

Sampled-Data Control Systems (036709)

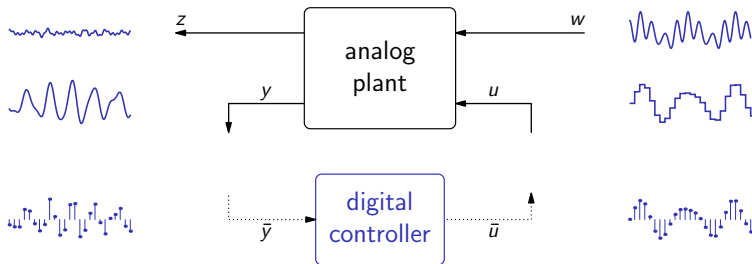
lecture no. 3

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Technion — IIT



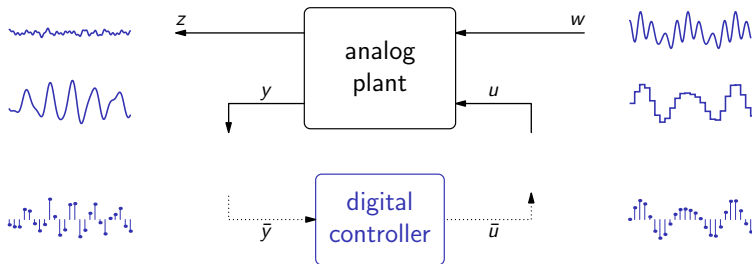
Sampled-data control systems



Provided **analog** and **digital** components are understood, the next question is

- how do they talk to each other?

Synchronizing continuous and discrete times



Times of the plant, $t \in \mathbb{R}$, and the controller, $i \in \mathbb{Z}$, are linked via the

- sequence of **sampling instances**, $\{t_i\}_{i \in \mathbb{Z}}$ (should be increasing),

which establishes a one-to-one correspondence between them. The intervals (t_i, t_{i+1}) are dubbed **intersample** time. We say that

- sampling is uniform (periodic) if $t_i = ih$ for a *sampling period* $h > 0$
- sampling is aperiodic if *sampling intervals* $h_i := t_{i+1} - t_i$ vary with i
- sampling is Zeno if there are infinitely many t_i 's in some finite $\mathbb{I} \subset \mathbb{R}$

Outline

A/D conversion in time domain

D/A conversion in time domain

A/D conversion in frequency domain

D/A conversion in frequency domain

Outline

A/D conversion in time domain

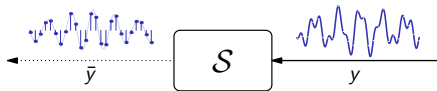
D/A conversion in time domain

A/D conversion in frequency domain

D/A conversion in frequency domain

Sampling

Analog-to-digital (A/D) converters, aka **samplers**, are devices transforming continuous-time signals into discrete sequences:



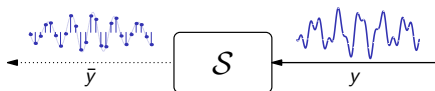
Conceptually, samplers are systems, $\mathcal{S} : y \mapsto \bar{y}$, just with

- different time variables for input and output signals.

Such systems are known as hybrid.

Remark: Dimensions of $y(t)$ and $\bar{y}[i]$ need not coincide in general.

Sampling: basic properties



$\mathcal{S} : y \mapsto \bar{y}$ is said to be

- **linear** if $\mathcal{S}(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 (\mathcal{S} y_1) + \alpha_2 (\mathcal{S} y_2)$ (superposition)

- shift invariant if

1. sampling is uniform (i.e., with a sampling period $h > 0$)
2. h -shift of y results in the unit shift of $\bar{y}$

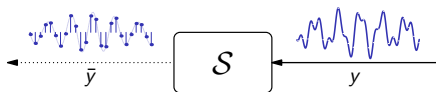
- causal if $\bar{y}[i]$ may only depend on $y(t)$ for $t \leq t_i$

- L_p -stable if

1. $y \in L_p \implies \bar{y} \in L_p$
2. $\|\bar{y}\|_p \leq \gamma \|y\|_p$ for all $y \in L_p$ and some $\gamma \geq 0$ independent of y .

- static (or memoryless) if $\bar{y}[i]$ may only depend on $y(t)$ for $t \in [t_{i-1}, t_i]$

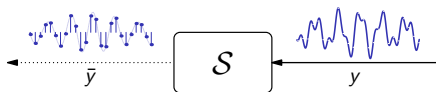
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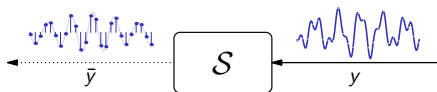
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$S : y \mapsto \bar{y}$ is said to be

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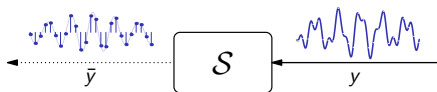
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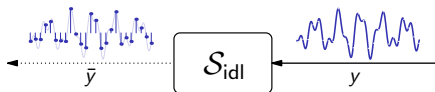
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The ideal sampler



Acts as

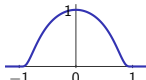
$$\bar{y}[i] = y(t_i)$$

and requires the *continuity* of $y(t)$ at sampling instances. S_{idl} is

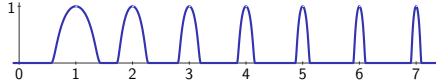
- linear
- shift invariant if $t_i = ih$
- causal
- L_∞ -stable
- L_2 -**un**stable, strangely enough
- static

The ideal sampler: L_2 stability

Bump function:

$$f_{\text{bump}}(t) = \begin{cases} e^{t^2/(t^2-1)} & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases} = \text{graph}$$


(smooth). Let now

$$y(t) = \sum_{i \in \mathbb{N}} f_{\text{bump}}\left(\frac{t-i}{(i+1)^{1.1}}\right) = \text{graph}$$


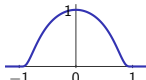
for which

$$\begin{aligned} \|y\|_2^2 &= \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} \left[f\left(\frac{t-i}{(i+1)^{1.1}}\right) \right]^2 dt = \sum_{i \in \mathbb{N}} \frac{1}{(i+1)^{1.1}} \int_{\mathbb{R}} [f(t-i)]^2 dt \\ &= \sum_{i \in \mathbb{N}} \frac{\|f_{\text{bump}}\|_2^2}{(i+1)^{1.1}} \approx 9.42516 < \infty, \end{aligned}$$

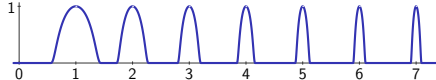
so $y \in L_2(\mathbb{R})$. If $t_i = i$, then $S_{\Delta t} y = q^{-1} x \notin \ell_2(\mathbb{Z})$. $\Rightarrow S_{\Delta t}$ is unstable.

The ideal sampler: L_2 stability

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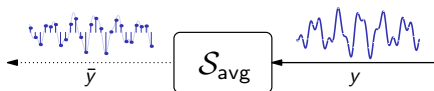
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so $y \in L_2(\mathbb{R})$. If $t_i = i$, then $\mathcal{S}_{\text{idl}} y = q^{-1} \mathbb{1} \notin \ell_2(\mathbb{Z})$, $\implies \mathcal{S}_{\text{idl}}$ is **unstable**.

The averaging sampler



Acts as

$$\bar{y}[i] = \frac{1}{\Delta} \int_{t_i - \Delta}^{t_i} y(s) ds$$

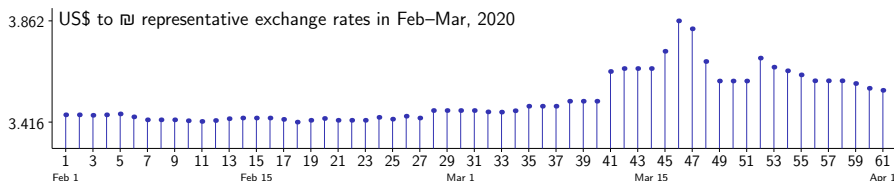
and does not require the continuity of $y(t)$ at sampling instances. $\mathcal{S}_{\text{avg}}$ is

- linear
- shift invariant if $t_i = ih$
- causal
- stable, both L_2 and L_∞
- static iff $\Delta \leq \inf_i h_i$

(homework)

As $\Delta \rightarrow 0$, $\mathcal{S}_{\text{avg}}$ approaches $\mathcal{S}_{\text{idl}}$.

The Bol sampler

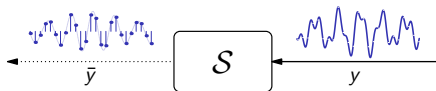


Each representative rate is calculated as

- average of randomly taken samples of banks rates in the last two hours prior publishing (either 3:15pm or 12:15pm), excluding those deviating from the sample average by more than two standard deviations,
- the same as in the previous day on Saturdays, Sundays, and holidays,
- exercising discretion in exceptional cases.

This sampler is nonlinear, time-varying, causal, static, and *perhaps* stable.

General linear shift-invariant sampler



With $t_i = ih$, it acts as

$$\bar{y}[i] = \int_{\mathbb{R}} \psi(ih - s) y(s) ds,$$

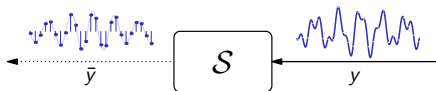
for a **sampling function** $\psi(t)$ (thus ψ). $\psi(ih)$ is the response of S , at the time instance i , to the input δ -impulse applied at the time instance $t = 0$.

Special cases:

S_{Dir} $\psi(t) = \delta(t)$

S_{avg} $\psi(t) = \frac{1(t) - 1(t - \Delta)}{\Delta}$

General linear shift-invariant sampler



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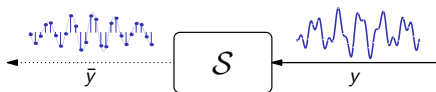
Special cases:

$\mathcal{S}_{\text{idl}}$: $\psi(t) = \delta(t)$

$\mathcal{S}_{\text{avg}}$: $\psi(t) = \frac{1}{\Delta} \text{rect}\left(\frac{t}{\Delta}\right) = (\mathbb{1}(t) - \mathbb{1}(t - \Delta)) / \Delta$

The graph shows a rectangular pulse function $\psi(t)$ with a height of $1/\Delta$ and a width of Δ , starting at $t=0$ and ending at $t=\Delta$. The horizontal axis is labeled t and has markers at 0 and Δh .

General LSI sampler: generation via $\mathcal{S}_{idl}$



Clearly,

$$\bar{y}[i] = \int_{\mathbb{R}} \psi(ih - s)y(s)ds$$

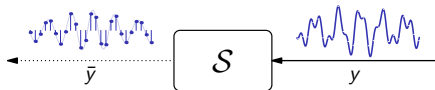
iff

$$\bar{y}[i] = y_f(ih), \quad \text{where } y_f(t) = \int_{\mathbb{R}} \psi(t - s)y(s)ds.$$

Hence, any generalized sampler is the

- cascade of an analog LTI filter F_ψ with the impulse response $\psi(t)$ and the ideal sampler.

General LSI sampler: generation via $\mathcal{S}_{idl}$



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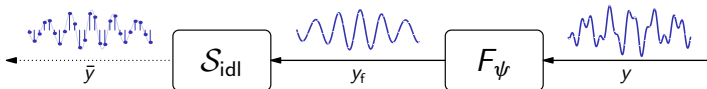
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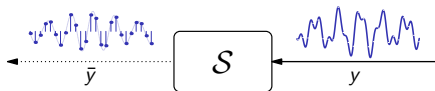
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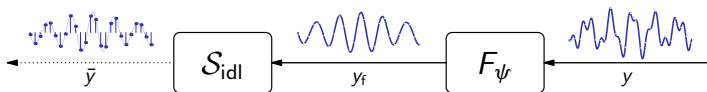
General LSI sampler: causality



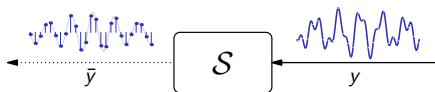
As

$$\bar{y}[i] = \int_{-\infty}^{ih} \psi(ih - s)y(s)ds + \int_{ih}^{\infty} \psi(ih - s)y(s)ds,$$

the causality of $\mathcal{S}$ is equivalent to $\psi(t) = 0$ for all $t < 0$, i.e. to that of F_ψ in



Static general LSI sampler



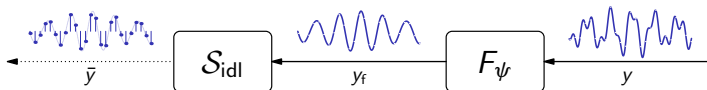
As

$$\bar{y}[i] = \int_{-\infty}^{(i-1)h} \psi(ih - s)y(s)ds + \int_{(i-1)h}^{ih} \psi(ih - s)y(s)ds + \int_{ih}^{\infty} \psi(ih - s)y(s)ds,$$

is static iff

$$\psi(t) = 0 \text{ whenever } t \notin [0, h],$$

which normally corresponds to a causal FIR (finite impulse response) F_ψ in



Outline

A/D conversion in time domain

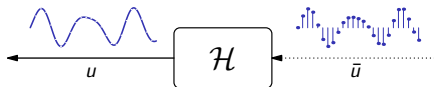
D/A conversion in time domain

A/D conversion in frequency domain

D/A conversion in frequency domain

Holds

Digital-to-analog (D/A) converters, aka **holds** or interpolators, are devices transforming discrete sequences into continuous-time signals:



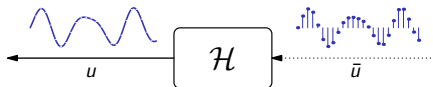
Conceptually, holds are systems, $\mathcal{H} : \bar{u} \mapsto u$, just with

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Such systems are also known as hybrid.

Remark: Dimensions of $\bar{u}[i]$ and $u(t)$ need not coincide in general.

Hold: basic properties



$\mathcal{H} : \bar{u} \mapsto u$ is said to be

- **linear** if $\mathcal{H}(\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2) = \alpha_1(\mathcal{H} \bar{u}_1) + \alpha_2(\mathcal{H} \bar{u}_2)$ (superposition)

— shift invariant if

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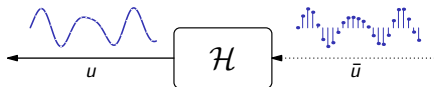
— causal if $u(t)$ may only depend on $\bar{u}[i]$ for i such that $t_i \leq t$

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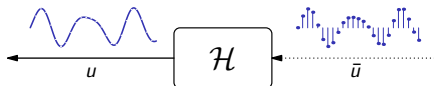
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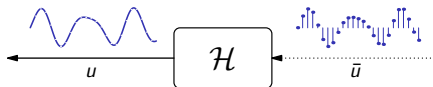
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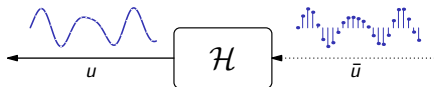
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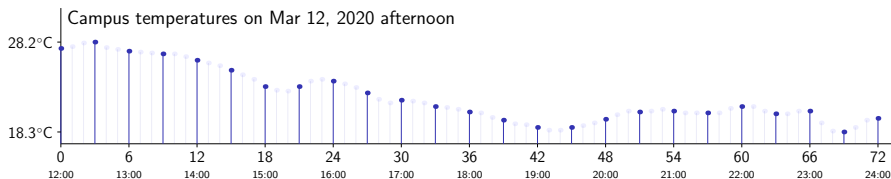
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Reconstruction problem

Return to the temperature measurements, but now assume that

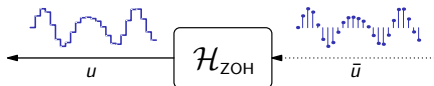
- all measurements but those taken every 0.5 hour are lost:



The question is

- how to reconstruct missing values (and even those never measured)?

The zero-order hold



Acts as

$$u(t) = \bar{u}[i], \quad \forall t \in [t_i, t_{i+1})$$

It is

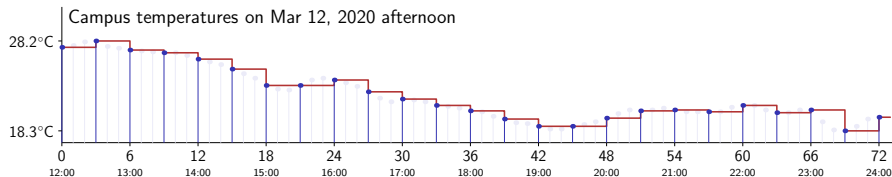
- linear
- shift invariant if $t_i = ih$
- causal
- stable, both L_∞ , as $\|u\|_\infty = \|\bar{u}\|_\infty$, and L_2 , as

$$\|u\|_2^2 = \int_{\mathbb{R}} \|u(t)\|^2 dt = \sum_{i \in \mathbb{Z}} \int_{t_i}^{t_{i+1}} \|u(t)\|^2 dt = \sum_{i \in \mathbb{Z}} h_i \|\bar{u}[i]\|^2 \leq \left(\sup_i h_i \right) \|\bar{u}\|_2^2$$

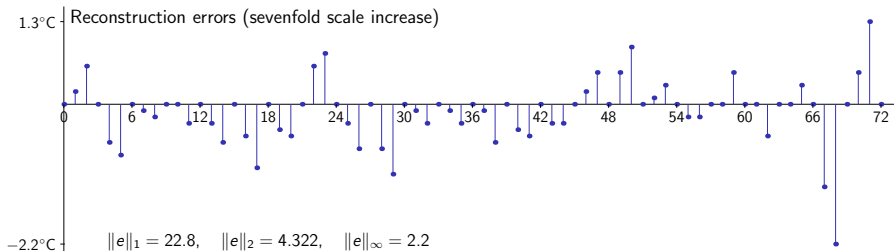
- static

The zero-order hold: example

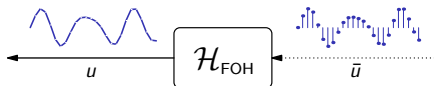
Continuous-time temperature reconstructed by $\mathcal{H}_{\text{ZOH}}$ is



and the errors at points that we can verify are



The first-order hold



Acts as

$$u(t) = \frac{t_{i+1} - t}{h_i} \bar{u}[i] + \frac{t - t_i}{h_i} \bar{u}[i + 1], \quad \forall t \in [t_i, t_{i+1}]$$

(linear interpolation between $\bar{u}[i]$ and $\bar{u}[i + 1]$ for each $i \in \mathbb{Z}$). It is

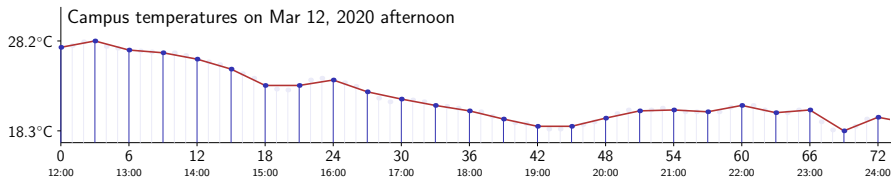
- linear
- shift invariant if $t_i = ih$
- **non**-causal

(delayed first-order hold, $u(t) = \frac{t_{i+1} - t}{h_i} \bar{u}[i - 1] + \frac{t - t_i}{h_i} \bar{u}[i]$, is causal)

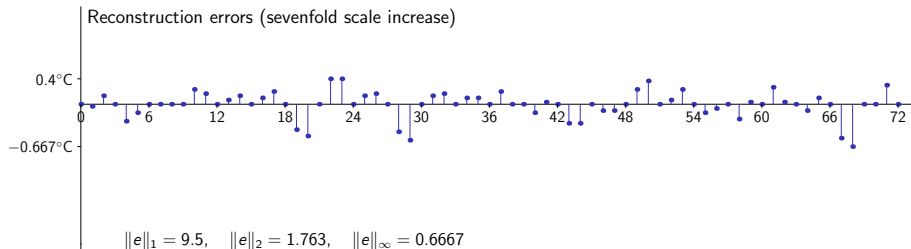
- stable, both L_∞ and L_2
- non-static (i.e. dynamic)

The first-order hold: example

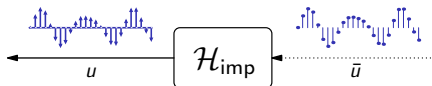
Continuous-time temperature reconstructed by $\mathcal{H}_{\text{FOH}}$ is



and the errors at points that we can verify are



The impulse hold



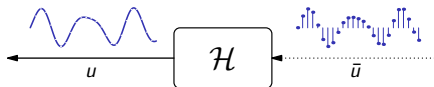
Acts as

$$u(t) = \delta(t - t_i) \bar{u}[i], \quad \forall t \in \mathbb{R}$$

It is

- linear
- shift invariant if $t_i = ih$
- causal
- unstable
- static
- not quite practical by itself...

General linear shift-invariant hold



With $t_i = ih$, it acts as

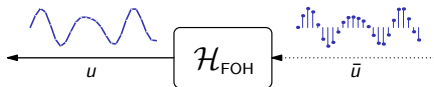
$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$$

for a hold function $\phi(t)$ (thus phi). $\phi(t)$ is the response of $\mathcal{H}$, at the time instance t , to the unit pulse applied at the time instance $i = 0$.

$$\phi(t) = \mathbb{1}(t) - \mathbb{1}(t - h) =$$


Figure 1.1 consists of two vertically aligned plots. The top plot shows a discrete-time signal $\bar{u}[i]$ as a sequence of blue dots on a horizontal axis labeled i . The dots are located at integer values of i from -5 to 5. The signal values are: $\bar{u}[-5] = 1$, $\bar{u}[-4] = 0.5$, $\bar{u}[-3] = 1$, $\bar{u}[-2] = 0$, $\bar{u}[-1] = 1$, $\bar{u}[0] = 0.5$, $\bar{u}[1] = 1$, $\bar{u}[2] = 1$, $\bar{u}[3] = 1$, $\bar{u}[4] = 0.5$, and $\bar{u}[5] = 1$. Dashed lines connect the dots at $i = -5, -4, -3, -2$ to the axis, and at $i = 1, 2, 3, 4, 5$ to the axis. The bottom plot shows the continuous-time reconstruction $u(t)$ as a blue step function on a horizontal axis labeled t . The function is constant at the values of the dots in the top plot over intervals of length h . The steps occur at $t = -5h, -4h, -3h, -2h, -h, 0, h, 2h, 3h, 4h, 5h$. Dashed lines indicate the boundaries of these intervals.

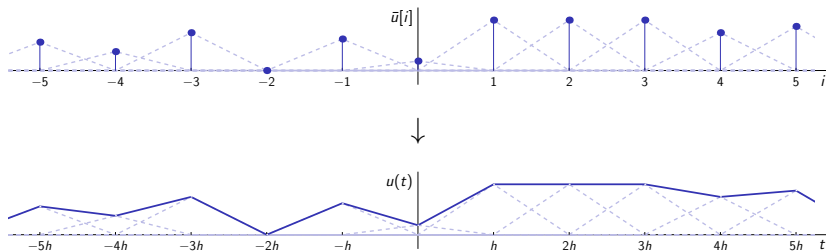
$\mathcal{H}_{\text{FOH}}$ as a general LSI hold



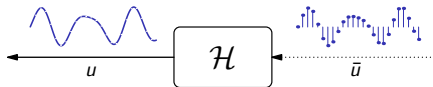
is a general hold with the hold function

$$\phi(t) = \frac{t+h}{h} \mathbb{1}(t+h) - \frac{2t}{h} \mathbb{1}(t) + \frac{t-h}{h} \mathbb{1}(t-h) = \begin{array}{c} 1 \\ \triangle \\ -h \quad 0 \quad h \\ t \end{array}$$

The D/A convolution $\sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$ acts in this case as



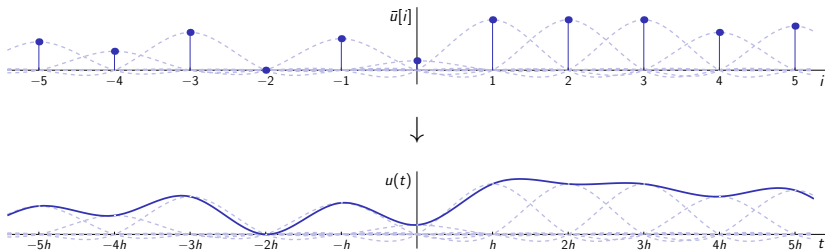
Cardinal cubic splines as a general LSI hold



is a general hold with the hold function

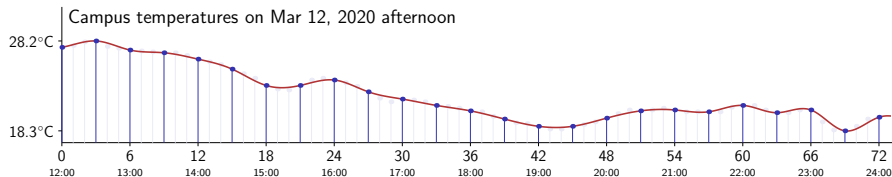
$$\phi(t) =$$

The D/A convolution $\sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$ acts in this case as

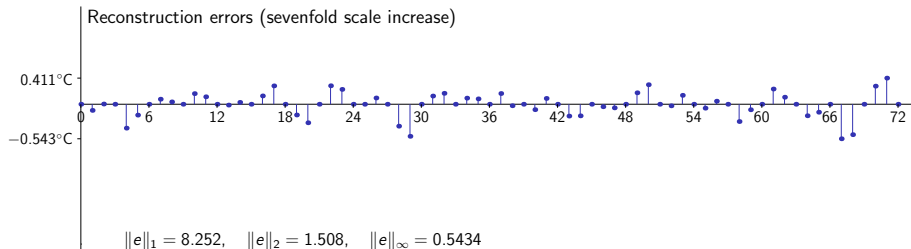


Cardinal cubic splines: example

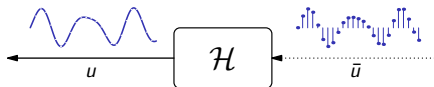
Continuous-time temperature reconstructed by cardinal cubic splines is



and the errors at points that we can verify are



General LSI hold: generation via $\mathcal{H}_{\text{imp}}$



It can be shown that

$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$$

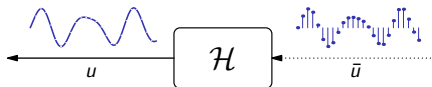
iff

$$u(t) = \int_{\mathbb{R}} \phi(t - s) u_{\text{it}}(s) ds, \quad \text{where } u_{\text{it}}(s) = \sum_{i \in \mathbb{Z}} \delta(s - ih) \bar{u}[i].$$

Hence, any generalized sampler is the

- cascade of the impulse hold and an analog LTI filter F_{ϕ} with the impulse response $\phi(t)$:

General LSI hold: generation via $\mathcal{H}_{\text{imp}}$



It can be shown that

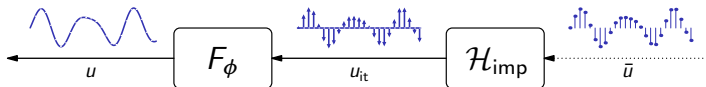
$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i]$$

iff

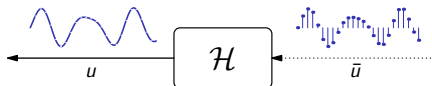
$$u(t) = \int_{\mathbb{R}} \phi(t - s) u_{\text{it}}(s) ds, \quad \text{where } u_{\text{it}}(s) = \sum_{i \in \mathbb{Z}} \delta(s - ih) \bar{u}[i].$$

Hence, any generalized sampler is the

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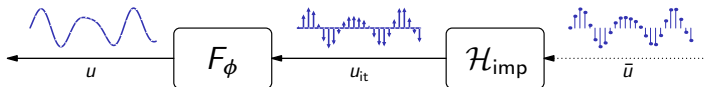
General LSI hold: causality



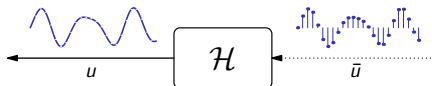
As

$$u(t) = \sum_{j=-\infty}^i \phi(t-jh) \bar{u}[j] + \sum_{j=i+1}^{\infty} \phi(t-jh) \bar{u}[j], \quad \forall t \in [ih, (i+1)h]$$

the causality of $\mathcal{H}$ is equivalent to $\phi(t) = 0$ for all $t < 0$, i.e. to that of F_ϕ in



Static general LSI hold



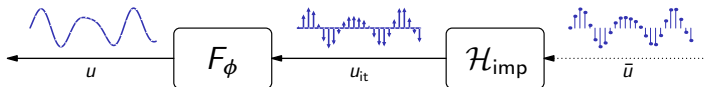
As

$$u(t) = \sum_{j=-\infty}^{i-1} \phi(t - jh) \bar{u}[j] + \phi(t - ih) \bar{u}[i] + \sum_{j=i+1}^{\infty} \phi(t - jh) \bar{u}[j]$$

for all $t \in [ih, (i+1)h]$, a hold is static iff

$$\phi(t) = 0 \text{ whenever } t \notin [0, h],$$

which normally corresponds to a causal FIR (finite impulse response) F_ϕ in



Outline

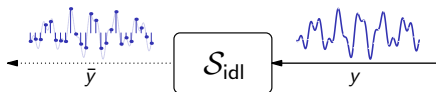
A/D conversion in time domain

D/A conversion in time domain

A/D conversion in frequency domain

D/A conversion in frequency domain

What do we lose by sampling analog signals ?



From the time-domain relation

$$\bar{y}[i] = y(ih)$$

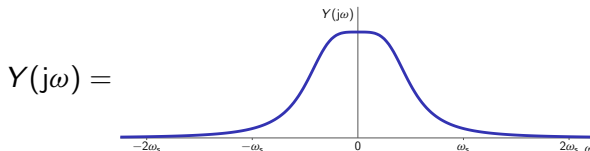
we know that all **intersample information** about $y(t)$ is **lost**. But

- to what extent is it important (if at all) ?

To answer this kind of questions, frequency-domain analysis is indispensable.

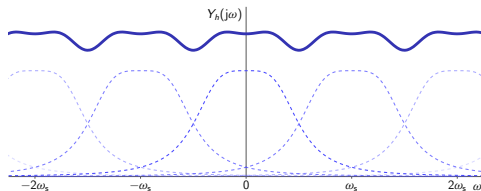
A weird function

Consider an analog signal $y(t)$ with the spectrum $Y(j\omega)$, e.g.



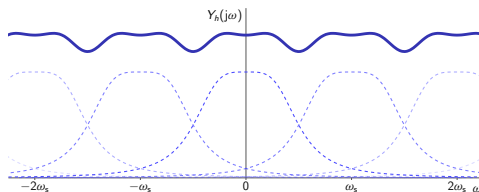
and define, for some $h > 0$, the function

$$Y_h(j\omega) := \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\omega + \omega_s i)) =$$



where $\omega_s := 2\pi/h$ (in rad/sec). Note that the mapping $Y \mapsto Y_h$ is linear.

A weird function: Fourier series



As $Y_h(j\omega)$ is ω_s -periodic, we can bring in its Fourier series expansion

$$Y_h(j\omega) = \sum_{i \in \mathbb{Z}} c_i e^{j \frac{2\pi}{\omega_s} i \omega} = \sum_{i \in \mathbb{Z}} c_i e^{j \omega h i},$$

where Fourier coefficients are calculated as

$$c_i = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} Y_h(j\omega) e^{-j \omega h i} d\omega.$$

A weird function: Fourier coefficients

With some extra efforts:

$$\begin{aligned} c_i &= \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} \frac{1}{h} \sum_{k \in \mathbb{Z}} Y(j(\omega + \omega_s k)) e^{-j\omega h i} d\omega \\ &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{-\omega_s/2}^{\omega_s/2} Y(j(\omega + \omega_s k)) e^{-j(\omega + \omega_s k) h i} d\omega \\ &= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{-\omega_s/2 + \omega_s k}^{\omega_s/2 + \omega_s k} Y(j\omega) e^{-j\omega h i} d\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} Y(j\omega) e^{-j\omega h i} d\omega \quad \left(\text{remember, } y(t) = \frac{1}{2\pi} \int_{\mathbb{R}} Y(j\omega) e^{j\omega t} d\omega \right) \\ &= y(-ih). \end{aligned}$$

A weird function: Fourier series (contd)

Thus, we end up with

$$Y_h(j\omega) = \sum_{i \in \mathbb{Z}} y(-ih)e^{j\omega hi} = \sum_{i \in \mathbb{Z}} y(ih)e^{-j\omega hi}.$$

Compare it with the discrete-time Fourier transform of $\bar{y} = \mathcal{S}_{\text{id}} y$,

$$\bar{Y}(e^{j\theta}) = \sum_{i \in \mathbb{Z}} \bar{y}[i]e^{-j\theta i} = \sum_{i \in \mathbb{Z}} y(ih)e^{-j\theta i}.$$

We can therefore say that

$$Y_h(j\omega) = \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\omega + \omega_s i)), \quad \text{where } \omega_s = \frac{2\pi}{h} \text{ (the sampling frequency)}$$

is the DTFT of the sampled signal $\bar{y}$ modulo scaling, $\theta = \omega h$, i.e.

$$\bar{Y}(e^{j\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\theta/h + \omega_s i)) = Y_h(j\theta/h).$$

A weird function: Fourier series (contd)

Thus, we end up with

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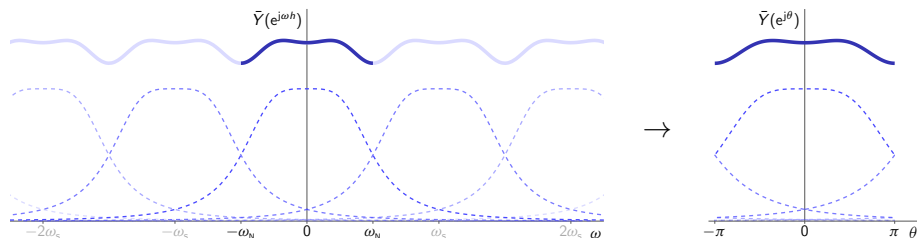
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Spectrum of sampled signal

We thus end up with

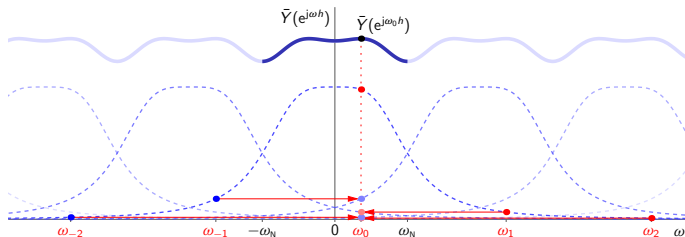


The frequency

$$\omega_N := \frac{\omega_s}{2} = \frac{\pi}{h}$$

is called the **Nyquist frequency** associated with the sampling period h (it is measured in rad/sec if h is in sec).

Spectrum of sampled signal: aliasing



Thus, the spectrum of $\bar{y}$ at each frequency $\theta_0 = \omega_0 h$ is a

- blend of analog frequency responses at $\omega_i := \omega_0 + \omega_s i$, $\forall i \in \mathbb{Z}$.

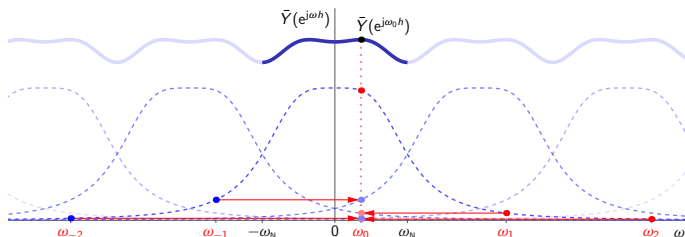
In other words,

- every discrete frequency $\theta_0 \in [-\pi, \pi]$ is an **alias** of all ω_i , $i \in \mathbb{Z}$.

This phenomenon is dubbed **aliasing**, with respect to the **base frequency** ω_0 .

Aliasing means information loss, we can no longer tell $Y(\omega_i)$ from $Y(\omega_j)$ in their effect on $Y(e^{j\theta_0})$ (unless we know their dependencies)

Spectrum of sampled signal: aliasing



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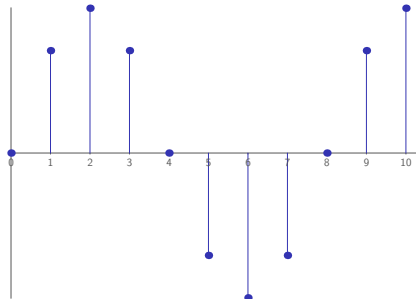
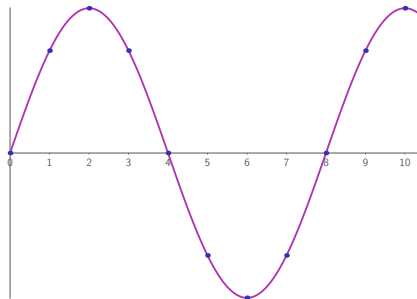
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Aliasing: example

Consider signals $y_1(t) = \sin\left(\frac{\pi}{4}t\right)$

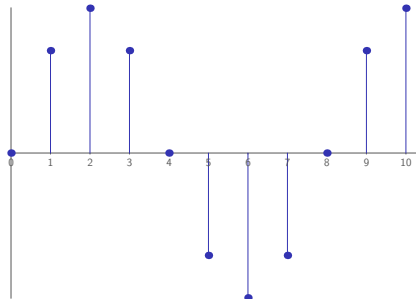
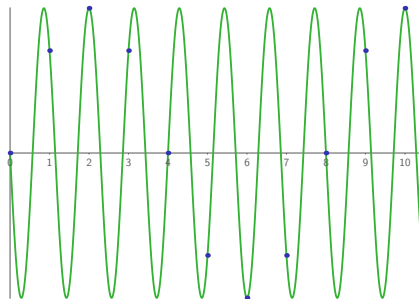
sampled at $h = 1$:



Aliasing: example

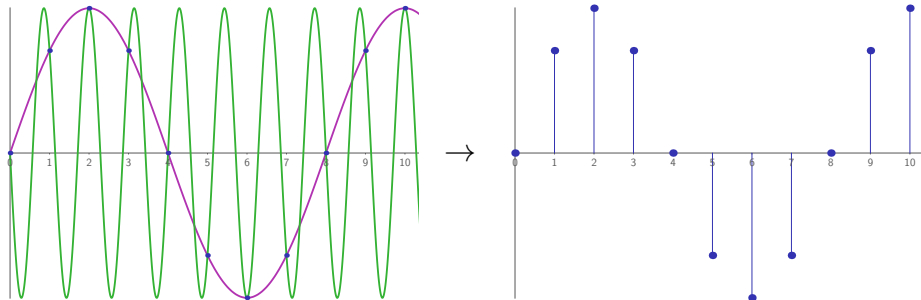
Consider signals

$y_2(t) = \sin\left(-\frac{7\pi}{4}t\right)$ sampled at $h = 1$:



Aliasing: example

Consider signals $y_1(t) = \sin\left(\frac{\pi}{4}t\right)$ and $y_2(t) = \sin\left(-\frac{7\pi}{4}t\right)$ sampled at $h = 1$:

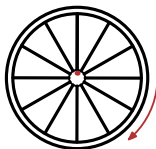


Sampling frequency is $\omega_s = 2\pi$, so that

— both $\omega_0 = \frac{\pi}{4}$ and $\omega_{-1} = -\frac{7\pi}{4} = \omega_0 - \omega_s$ have aliases at $\theta_0 = \omega_0$ and, consequently, produce the same sampled signal.

Aliasing: non-control examples

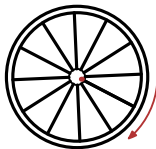
Wagon-wheel effect:



(shot with 12 FPS frame rate)

Aliasing: non-control examples

Wagon-wheel effect:



(shot with 12 FPS frame rate)

Moiré pattern:

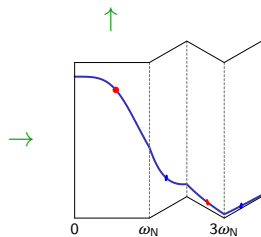
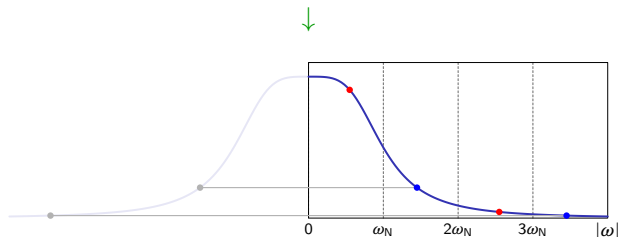
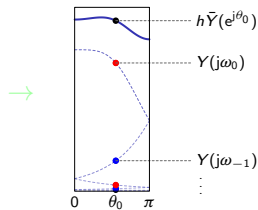
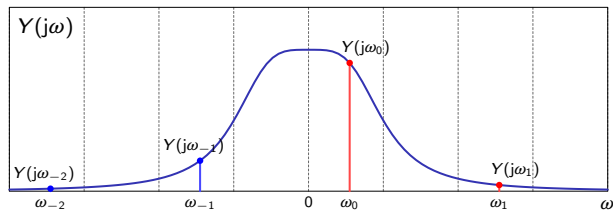


downsampling
→



Frequency folding

If $Y(j\omega) = Y(-j\omega) \in \mathbb{R}, \forall \omega$, the spectrum of its sampled version, $\bar{Y}(e^{j\theta})$, can be constructed also via the following **folding** procedure:



L_2 -instability of the ideal sampler, again

Let $y(t)$ be a signal with

$$Y(j\omega) = \frac{\sqrt{2}}{\sqrt{\omega^2 + 1}}.$$

By Parseval,

$$\|y\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2}{\omega^2 + 1} d\omega = 1,$$

so y is unit-energy (so, bounded) signal. Frequency response of $\bar{y} = \mathcal{S}_{\text{id}} y$ is

$$\bar{Y}(e^{j\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \frac{\sqrt{2}}{\sqrt{(\theta/h + 2\pi/h i)^2 + 1}} = \sum_{i \in \mathbb{Z}} \frac{\sqrt{2}}{\sqrt{(\theta + 2\pi i)^2 + h^2}} = \infty$$

for every $\theta \in [-\pi, \pi]$. This means that $\bar{y}$ is unbounded, i.e. that the ideal sampler $\mathcal{S}_{\text{id}}$ is L_2 -unstable (yet another proof of that)

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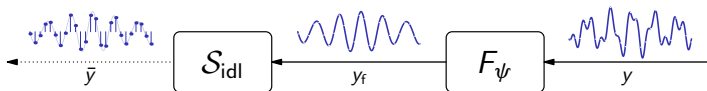
$$\bar{Y}(e^{j\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \frac{\sqrt{2}}{\sqrt{(\theta/h + 2\pi/h i)^2 + 1}} = \sum_{i \in \mathbb{Z}} \frac{\sqrt{2}}{\sqrt{(\theta + 2\pi i)^2 + h^2}} = \infty$$

for every $\theta \in [-\pi, \pi]$. This means that $\bar{y}$ is unbounded, i.e. that

- the ideal sampler $\mathcal{S}_{\text{idl}}$ is L_2 -unstable (yet another proof of that)

Remark: $\mathcal{S}_{\text{idl}}$ does produce finite-energy discrete signals from analog inputs, whose spectra decay faster than $1/|\omega|$ at high frequencies.

General LSI sampler



Just in two steps:

1. filter y by the LTI F_ψ , done frequency-by-frequency
2. apply the analysis above to $Y_f(j\omega) = F_\psi(j\omega)Y(j\omega)$ instead of $Y(j\omega)$

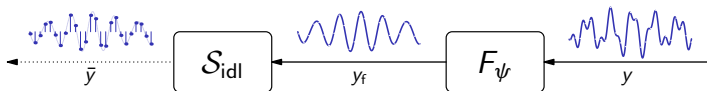
Example

For the averaging sampler, whose $\psi(t) = (1(t) - 1(t - \Delta))/\Delta$,

$$F_\psi(s) = \frac{1 - e^{-\Delta s}}{\Delta s} \longrightarrow$$

This is a low-pass filter, whose high-frequency gain decays as $1/|\omega|$. Hence, S_{avg} is L_2 -stable.

General LSI sampler



Just in two steps:

1. filter y by the LTI F_ψ , done frequency-by-frequency
2. apply the analysis above to $Y_f(j\omega) = F_\psi(j\omega)Y(j\omega)$ instead of $Y(j\omega)$

Example

For the averaging sampler, whose $\psi(t) = (\mathbb{1}(t) - \mathbb{1}(t - \Delta))/\Delta$,

$$F_\psi(s) = \frac{1 - e^{-\Delta s}}{\Delta s} \quad \longrightarrow \quad \text{Plot of } |F_\psi(j\omega)|$$

This is a low-pass filter, whose high-frequency gain decays as $1/|\omega|$. Hence, $\mathcal{S}_{\text{avg}}$ is L_2 -stable.



Outline

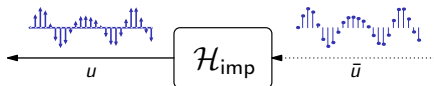
A/D conversion in time domain

D/A conversion in time domain

A/D conversion in frequency domain

D/A conversion in frequency domain

Spectrum of signals reconstructed by $\mathcal{H}_{\text{imp}}$



The Fourier transform of this $u(t)$ is

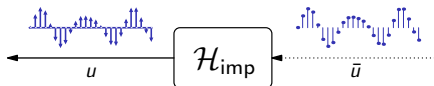
$$\begin{aligned}
 U(j\omega) &= \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{i \in \mathbb{Z}} \delta(t - ih) \bar{u}[i] e^{-j\omega t} dt \\
 &= \sum_{i \in \mathbb{Z}} \int_{-\infty}^{\infty} \delta(t - ih) e^{-j\omega t} dt \bar{u}[i] = \sum_{i \in \mathbb{Z}} \bar{u}[i] e^{-j(\omega h)i} \\
 &= \bar{U}(e^{j\omega h}) = \bar{U}(e^{j\theta})|_{\theta=\omega h}.
 \end{aligned}$$

Because $\bar{U}(e^{j\omega h})$ is a $(2\pi/h)$ -periodic function of ω ,

$\mathcal{H}_{\text{imp}}$ merely clones the spectrum of $\bar{u}$,

whose frequency θ -axis in $[-\pi, \pi]$ is scaled to fit the ω -axis in $[-\omega_N, \omega_N]$.

Spectrum of signals reconstructed by $\mathcal{H}_{\text{imp}}$



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 U(j\omega) &= \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{i \in \mathbb{Z}} \delta(t - ih) \bar{u}[i] e^{-j\omega t} dt \\
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 \end{aligned}$$

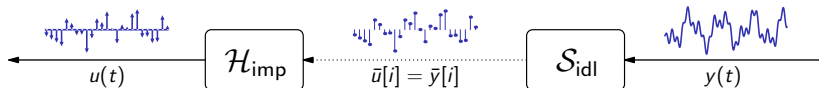
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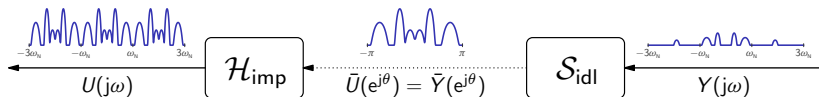
whose frequency θ -axis in $[-\pi, \pi]$ is scaled to fit the ω -axis in $[-\omega_N, \omega_N]$.

Sample-and-hold circuit of the form $\mathcal{H}_{\text{imp}}\mathcal{S}_{\text{idl}}$

In time domain:



In frequency domain:



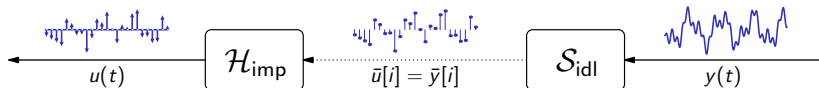
The resulting

$U(j\omega)$ is exactly the "weird signal" used to derive $\bar{Y}(e^{j\theta})$, i.e.

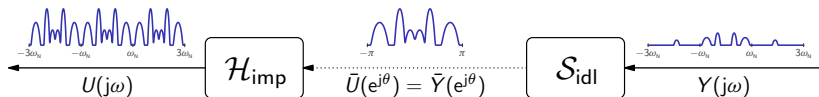
$$U(j\omega) = Y_h(j\omega) = \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\omega + \omega_H i)).$$

Sample-and-hold circuit of the form $\mathcal{H}_{\text{imp}}\mathcal{S}_{\text{idl}}$

In time domain:



In frequency domain:

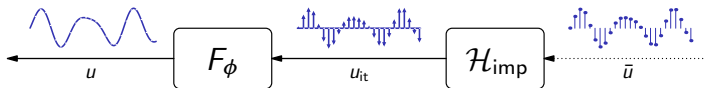


The resulting

- $U(j\omega)$ is exactly the “weird signal” used to derive $\bar{Y}(e^{j\theta})$, i.e.

$$U(j\omega) = Y_h(j\omega) := \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\omega + \omega_s i)).$$

General LSI hold



Just in two steps, again:

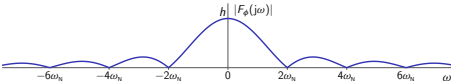
1. apply the analysis above to derive $U_{\text{it}}(j\omega)$
2. filter u_{it} by the LTI F_ϕ to end up with $U(j\omega) = F_\phi(j\omega)U_{\text{it}}(j\omega)$

General LSI hold: examples

$\mathcal{H}_{\text{ZOH}}$ has

$$\phi(t) = \mathbb{1}(t) - \mathbb{1}(t - h) = \begin{array}{c} 1 \\ \text{---} \\ 0 \quad h \quad t \end{array} \quad \Rightarrow \quad F_{\phi}(s) = \frac{1 - e^{-hs}}{s},$$

which corresponds to the low-pass

$$F_{\phi}(j\omega) = h \frac{\sin(\omega h/2)}{\omega h/2} e^{-j\omega h/2} =$$


$\mathcal{H}_{\text{FOH}}$ has

$$\phi(t) = \begin{array}{c} t \\ \text{---} \\ 0 \quad h \quad t \end{array} \quad \Rightarrow \quad F_{\phi}(s) = \frac{1}{h} \left(\frac{e^{sh/2} - e^{-hs/2}}{s} \right)^2,$$

which corresponds to the low-pass

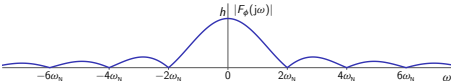
$$F_{\phi}(j\omega) = h \left(\frac{\sin(\omega h/2)}{\omega h/2} \right)^2 =$$

General LSI hold: examples

$\mathcal{H}_{\text{ZOH}}$ has

$$\phi(t) = \mathbb{1}(t) - \mathbb{1}(t - h) = \begin{array}{c} 1 \\ \text{---} \\ 0 \quad h \quad t \end{array} \quad \Rightarrow \quad F_{\phi}(s) = \frac{1 - e^{-hs}}{s},$$

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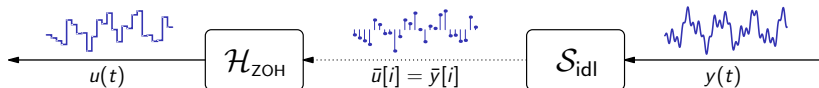
$$\phi(t) = \begin{array}{c} 1 \\ \text{---} \\ -h \quad 0 \quad h \quad t \end{array} = \frac{1}{h} \left(\begin{array}{c} 1 \\ \text{---} \\ -h/2 \quad h/2 \quad t \end{array} * \begin{array}{c} 1 \\ \text{---} \\ -h/2 \quad h/2 \quad t \end{array} \right) \quad \Rightarrow \quad F_{\phi}(s) = \frac{1}{h} \left(\frac{e^{sh/2} - e^{-hs/2}}{s} \right)^2,$$

which corresponds to the low-pass

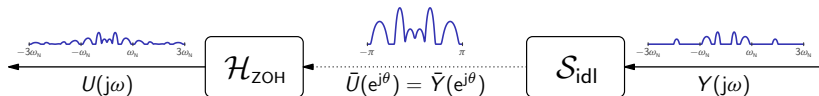
$$F_{\phi}(j\omega) = h \left(\frac{\sin(\omega h/2)}{\omega h/2} \right)^2 =$$


Sample-and-hold circuit of the form $\mathcal{H}_{\text{ZOH}}\mathcal{S}_{\text{idl}}$

In time domain:

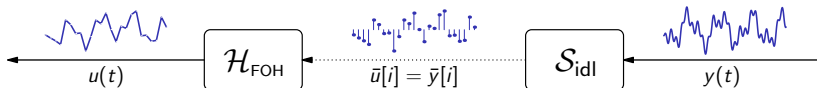


In frequency domain:

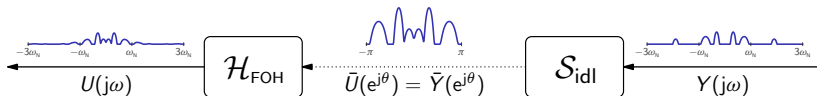


Sample-and-hold circuit of the form $\mathcal{H}_{\text{FOH}} \mathcal{S}_{\text{idl}}$

In time domain:

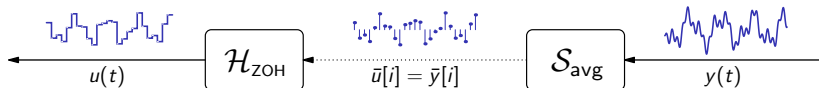


In frequency domain:

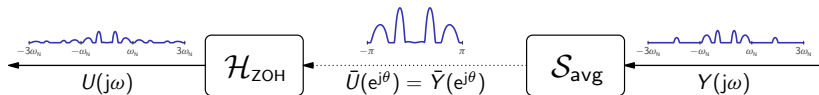


Sample-and-hold circuit of the form $\mathcal{H}_{\text{ZOH}} \mathcal{S}_{\text{avg}}$ with $\Delta = h$

In time domain:



In frequency domain:



where

$$F_{\psi}(s) = \frac{1 - e^{-hs}}{hs} \quad \longrightarrow \quad |F_{\psi}(j\omega)|$$

The magnitude response $|F_{\psi}(j\omega)|$ is shown as a plot. It is a periodic function with a main peak of 1 at $\omega = 0$ and smaller side lobes at $\omega = \pm 2\pi/\Delta, \pm 4\pi/\Delta, \pm 6\pi/\Delta$. The x-axis is labeled ω and the y-axis is labeled $|F_{\psi}(j\omega)|$.

is low pass, so aliased values are smaller.