\left(\frac{t-i}{(i+1)^{1.1}}\right) = \text{graph of a sum of bumps}$$

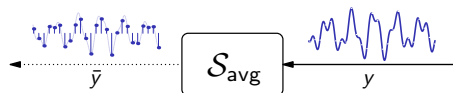
for which

$$\begin{aligned} \|y\|_2^2 &= \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} [f(\frac{t-i}{(i+1)^{1.1}})]^2 dt = \sum_{i \in \mathbb{N}} \frac{1}{(i+1)^{1.1}} \int_{\mathbb{R}} [f(t-i)]^2 dt \\ &= \sum_{i \in \mathbb{N}} \frac{\|f_{\text{bump}}\|_2^2}{(i+1)^{1.1}} \approx 9.42516 < \infty, \end{aligned}$$

so $y \in L_2(\mathbb{R})$. If $t_i = i$, then $\mathcal{S}_{\text{idl}} y = q^{-1} \mathbb{1} \notin \ell_2(\mathbb{Z})$, $\implies \mathcal{S}_{\text{idl}}$ is **unstable**.

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The averaging sampler



Acts as

$$\bar{y}[i] = \frac{1}{\Delta} \int_{t_i - \Delta}^{t_i} y(s) ds$$

and does not require the continuity of $y(t)$ at sampling instances. S_{avg} is

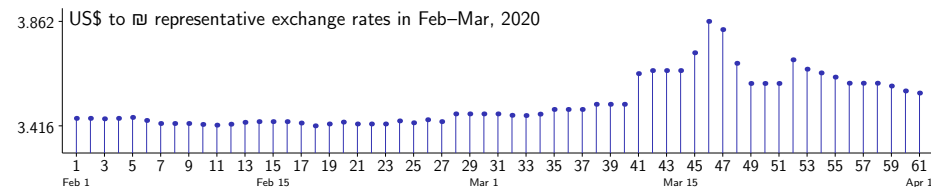
- linear
- shift invariant if $t_i = ih$
- causal
- stable, both L_2 and L_∞
- static iff $\Delta \leq \inf_i h_i$

(homework)

As $\Delta \rightarrow 0$, S_{avg} approaches S_{idl} .

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The Bol sampler



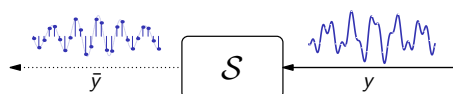
Each representative rate is calculated as

- average of randomly taken samples of banks rates in the last two hours prior publishing (either 3:15pm or 12:15pm), excluding those deviating from the sample average by more than two standard deviations,
- the same as in the previous day on Saturdays, Sundays, and holidays,
- exercising discretion in exceptional cases.

This sampler is nonlinear, time-varying, causal, static, and *perhaps* stable.

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General linear shift-invariant sampler



With $t_i = ih$, it acts as

$$\bar{y}[i] = \int_{\mathbb{R}} \psi(ih - s) y(s) ds,$$

for a **sampling function** $\psi(t)$ (thus psi). $\psi(ih)$ is the response of S , at the time instance i , to the input δ -impulse applied at the time instance $t = 0$.

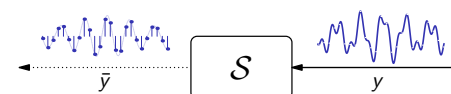
Special cases:

$$S_{idl}: \psi(t) = \delta(t)$$

$$S_{avg}: \psi(t) = \frac{1}{\Delta} \text{rect}\left(\frac{t}{\Delta}\right) = (\mathbb{1}(t) - \mathbb{1}(t - \Delta)) / \Delta$$

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General LSI sampler: generation via S_{idl}



Clearly,

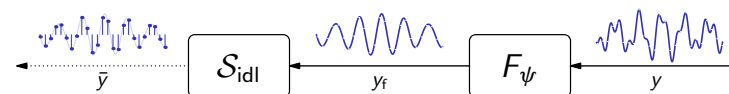
$$\bar{y}[i] = \int_{\mathbb{R}} \psi(ih - s) y(s) ds$$

iff

$$\bar{y}[i] = y_f(ih), \quad \text{where } y_f(t) = \int_{\mathbb{R}} \psi(t - s) y(s) ds.$$

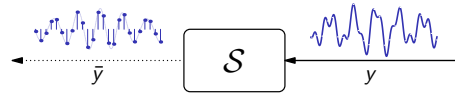
Hence, any generalized sampler is the

- cascade of an analog LTI filter F_ψ with the impulse response $\psi(t)$ and the ideal sampler:



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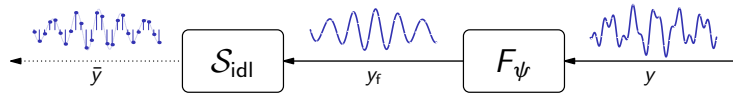
General LSI sampler: causality



As

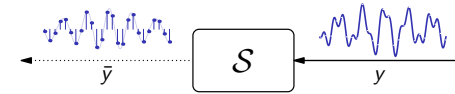
$$\bar{y}[i] = \int_{-\infty}^{ih} \psi(ih - s)y(s)ds + \int_{ih}^{\infty} \psi(ih - s)y(s)ds,$$

the causality of $\mathcal{S}$ is equivalent to $\psi(t) = 0$ for all $t < 0$, i.e. to that of F_ψ in



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Static general LSI sampler



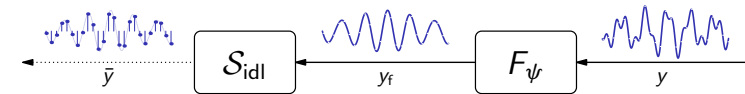
As

$$\begin{aligned} \bar{y}[i] = & \int_{-\infty}^{(i-1)h} \psi(ih - s)y(s)ds \\ & + \int_{(i-1)h}^{ih} \psi(ih - s)y(s)ds + \int_{ih}^{\infty} \psi(ih - s)y(s)ds, \end{aligned}$$

$\mathcal{S}$ is static iff

$$\psi(t) = 0 \text{ whenever } t \notin [0, h],$$

which normally corresponds to a causal FIR (finite impulse response) F_ψ in



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Outline

A/D conversion in time domain

D/A conversion in time domain

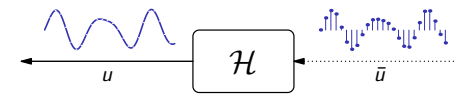
A/D conversion in frequency domain

D/A conversion in frequency domain

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Holds

Digital-to-analog (D/A) converters, aka **holds** or interpolators, are devices transforming discrete sequences into continuous-time signals:



Conceptually, holds are systems, $\mathcal{H} : \bar{u} \mapsto u$, just with

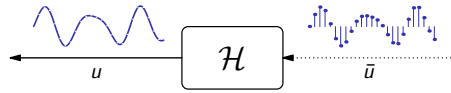
- different time variables for input and output signals.

Such systems are also known as hybrid.

Remark: Dimensions of $\bar{u}[i]$ and $u(t)$ need not coincide in general.

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Hold: basic properties



$\mathcal{H} : \bar{u} \mapsto u$ is said to be

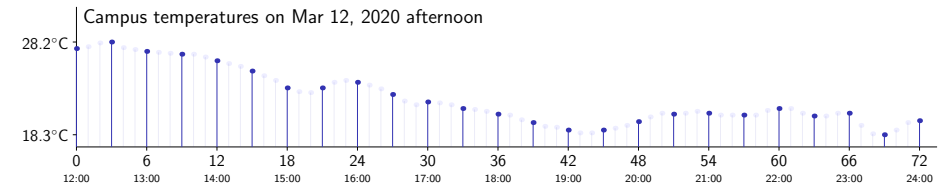
- **linear** if $\mathcal{H}(\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2) = \alpha_1(\mathcal{H}\bar{u}_1) + \alpha_2(\mathcal{H}\bar{u}_2)$ (superposition)
- **shift invariant** if
 1. sampling is uniform (say, with a sampling period $h > 0$)
 2. the unit shift of $\bar{u}$ results in the h -shift of u
- **causal** if $u(t)$ may only depend on $\bar{u}[i]$ for i such that $t_i \leq t$
- **ℓ_p -stable** if
 1. $\bar{u} \in \ell_p \implies u \in L_p$
 2. $\|u\|_p \leq \gamma \|\bar{u}\|_p$ for all $\bar{u} \in \ell_p$ and some $\gamma \geq 0$ independent of $\bar{u}$.
- **static** (or memoryless) if $u(t)$ for $t \in [t_i, t_{i+1}]$ may only depend on $\bar{u}[i]$

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Reconstruction problem

Return to the temperature measurements, but now assume that

- all measurements but those taken every 0.5 hour are lost:

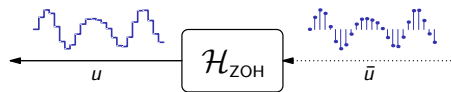


The question is

- how to reconstruct missing values (and even those never measured)?

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The zero-order hold



Acts as

$$u(t) = \bar{u}[i], \quad \forall t \in [t_i, t_{i+1})$$

It is

- linear
- shift invariant if $t_i = ih$
- causal
- stable, both L_∞ , as $\|u\|_\infty = \|\bar{u}\|_\infty$, and L_2 , as

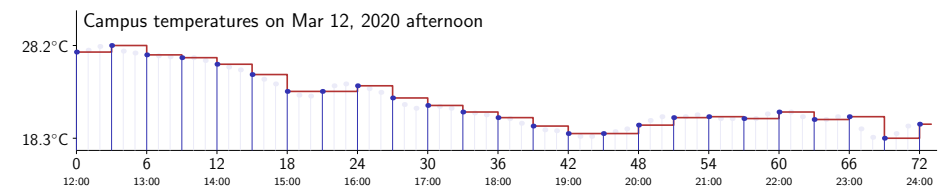
$$\|u\|_2^2 = \int_{\mathbb{R}} \|u(t)\|^2 dt = \sum_{i \in \mathbb{Z}} \int_{t_i}^{t_{i+1}} \|u(t)\|^2 dt = \sum_{i \in \mathbb{Z}} h_i \|\bar{u}[i]\|^2 \leq \left(\sup_i h_i \right) \|\bar{u}\|_2^2$$

- static

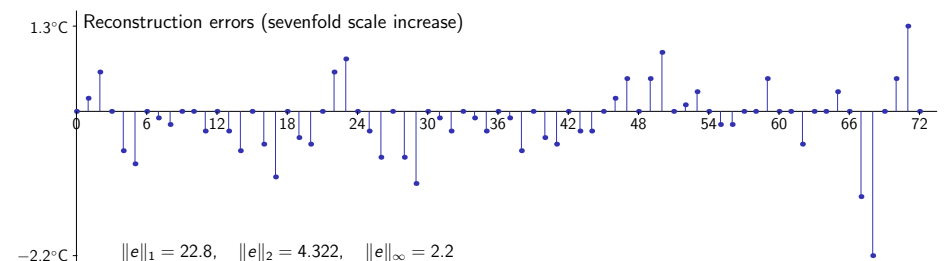
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The zero-order hold: example

Continuous-time temperature reconstructed by $\mathcal{H}_{\text{ZOH}}$ is

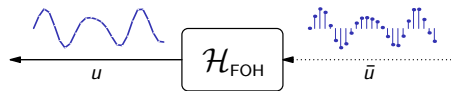


and the errors at points that we can verify are



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The first-order hold



Acts as

$$u(t) = \frac{t_{i+1} - t}{h_i} \bar{u}[i] + \frac{t - t_i}{h_i} \bar{u}[i + 1], \quad \forall t \in [t_i, t_{i+1}]$$

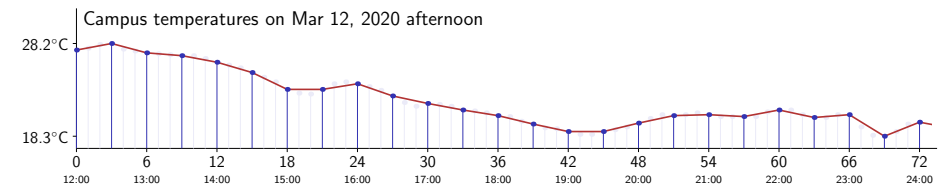
(linear interpolation between $\bar{u}[i]$ and $\bar{u}[i + 1]$ for each $i \in \mathbb{Z}$). It is

- linear
- shift invariant if $t_i = ih$
- **non-causal**
(delayed first-order hold, $u(t) = \frac{t_{i+1} - t}{h_i} \bar{u}[i - 1] + \frac{t - t_i}{h_i} \bar{u}[i]$, is causal)
- stable, both L_∞ and L_2
- non-static (i.e. dynamic)

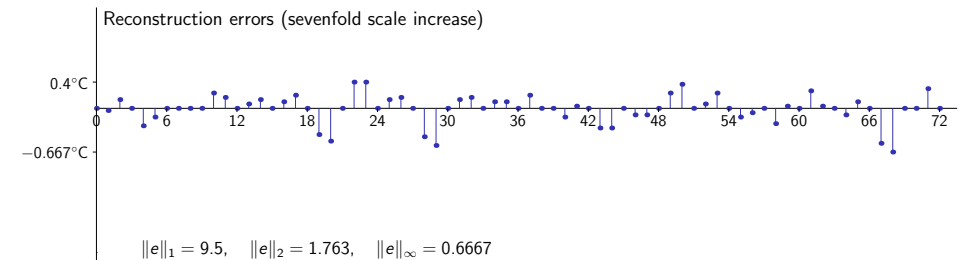
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The first-order hold: example

Continuous-time temperature reconstructed by $\mathcal{H}_{\text{FOH}}$ is

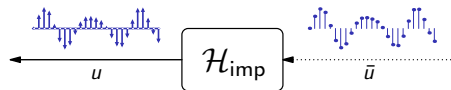


and the errors at points that we can verify are



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The impulse hold



Acts as

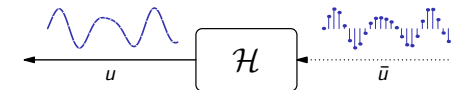
$$u(t) = \delta(t - t_i) \bar{u}[i], \quad \forall t \in \mathbb{R}$$

It is

- linear
- shift invariant if $t_i = ih$
- causal
- unstable
- static
- not quite practical by itself...

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General linear shift-invariant hold



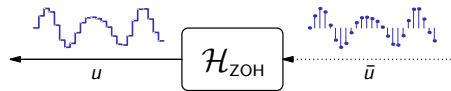
With $t_i = ih$, it acts as

$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$$

for a **hold function** $\phi(t)$ (thus phi). $\phi(t)$ is the response of $\mathcal{H}$, at the time instance t , to the unit pulse applied at the time instance $i = 0$.

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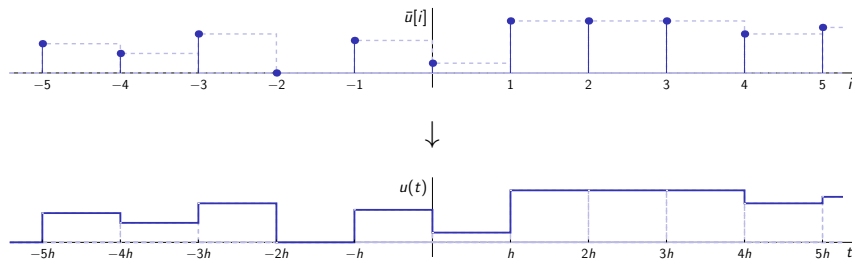
$\mathcal{H}_{\text{ZOH}}$ as a general LSI hold



is a general hold with the hold function

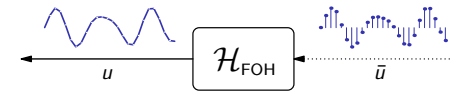
$$\phi(t) = \mathbb{1}(t) - \mathbb{1}(t - h) = \begin{cases} 1 & 0 \leq t < h \\ 0 & \text{otherwise} \end{cases}$$

The D/A convolution $\sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$ acts in this case as



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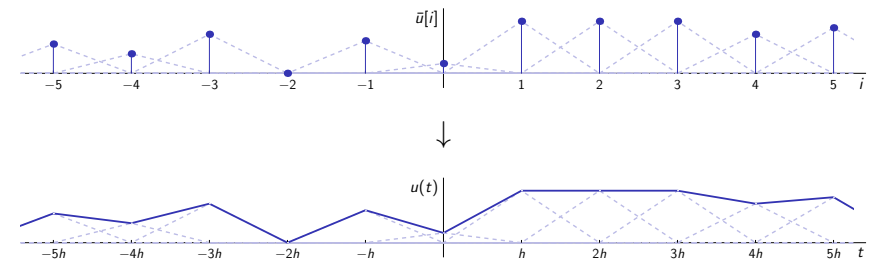
$\mathcal{H}_{\text{FOH}}$ as a general LSI hold



is a general hold with the hold function

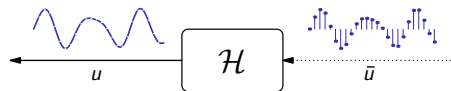
$$\phi(t) = \frac{t+h}{h} \mathbb{1}(t+h) - \frac{2t}{h} \mathbb{1}(t) + \frac{t-h}{h} \mathbb{1}(t-h) = \begin{cases} \frac{t+h}{h} & -h \leq t < 0 \\ -\frac{2t}{h} & 0 \leq t < h \\ \frac{t-h}{h} & h \leq t < 2h \\ 0 & \text{otherwise} \end{cases}$$

The D/A convolution $\sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$ acts in this case as



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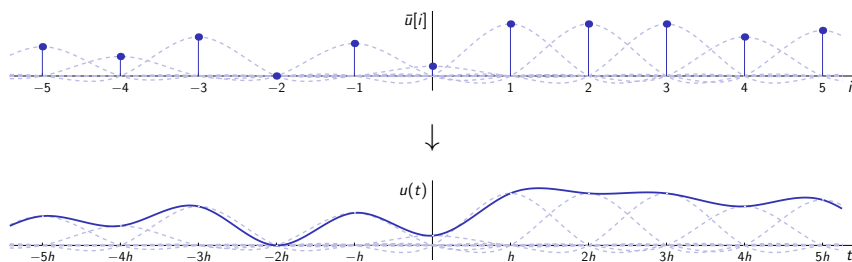
Cardinal cubic splines as a general LSI hold



is a general hold with the hold function

$$\phi(t) = \begin{cases} \frac{1}{6} \left(\frac{t}{h} \right)^3 - \frac{1}{2} \left(\frac{t}{h} \right)^2 + \frac{1}{6} \left(\frac{t}{h} \right) & -h \leq t < 0 \\ \frac{1}{6} \left(\frac{t}{h} \right)^3 - \frac{1}{2} \left(\frac{t}{h} \right)^2 + \frac{1}{6} \left(\frac{t}{h} \right) & 0 \leq t < h \\ 0 & \text{otherwise} \end{cases}$$

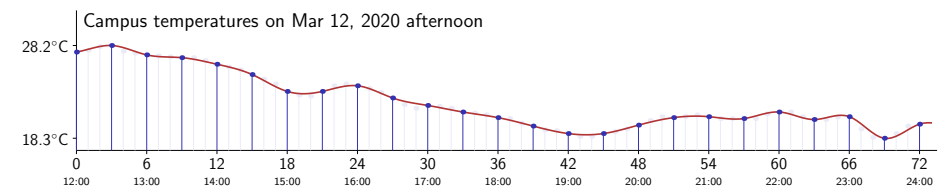
The D/A convolution $\sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$ acts in this case as



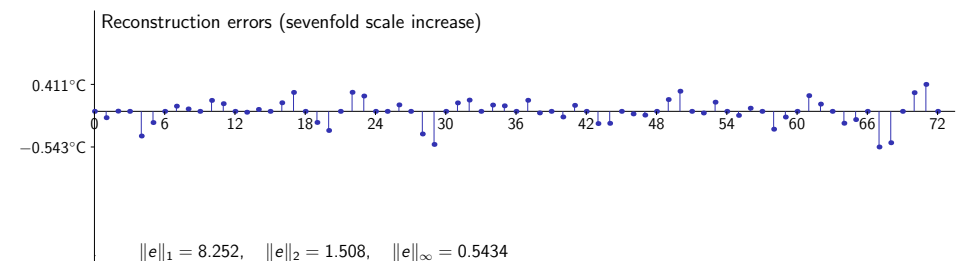
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Cardinal cubic splines: example

Continuous-time temperature reconstructed by cardinal cubic splines is

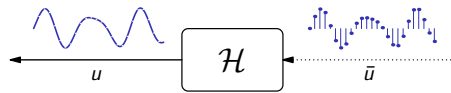


and the errors at points that we can verify are



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General LSI hold: generation via $\mathcal{H}_{\text{imp}}$



It can be shown that

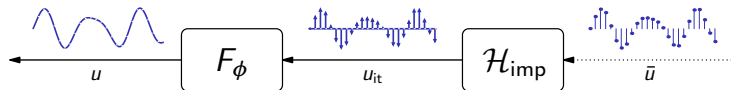
$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$$

iff

$$u(t) = \int_{\mathbb{R}} \phi(t - s) u_{\text{it}}(s) ds, \quad \text{where } u_{\text{it}}(s) = \sum_{i \in \mathbb{Z}} \delta(s - ih) \bar{u}[i].$$

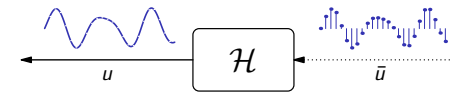
Hence, any generalized sampler is the

- cascade of the impulse hold and an analog LTI filter F_ϕ with the impulse response $\phi(t)$:



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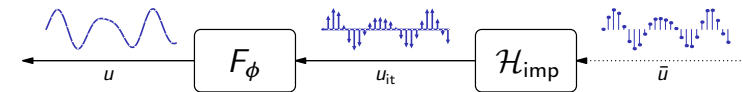
General LSI hold: causality



As

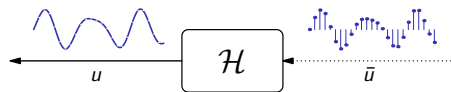
$$u(t) = \sum_{j=-\infty}^i \phi(t - jh) \bar{u}[j] + \sum_{j=i+1}^{\infty} \phi(t - jh) \bar{u}[j], \quad \forall t \in [ih, (i+1)h]$$

the causality of $\mathcal{H}$ is equivalent to $\phi(t) = 0$ for all $t < 0$, i.e. to that of F_ϕ in



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Static general LSI hold



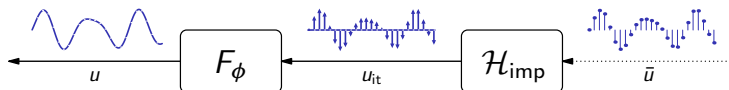
As

$$u(t) = \sum_{j=-\infty}^{i-1} \phi(t - jh) \bar{u}[j] + \phi(t - ih) \bar{u}[i] + \sum_{j=i+1}^{\infty} \phi(t - jh) \bar{u}[j]$$

for all $t \in [ih, (i+1)h]$, a hold is static iff

$$\phi(t) = 0 \text{ whenever } t \notin [0, h],$$

which normally corresponds to a causal FIR (finite impulse response) F_ϕ in



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Outline

A/D conversion in time domain

D/A conversion in time domain

A/D conversion in frequency domain

D/A conversion in frequency domain

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What do we lose by sampling analog signals?



From the time-domain relation

$$\bar{y}[i] = y(ih)$$

we know that all **intersample information** about $y(t)$ is **lost**. But

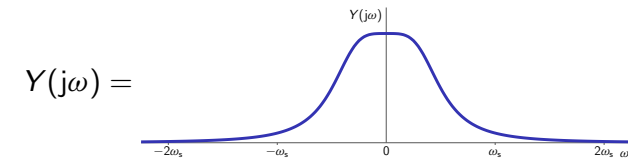
- to what extent is it important (if at all)?

To answer this kind of questions, frequency-domain analysis is indispensable.

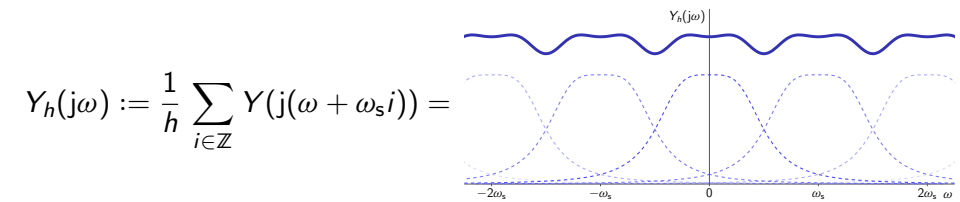
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A weird function

Consider an analog signal $y(t)$ with the spectrum $Y(j\omega)$, e.g.



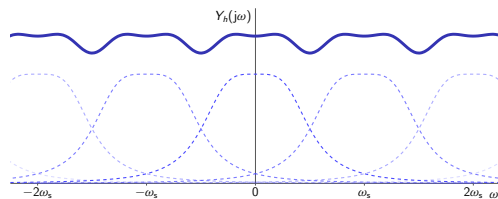
and define, for some $h > 0$, the function



where $\omega_s := 2\pi/h$ (in rad/sec). Note that the mapping $Y \mapsto Y_h$ is linear.

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A weird function: Fourier series



As $Y_h(j\omega)$ is ω_s -periodic, we can bring in its Fourier series expansion

$$Y_h(j\omega) = \sum_{i \in \mathbb{Z}} c_i e^{j \frac{2\pi}{\omega_s} i \omega} = \sum_{i \in \mathbb{Z}} c_i e^{j \omega h i},$$

where Fourier coefficients are calculated as

$$c_i = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} Y_h(j\omega) e^{-j \omega h i} d\omega.$$

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A weird function: Fourier coefficients

With some extra efforts:

$$\begin{aligned} c_i &= \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} \frac{1}{h} \sum_{k \in \mathbb{Z}} Y(j(\omega + \omega_s k)) e^{-j \omega h i} d\omega \\ &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{-\omega_s/2}^{\omega_s/2} Y(j(\omega + \omega_s k)) e^{-j(\omega + \omega_s k) h i} d\omega \\ &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{-\omega_s/2 + \omega_s k}^{\omega_s/2 + \omega_s k} Y(j\omega) e^{-j \omega h i} d\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} Y(j\omega) e^{-j \omega h i} d\omega \quad \left(\text{remember, } y(t) = \frac{1}{2\pi} \int_{\mathbb{R}} Y(j\omega) e^{j \omega t} d\omega \right) \\ &= y(-ih). \end{aligned}$$

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A weird function: Fourier series (contd)

Thus, we end up with

$$Y_h(j\omega) = \sum_{i \in \mathbb{Z}} y(-ih) e^{j\omega hi} = \sum_{i \in \mathbb{Z}} y(ih) e^{-j\omega hi}.$$

Compare it with the discrete-time Fourier transform of $\bar{y} = \mathcal{S}_{\text{idl}} y$,

$$\bar{Y}(e^{j\theta}) = \sum_{i \in \mathbb{Z}} \bar{y}[i] e^{-j\theta i} = \sum_{i \in \mathbb{Z}} y(ih) e^{-j\theta i}.$$

We can therefore say that

$$Y_h(j\omega) = \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\omega + \omega_s i)), \quad \text{where } \omega_s = \frac{2\pi}{h} \text{ (the sampling frequency)}$$

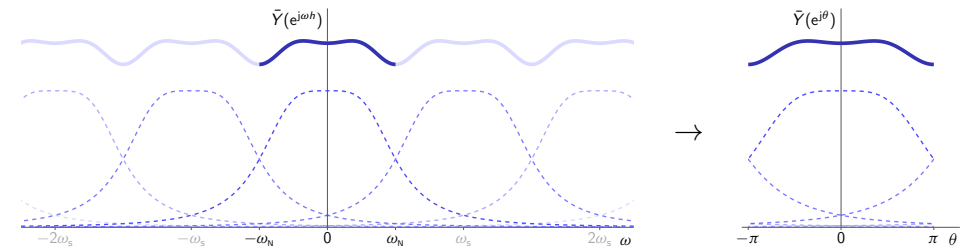
is the DTFT of the sampled signal $\bar{y}$ modulo scaling, $\theta = \omega h$, i.e.

$$\bar{Y}(e^{j\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\theta/h + \omega_s i)) =: Y_h(j\theta/h).$$

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Spectrum of sampled signal

We thus end up with



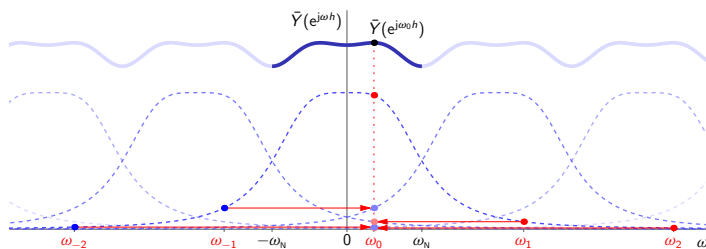
The frequency

$$\omega_N := \frac{\omega_s}{2} = \frac{\pi}{h}$$

is called the **Nyquist frequency** associated with the sampling period h (it is measured in rad/sec if h is in sec).

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Spectrum of sampled signal: aliasing



Thus, the spectrum of $\bar{y}$ at each frequency $\theta_0 = \omega_0 h$ is a

- blend of analog frequency responses at $\omega_i := \omega_0 + \omega_s i$, $\forall i \in \mathbb{Z}$.

In other words,

- every discrete frequency $\theta_0 \in [-\pi, \pi]$ is an **alias** of all ω_i , $i \in \mathbb{Z}$.

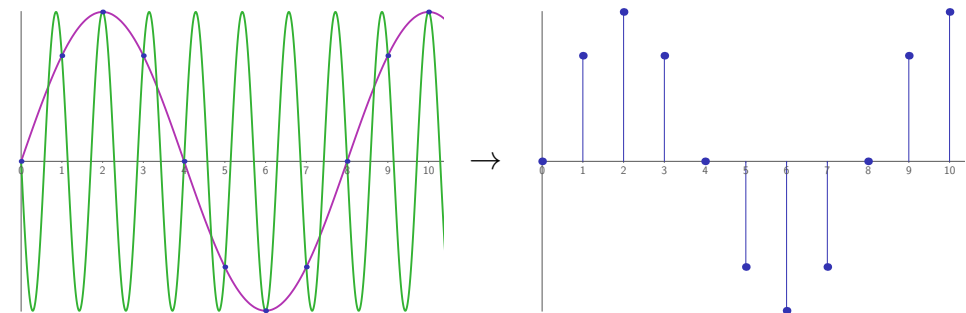
This phenomenon is dubbed **aliasing**, with respect to the **base frequency** ω_0 .

Aliasing means **information loss**, we can no longer tell $Y(j\omega_i)$ from $Y(j\omega_j)$ in their effect on $\bar{Y}(e^{j\theta_0})$ (unless we know their dependencies).

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Aliasing: example

Consider signals $y_1(t) = \sin(\frac{\pi}{4}t)$ and $y_2(t) = \sin(-\frac{7\pi}{4}t)$ sampled at $h = 1$:



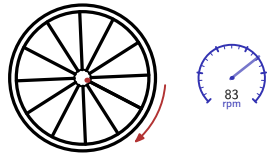
Sampling frequency is $\omega_s = 2\pi$, so that

- both $\omega_0 = \frac{\pi}{4}$ and $\omega_{-1} = -\frac{7\pi}{4} = \omega_0 - \omega_s$ have aliases at $\theta_0 = \omega_0$ and, consequently, produce the same sampled signal.

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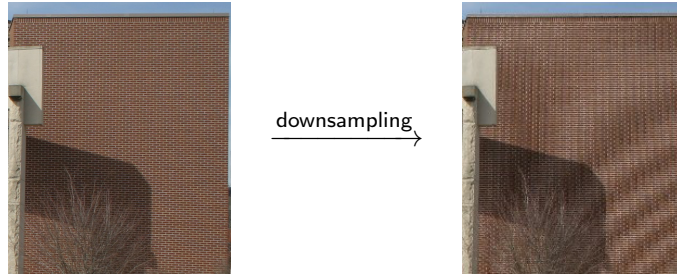
Aliasing: non-control examples

Wagon-wheel effect:



(shot with 12 FPS frame rate)

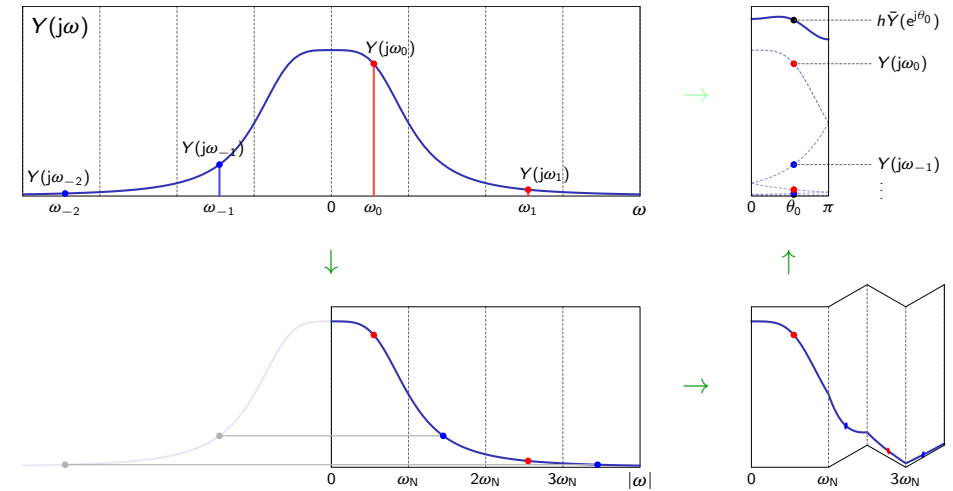
Moiré pattern:



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Frequency folding

If $Y(j\omega) = Y(-j\omega) \in \mathbb{R}, \forall \omega$, the spectrum of its sampled version, $\bar{Y}(e^{j\theta})$, can be constructed also via the following **folding** procedure:



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L_2 -instability of the ideal sampler, again

Let $y(t)$ be a signal with

$$Y(j\omega) = \frac{\sqrt{2}}{\sqrt{\omega^2 + 1}}.$$

By Parseval,

$$\|y\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2}{\omega^2 + 1} d\omega = 1,$$

so y is unit-energy (so, bounded) signal. Frequency response of $\bar{y} = \mathcal{S}_{\text{idl}} y$ is

$$\bar{Y}(e^{j\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \frac{\sqrt{2}}{\sqrt{(\theta/h + 2\pi/h i)^2 + 1}} = \sum_{i \in \mathbb{Z}} \frac{\sqrt{2}}{\sqrt{(\theta + 2\pi i)^2 + h^2}} = \infty$$

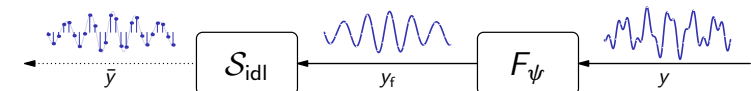
for every $\theta \in [-\pi, \pi]$. This means that $\bar{y}$ is unbounded, i.e. that

- the ideal sampler $\mathcal{S}_{\text{idl}}$ is L_2 -unstable (yet another proof of that)

Remark: $\mathcal{S}_{\text{idl}}$ does produce finite-energy discrete signals from analog inputs, whose spectra decay faster than $1/|\omega|$ at high frequencies.

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General LSI sampler



Just in two steps:

- filter y by the LTI F_ψ , done frequency-by-frequency
- apply the analysis above to $Y_f(j\omega) = F_\psi(j\omega)Y(j\omega)$ instead of $Y(j\omega)$

Example

For the averaging sampler, whose $\psi(t) = (\mathbb{1}(t) - \mathbb{1}(t - \Delta))/\Delta$,

$$F_\psi(s) = \frac{1 - e^{-\Delta s}}{\Delta s} \rightarrow \frac{1}{\Delta} \frac{1 - e^{-j\omega\Delta}}{j\omega} \rightarrow \frac{1}{\Delta} \frac{2 \sin(\omega\Delta/2)}{\omega}$$

This is a low-pass filter, whose high-frequency gain decays as $1/|\omega|$. Hence, $\mathcal{S}_{\text{avg}}$ is L_2 -stable. ✓

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Outline

A/D conversion in time domain

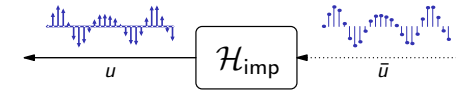
D/A conversion in time domain

A/D conversion in frequency domain

D/A conversion in frequency domain

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Spectrum of signals reconstructed by $\mathcal{H}_{\text{imp}}$



The Fourier transform of this $u(t)$ is

$$\begin{aligned} U(j\omega) &= \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{i \in \mathbb{Z}} \delta(t - ih) \bar{u}[i] e^{-j\omega t} dt \\ &= \sum_{i \in \mathbb{Z}} \int_{-\infty}^{\infty} \delta(t - ih) e^{-j\omega t} dt \bar{u}[i] = \sum_{i \in \mathbb{Z}} \bar{u}[i] e^{-j(\omega h)i} \\ &= \bar{U}(e^{j\omega h}) = \bar{U}(e^{j\theta})|_{\theta=\omega h}. \end{aligned}$$

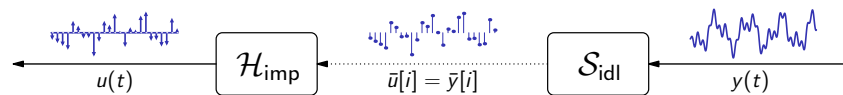
Because $\bar{U}(e^{j\omega h})$ is a $(2\pi/h)$ -periodic function of ω ,

— $\mathcal{H}_{\text{imp}}$ merely clones the spectrum of $\bar{u}$,
whose frequency θ -axis in $[-\pi, \pi]$ is scaled to fit the ω -axis in $[-\omega_N, \omega_N]$.

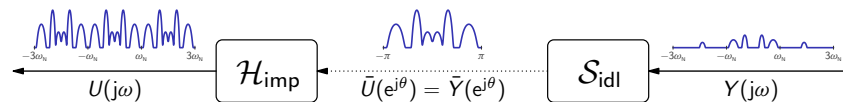
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Sample-and-hold circuit of the form $\mathcal{H}_{\text{imp}} \mathcal{S}_{\text{idl}}$

In time domain:



In frequency domain:



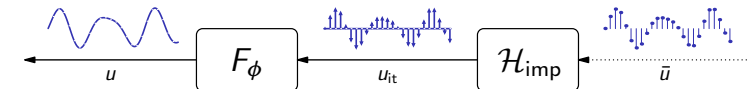
The resulting

— $U(j\omega)$ is exactly the “weird signal” used to derive $\bar{Y}(e^{j\theta})$, i.e.

$$U(j\omega) = Y_h(j\omega) := \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\omega + \omega_s i)).$$

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General LSI hold



Just in two steps, again:

1. apply the analysis above to derive $U_{\text{it}}(j\omega)$
2. filter u_{it} by the LTI F_ϕ to end up with $U(j\omega) = F_\phi(j\omega) U_{\text{it}}(j\omega)$

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General LSI hold: examples

$\mathcal{H}_{\text{ZOH}}$ has

$$\phi(t) = \mathbb{1}(t) - \mathbb{1}(t-h) = \begin{array}{c} \text{1} \\ \text{0} \quad h \quad t \end{array} \Rightarrow F_{\phi}(s) = \frac{1 - e^{-hs}}{s},$$

which corresponds to the low-pass

$$F_{\phi}(j\omega) = h \frac{\sin(\omega h/2)}{\omega h/2} e^{-j\omega h/2} = \begin{array}{c} h |F_{\phi}(j\omega)| \\ -6\omega_{Ny} \quad -4\omega_{Ny} \quad -2\omega_{Ny} \quad 0 \quad 2\omega_{Ny} \quad 4\omega_{Ny} \quad 6\omega_{Ny} \quad \omega \end{array}$$

$\mathcal{H}_{\text{FOH}}$ has

$$\phi(t) = \begin{array}{c} \text{1} \\ -h \quad 0 \quad h \quad t \end{array} = \frac{1}{h} \left(\begin{array}{c} \text{1} \\ -h/2 \quad h/2 \quad t \end{array} * \begin{array}{c} \text{1} \\ -h/2 \quad h/2 \quad t \end{array} \right) \Rightarrow F_{\phi}(s) = \frac{1}{h} \left(\frac{e^{sh/2} - e^{-hs/2}}{s} \right)^2,$$

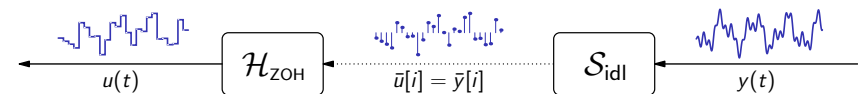
which corresponds to the low-pass

$$F_{\phi}(j\omega) = h \left(\frac{\sin(\omega h/2)}{\omega h/2} \right)^2 = \begin{array}{c} h |F_{\phi}(j\omega)| \\ -6\omega_{Ny} \quad -4\omega_{Ny} \quad -2\omega_{Ny} \quad 0 \quad 2\omega_{Ny} \quad 4\omega_{Ny} \quad 6\omega_{Ny} \quad \omega \end{array}$$

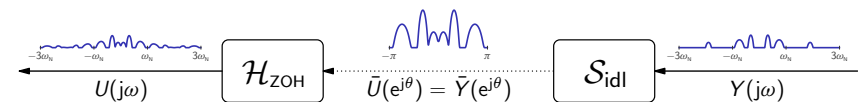
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Sample-and-hold circuit of the form $\mathcal{H}_{\text{ZOH}}\mathcal{S}_{\text{idl}}$

In time domain:



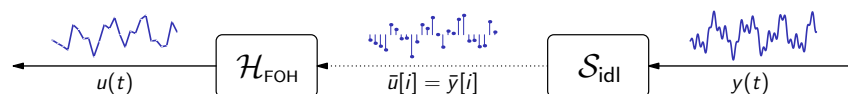
In frequency domain:



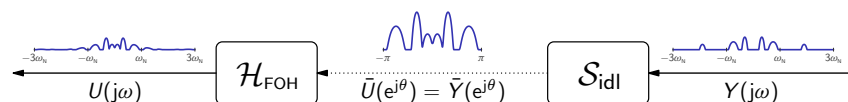
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Sample-and-hold circuit of the form $\mathcal{H}_{\text{FOH}}\mathcal{S}_{\text{idl}}$

In time domain:



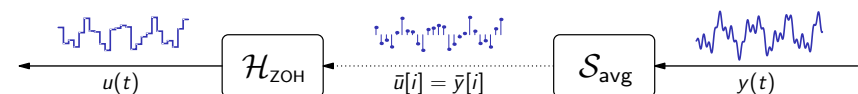
In frequency domain:



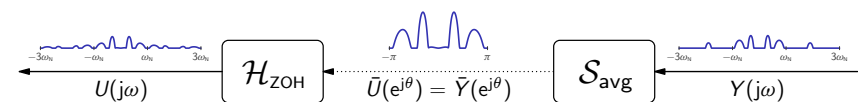
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Sample-and-hold circuit of the form $\mathcal{H}_{\text{ZOH}}\mathcal{S}_{\text{avg}}$ with $\Delta = h$

In time domain:



In frequency domain:



where

$$F_{\psi}(s) = \frac{1 - e^{-hs}}{hs} \rightarrow \begin{array}{c} 1 |F_{\psi}(j\omega)| \\ -6\pi/\Delta \quad -4\pi/\Delta \quad -2\pi/\Delta \quad 0 \quad 2\pi/\Delta \quad 4\pi/\Delta \quad 6\pi/\Delta \quad \omega \end{array}$$

is low pass, so aliased values are smaller.

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