

Sampled-Data Control Systems (036709)

lecture no. 2

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT



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Outline

Discrete signals

Discrete signals in the frequency domain

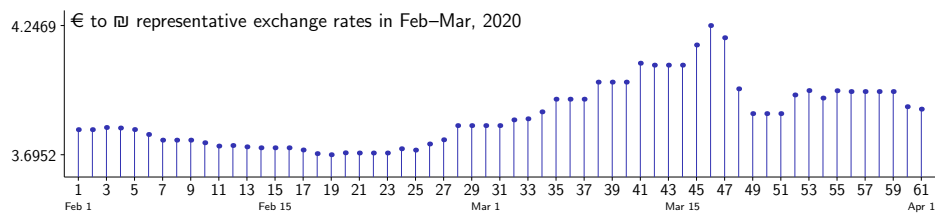
Discrete signals in the z-domain

Discrete systems

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Signals

reflect evolving information:



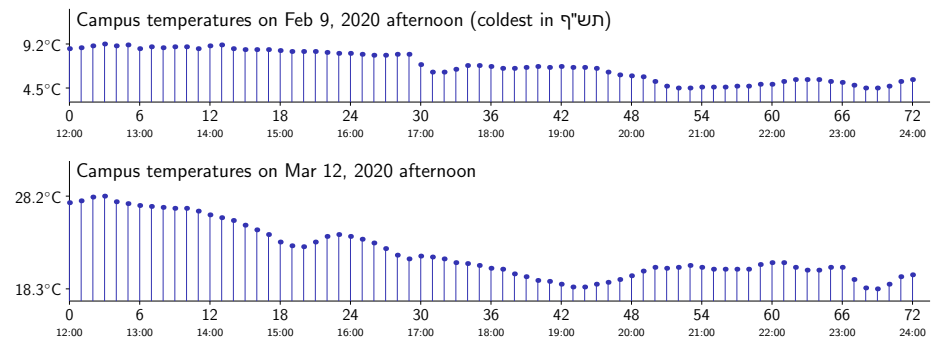
Mathematically, signals are

- functions of independent variables, like time
- discrete signals are functions $\mathbb{Z} \supset \mathbb{I} \rightarrow \mathbb{F}^n$ ($\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$)
- denoted as $f[t]$ (square brackets differentiate from analog signals)

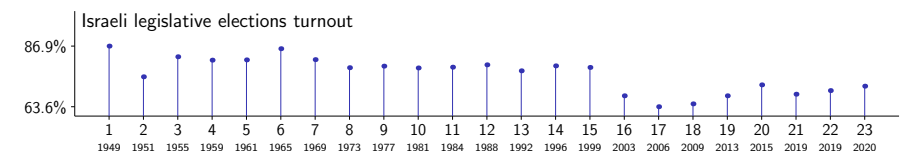
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Examples of discrete signals

Some are discretized versions of analog signals:

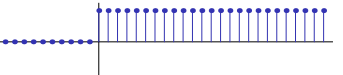
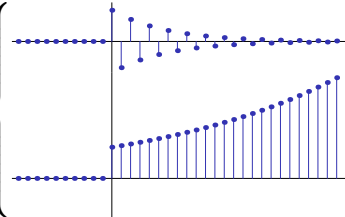
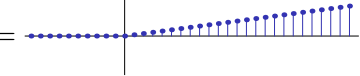
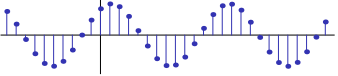


Some are intrinsically discrete:



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Basic discrete signals

- pulse: $f[t] = \delta[t] =$ 
- step: $f[t] = \mathbb{1}[t] =$ 
- exponential: $f[t] = \lambda^t \mathbb{1}[t] =$  $\begin{cases} \text{if } |\lambda| < 1 \\ \text{if } |\lambda| > 1 \end{cases}$
- ramp: $f[t] = t \mathbb{1}[t] =$ 
- sinusoid¹: $f[t] = \sin[\omega t + \phi] =$ 

¹Periodic, not necessarily with $T = 2\pi/\omega$, only if $\omega = 2\pi\alpha$ for some $\alpha \in \mathbb{Q}$ (rational).

Size matters

Spaces (boundedness measures):

$$\ell_1(\mathbb{Z}) := \left\{ f : \mathbb{Z} \rightarrow \mathbb{F} \mid \|f\|_1 := \sum_{t \in \mathbb{Z}} |f[t]| < \infty \right\}$$

$$\ell_2(\mathbb{Z}) := \left\{ f : \mathbb{Z} \rightarrow \mathbb{F} \mid \|f\|_2 := \left(\sum_{t \in \mathbb{Z}} |f[t]|^2 \right)^{1/2} < \infty \right\} \quad (\text{bounded energy})$$

$$\ell_\infty(\mathbb{Z}) := \left\{ f : \mathbb{Z} \rightarrow \mathbb{F} \mid \|f\|_\infty := \sup_{t \in \mathbb{Z}} |f[t]| < \infty \right\} \quad (\text{bounded amplitude})$$

with $\ell_1(\mathbb{Z}) \subset \ell_2(\mathbb{Z}) \subset \ell_\infty(\mathbb{Z})$ and $\|f\|_1 \geq \|f\|_2 \geq \|f\|_\infty$.

Examples:

- $f[t] = \lambda^t \mathbb{1}[t]$ belongs to $\begin{cases} \ell_1, \ell_2, \text{ and } \ell_\infty & \text{if } |\lambda| < 1 \\ \text{only } \ell_\infty & \text{if } |\lambda| = 1 \\ \text{none of them} & \text{if } |\lambda| > 1 \end{cases}$
- $f[t] = \frac{1}{t+1} \mathbb{1}[t]$ belongs to ℓ_2 and ℓ_∞ , but not to ℓ_1

Shift operator

Let $f : \mathbb{Z} \rightarrow \mathbb{F}$. Define the **forward-shift operator** q as satisfying

$$qf[t] = f[t+1], \quad \forall t \in \mathbb{Z}.$$

Naturally, $q^i f[t] = f[t+i]$ and the **backward-shift operator** q^{-1} satisfies

$$q^{-1}f[t] = f[t-1], \quad \forall t \in \mathbb{Z}$$

Shifts do not change norms, viz. $\|q^i f\|_\bullet = \|f\|_\bullet$ for all bounded f and $i \in \mathbb{Z}$ (if $f : \mathbb{Z}_+ \rightarrow \mathbb{F}$, then the statement above is true only if $i \in \mathbb{Z}_+$).

Difference operator

The difference operator $q - 1$, for which

$$(q - 1)f[t] = f[t+1] - f[t],$$

may be thought of as the discrete counterpart of the derivative (and so may the backward difference $1 - q^{-1} = (q - 1)q^{-1}$).

We then say that

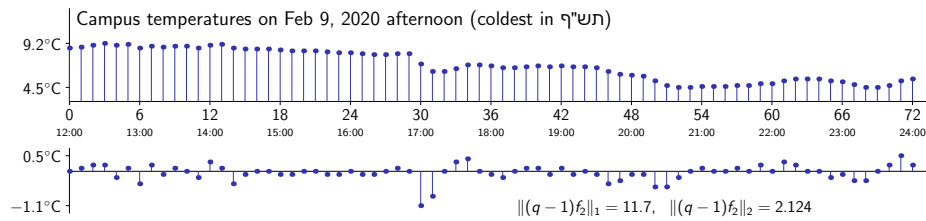
- f_1 is faster (slower) than f_2 if

$$\|(q - 1)f_1\|_\bullet > \|(q - 1)f_2\|_\bullet \quad (\|(q - 1)f_1\|_\bullet < \|(q - 1)f_2\|_\bullet)$$

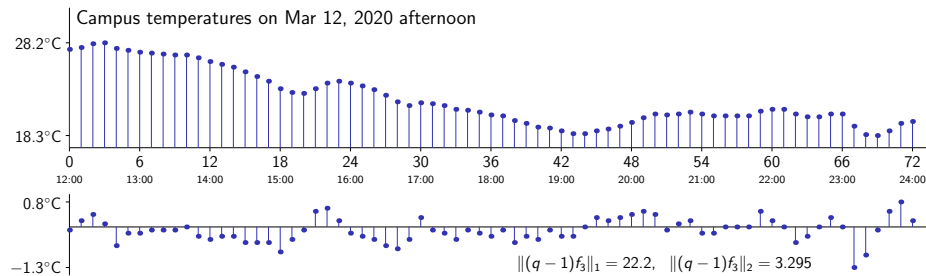
in whatever appropriate signal norm. This measure should be taken with a grain of salt though, only as a rough estimate.

Pace of signals: example

The signal



is slower than



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Outline

Discrete signals in the frequency domain

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Discrete signals as vectors

Unlike continuous-time signals, their discrete counterparts may be

- viewed as (almost, they may be infinite dimensional) plain vectors, namely

$$\llbracket f \rrbracket = \begin{bmatrix} \vdots \\ f[-2] \\ f[-1] \\ f[0] \\ f[1] \\ f[2] \\ \vdots \end{bmatrix}, \quad \text{e.g. } \llbracket \delta \rrbracket = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \text{and} \quad \llbracket 1 \rrbracket = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}.$$

Such a representation may be a convenient way to think of discrete signals, all vector manipulation rules / measures apply, even if vectors are infinite.

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Discrete-time harmonic signals

Signal

$$f[t] = \gamma e^{i\theta t} =$$
 $\gamma \in \mathbb{C}, \theta \in \mathbb{R}$

where θ is the frequency, $|\gamma|$ is the amplitude, and $\phi = \arg \gamma$ is the initial phase, is called the **discrete harmonic signal**. By Euler's formula,

$$\text{Re}(\gamma e^{j\theta t}) = |\gamma| \cos[\theta t + \phi] \quad \text{and} \quad \text{Im}(\gamma e^{j\theta t}) = |\gamma| \sin[\theta t + \phi].$$

Hence, the discrete harmonic signal may be thought of as a plain sinusoid.

Two qualitative deviations from the continuous-time case:

- $\gamma e^{i\theta t}$ might not be periodic (if $2\pi/\theta$ is irrational);
- because

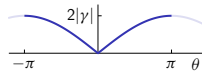
$$e^{j(\theta+2\pi i)t} = e^{j\theta t} e^{j2\pi it} = e^{j\theta t}, \quad \forall i \in \mathbb{Z}$$

we may only consider $\theta \in [-\pi, \pi]$ and the highest frequency is $|\theta| = \pi$.

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Pace of harmonic signals

If $f[t] = \gamma e^{j\theta t}$, then

$$|(q-1)f[t]| = |\gamma e^{j\theta t}(e^{j\theta} - 1)| = |\gamma| \sqrt{2 - 2\cos\theta} = 2|\gamma| \left| \sin\frac{\theta}{2} \right|$$


Because this is a strictly increasing function of $|\theta| \in [0, \pi]$, we conclude that

— $\gamma e^{j\theta_1 t}$ is faster (slower) than $\gamma e^{j\theta_2 t}$ if $|\theta_1| > |\theta_2|$ ($|\theta_1| < |\theta_2|$), provided both θ_1 and θ_2 are in $[-\pi, \pi]$. Thus, the fastest discrete harmonic signal is

$$\gamma e^{\pm j\pi t} = \gamma(-1)^t.$$

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Discrete-time Fourier transform (DTFT)

Given $f : \mathbb{Z} \rightarrow \mathbb{F}^n$, its discrete-time Fourier transform

$$\mathfrak{F}\{f\} = F(e^{j\theta}) := \sum_{t \in \mathbb{Z}} f[t] e^{-j\theta t},$$

for the angular frequency $\theta \in [-\pi, \pi]$ (in radians per step).

Remark: If the range of θ is extended to the whole $\mathbb{R}$, then $F(e^{j\theta})$ is 2π -periodic as a function of θ .

Strictly speaking, $\mathfrak{F}\{f\}$ exists as a function of θ only if

— $f \in \ell_1(\mathbb{Z})$ (or $f \in \ell_2(\mathbb{Z})$, if a weaker convergence is used).

Inverse DTFT:

$$\mathfrak{F}^{-1}\{F\} = f[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\theta}) e^{j\theta t} d\theta.$$

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DTFT: interpretation

It follows from

$$f[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\theta}) e^{j\theta t} d\theta.$$

that $f[t]$ is a superposition of elementary harmonic signals $e^{j\theta t}$. The signal

— $F(e^{j\theta})$ is the **frequency-domain** representation (or **spectrum**) of $f[t]$. $F(e^{j\theta_0})$ quantifies the contribution of $e^{j\theta_0 t}$ to $f[t]$.

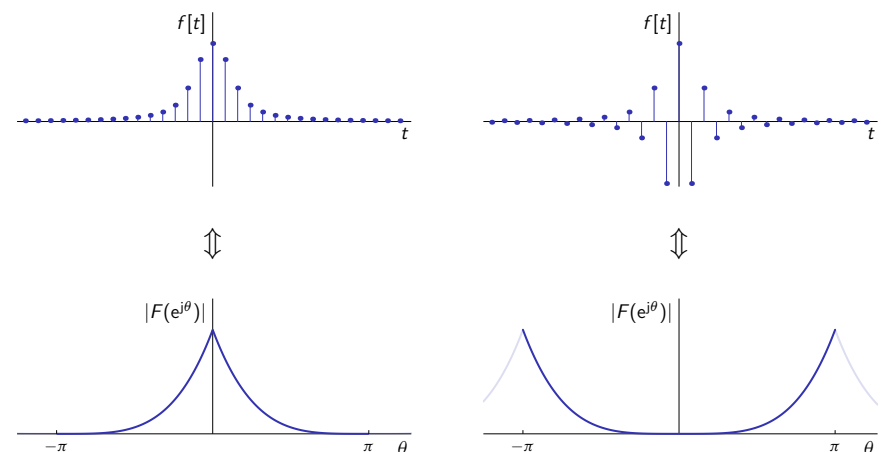
Hence, spectrum offers a

— viewpoint on f , where fast and slow components are separated.

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DTFT: interpretation (contd)

“slow” $f[t]$ ($\|(q-1)f\|_2 = 0.137$): “fast” $f[t]$ ($\|(q-1)f\|_2 = 0.652$):



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DTFTs in the vector form

The equality

$$F(e^{j\theta}) = \sum_{t \in \mathbb{Z}} f[t] e^{-j\theta t}$$

can be equivalently written as

$$F(e^{j\theta}) = v'(\theta) \llbracket f \rrbracket, \quad \text{where}^2 \quad v(\theta) := \begin{bmatrix} \vdots \\ e^{-2j\theta} \\ e^{-j\theta} \\ 1 \\ e^{j\theta} \\ e^{2j\theta} \\ \vdots \end{bmatrix} \quad \text{and} \quad \llbracket f \rrbracket = \begin{bmatrix} \vdots \\ f[-2] \\ f[-1] \\ f[0] \\ f[1] \\ f[2] \\ \vdots \end{bmatrix}.$$

As a matter of fact,

- $v(\theta) = \llbracket f \rrbracket$ for the harmonic $f[t] = e^{j\theta t}$.

²Mind that $(\cdot)'$ means complex-conjugate transpose.

DTFTs of some discrete signals

$f[t]$	$F(e^{j\theta})$	condition
$\delta[t] = \text{[impulse at } t=0]$	1	
$\mathbb{1}[t] = \text{[unit step]}$	$\frac{1}{1 - e^{-j\theta}} + \pi \sum_{i \in \mathbb{Z}} \delta(\theta - 2\pi i)$	
$\lambda^t \mathbb{1}[t] = \text{[decaying exponential]}$	$\frac{1}{1 - \lambda e^{-j\theta}}$	$ \lambda < 1$
$e^{j\theta_0 t} = \text{[complex exponential]}$	$2\pi \sum_{i \in \mathbb{Z}} \delta(\theta - \theta_0 - 2\pi i)$	

Properties of DTFT

Linearity: for all constants α_1 and α_2 ,

$$\begin{aligned} \mathfrak{F}\{\alpha_1 f_1 + \alpha_2 f_2\} &= \sum_{t \in \mathbb{Z}} (\alpha_1 f_1[t] + \alpha_2 f_2[t]) e^{-j\theta t} \\ &= \alpha_1 \sum_{t \in \mathbb{Z}} f_1[t] e^{-j\theta t} + \alpha_2 \sum_{t \in \mathbb{Z}} f_2[t] e^{-j\theta t} \\ &= \alpha_1 \mathfrak{F}\{f_1\} + \alpha_2 \mathfrak{F}\{f_2\}, \end{aligned}$$

provided both $\mathfrak{F}\{f_1\}$ and $\mathfrak{F}\{f_2\}$ exist.

Time shift: for every $\tau \in \mathbb{Z}$,

$$\begin{aligned} \mathfrak{F}\{q^\tau f\} &= \sum_{t \in \mathbb{Z}} f[t + \tau] e^{-j\theta t} = \sum_{t \in \mathbb{Z}} f[t] e^{-j\theta(t-\tau)} = e^{j\theta\tau} \sum_{t \in \mathbb{Z}} f[t] e^{-j\theta t} \\ &= e^{j\theta\tau} \mathfrak{F}\{f\}. \end{aligned}$$

Properties of DTFT

Time reversal: if $g[t] = f[-t]$, then

$$\mathfrak{F}\{g\}(e^{j\theta}) = \sum_{t \in \mathbb{Z}} f[-t] e^{-j\theta t} = \sum_{t \in \mathbb{Z}} f[t] e^{j\theta t} = \mathfrak{F}\{f\}(e^{-j\theta}).$$

Convolution: for given f and g , define their convolution sum $f * g$ as

$$(f * g)[t] := \sum_{s \in \mathbb{Z}} f[t - s] g[s] = \sum_{s \in \mathbb{Z}} f[s] g[t - s].$$

Assuming $\mathfrak{F}\{f\}$ and $\mathfrak{F}\{g\}$ exist,

$$\begin{aligned} \mathfrak{F}\{f * g\} &= \sum_{t \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} f[s] g[t - s] e^{-j\theta t} = \sum_{s \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} f[s] e^{-j\theta s} g[t - s] e^{-j\theta(t-s)} \\ &= \sum_{s \in \mathbb{Z}} f[s] e^{-j\theta s} \sum_{t \in \mathbb{Z}} g[t - s] e^{-j\theta(t-s)} = \sum_{s \in \mathbb{Z}} f[s] e^{-j\theta s} \sum_{t \in \mathbb{Z}} g[t] e^{-j\theta t} \\ &= \mathfrak{F}\{f\} \mathfrak{F}\{g\}. \end{aligned}$$

Properties of DTFT: Parseval

Because $F(e^{j\theta}) = v'(\theta)[f]$,

$$\overline{F(e^{j\theta})}F(e^{j\theta}) = [f]'v(\theta)v'(\theta)[f],$$

where $v_i(\theta) = e^{j\theta i}$ for all $i \in \mathbb{Z}$. Now,

$$\begin{aligned} v(\theta)v'(\theta) &= \begin{bmatrix} \vdots \\ e^{-j\theta} \\ 1 \\ e^{j\theta} \\ \vdots \end{bmatrix} \begin{bmatrix} \dots & e^{j\theta} & 1 & e^{-j\theta} & \dots \end{bmatrix} \\ &= \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 1 & e^{-j\theta} & e^{-2j\theta} & \dots \\ \dots & e^{j\theta} & 1 & e^{-j\theta} & \dots \\ \dots & e^{2j\theta} & e^{j\theta} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} = [e^{j\theta(i-j)}]_{i,j \in \mathbb{Z}}. \end{aligned}$$

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Properties of DTFT: Parseval (contd)

Taking into account that

$$\int_{-\pi}^{\pi} e^{j\theta k} d\theta = \begin{cases} 2\pi & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \implies \int_{-\pi}^{\pi} v(\theta)v'(\theta) d\theta = 2\pi I,$$

we have:

$$\begin{aligned} \int_{-\pi}^{\pi} |F(e^{j\theta})|^2 d\theta &= \int_{-\pi}^{\pi} [f]'v(\theta)v'(\theta)[f] d\theta = [f]' \int_{-\pi}^{\pi} v(\theta)v'(\theta) d\theta [f] \\ &= 2\pi [f]'[f] = 2\pi \sum_{t \in \mathbb{Z}} |f[t]|^2 \end{aligned}$$

and we end up with Parseval's equality:

$$\|f\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{j\theta})|^2 d\theta =: \|F\|_2^2.$$

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Discrete signals in the frequency domain

Discrete signals in the z-domain

Discrete systems

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z-transform

Given $f : \mathbb{Z} \rightarrow \mathbb{F}^n$, its (bilateral or two-sided) z-transform

$$\mathfrak{Z}\{f\} = F(z) := \sum_{t \in \mathbb{Z}} f[t]z^{-t}$$

for all $z \in \mathbb{C}$ for which the series converges. This transform

- ☺ applies to a substantially wider class of signals than DTFT
- ☺ facilitates tracing stability and causality in the transformed domain
- ☹ but does not have a clear interpretation, unlike DTFT

History:

- Witold Hurewicz (1947), notation was already “z”
- Yakov Z. Tsytkin (1950), called it discrete Laplace transform
- John R. Ragazzini & Lotfi A. Zadeh (1952), coined the term

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RoC of z-transform

The set of $z \in \mathbb{C}$ for which $\sum_{t \in \mathbb{Z}} f[t]z^{-t}$ converges is known as the **region of convergence (RoC)** of the z-transform. Technically,

- z is in the RoC of f if $f[t]z^{-t}$ belongs to $\ell_1(\mathbb{Z})$,

i.e. if $\sum_{t \in \mathbb{Z}} |f[t]| \cdot |z|^{-t} < \infty$. If

- $|z| < 1$, this requires that f does not grow “too fast” as $t \rightarrow +\infty$ and decays “sufficiently fast” as $t \rightarrow -\infty$
- $|z| > 1$, this requires that f does not grow “too fast” as $t \rightarrow -\infty$ and decays “sufficiently fast” as $t \rightarrow +\infty$

Less draconian if $f[k]$ is bounded for all k and

- $f[t] = 0$ for all $t \notin \mathbb{Z}_{t_1..t_2}$, then $\text{RoC} = \mathbb{C}$
- $f[t] = 0$ for all $t < 0$, then *typically* $\text{RoC} = \mathbb{C} \setminus \bar{\mathbb{D}}_\alpha$ for some $\alpha > 0$
- $f[t] = 0$ for all $t \geq 0$, then *typically* $\text{RoC} = \mathbb{D}_\alpha$ for some $\alpha > 0$

Remark: $F(z)$ can be defined outside its RoC via the (dirty) trick of *analytic continuation*.

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z-transforms of some discrete signals

$f[t]$	$F(z)$	RoC
$\delta[t] = \begin{array}{c} \uparrow \\ \text{---} \end{array}$	1	$\mathbb{C}$
$\mathbb{1}[t] = \begin{array}{c} \text{---} \\ \text{---} \end{array}$	$\frac{z}{z-1}$	$\mathbb{C} \setminus \bar{\mathbb{D}}_1$
$t\mathbb{1}[t] = \begin{array}{c} \text{---} \\ \text{---} \end{array}$	$\frac{z}{(z-1)^2}$	$\mathbb{C} \setminus \bar{\mathbb{D}}_1$
$\sin[\theta t + \phi]\mathbb{1}[t] = \begin{array}{c} \text{---} \\ \text{---} \end{array}$	$\frac{\sin(\phi)z^2 + \sin(\theta - \phi)z}{z^2 - 2\cos(\theta)z + 1}$	$\mathbb{C} \setminus \bar{\mathbb{D}}_1$
$\lambda^t \mathbb{1}[t] = \begin{array}{c} \text{---} \\ \text{---} \end{array}$	$\frac{z}{z-\lambda}$	$\mathbb{C} \setminus \bar{\mathbb{D}}_{ \lambda }$
$-\lambda^t \mathbb{1}[-1-t] = \begin{array}{c} \text{---} \\ \text{---} \end{array}$	$\frac{z}{z-\lambda}$	$\mathbb{D}_{ \lambda }$
$\lambda^t \mathbb{1}[t] + \mu^t \mathbb{1}[-1-t] = \begin{array}{c} \text{---} \\ \text{---} \end{array}$	$\frac{z}{z-\lambda} - \frac{z}{z-\mu}$	$\underbrace{\mathbb{D}_{ \mu } \setminus \bar{\mathbb{D}}_{ \lambda }}_{\text{if } \lambda < \mu }$

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Properties of z transform

Counterparts of properties of DTFT, proofs are similar (or straightforward):

Linearity: $\mathcal{Z}\{\alpha_1 f_1 + \alpha_2 f_2\} = \alpha_1 \mathcal{Z}\{f_1\} + \alpha_2 \mathcal{Z}\{f_2\}$, for all constants α_1 and α_2 , safe in $\text{RoC}_{f_1} \cap \text{RoC}_{f_2}$.

Time shift: $\mathcal{Z}\{q^\tau f\} = z^\tau \mathcal{Z}\{f\}$ for all $\tau \in \mathbb{Z}$.

Remark: By the shift property, z can be interpreted as the transformed-domain counterpart of the forward shift q and, also with linearity, $(z-1)$ —as that of $q-1$, i.e. the derivative.

Convolution: $\mathcal{Z}\{f * g\} = \mathcal{Z}\{f\}\mathcal{Z}\{g\}$ safe in $\text{RoC}_f \cap \text{RoC}_g$.

Time reversal: if $g[t] = f[-t]$, then $\mathcal{Z}\{g\}(z) = \mathcal{Z}\{f\}(z^{-1})$ in $1/\text{RoC}_f$.

z-scaling: if $g[t] = \lambda^t f[t]$, then $\mathcal{Z}\{g\}(z) = \mathcal{Z}\{f\}(z/\lambda)$ in $|\lambda|\text{RoC}_f$.

Upsampling: if $g[t] = \begin{cases} f[s] & \text{if } t = vs \\ 0 & \text{if } t \neq vs \end{cases}$, then $\mathcal{Z}\{g\}(z) = \mathcal{Z}\{f\}(z^v)$ in RoC_f .

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Properties of z transform (contd)

If $f[t] = 0$ for all $t < 0$, then the following two results hold true.

The initial value theorem:

$$f[0] = \lim_{z \rightarrow \infty} F(z).$$

Proof: follows directly from $F(z) = f[0] + f[1]z^{-1} + f[2]z^{-2} + \dots$ □

The final value theorem: if the limit in the left-hand side below exists, then

$$\lim_{t \rightarrow \infty} f[t] = \lim_{z \rightarrow 1} (z-1)F(z).$$

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Discrete signals in the z-domain

Discrete systems

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I/O systems

Discrete systems are understood as signal processors, i.e. mappings $u \mapsto y$ and their action is written as $y = Gu$ (not a product!). Signal

- u is known as the input
- y is known as the output

Mathematical models are still

- descriptions of systems in a mathematical language.

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(Very) naïve epidemic spread model

Let

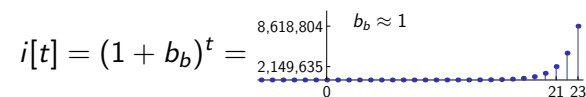
- $i[t]$ be the number of infectives in the population at day t (output)
- $b_b > 0$ is the number of infected by each infective per day, on average

The model:

$$i[t + 1] = i[t] + b_b i[t] = (1 + b_b) i[t].$$

Epidemic development:

1. if there is no input, $i[t + 1] = i[t] = 0$,
2. if there is an input (patient zero), then $i[0] = 1$ (so $u = \delta$) and



i.e. for this b_b in 23 days the whole population is infected (and counting).

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SIR epidemic spread model

Let

- $s[t]$ be the number of susceptibles in the population at day t
- $i[t]$ be the number of infectives in the population at day t
- $r[t]$ be the number of removed in the population at day t

The SIR model:

$$\begin{cases} s[t + 1] = s[t] - b s[t] i[t] & s[0] = s_0 & \text{(total population)} \\ i[t + 1] = i[t] + (b s[t] - a) i[t] & i[0] = 1 & \text{(patient zero, input)} \\ r[t + 1] = r[t] + a i[t] & r[0] = 0 \end{cases}$$

for some infection rate $b > 0$ and recovery rate $a > 0$. Epidemic only if

$$R_0 := \frac{b s_0}{a} > 1 \implies \begin{Bmatrix} s[t] \\ i[t] \end{Bmatrix} = \begin{matrix} 8,618,804 \\ 1,679,769 \end{matrix} \quad R_0 = 2, b_b \approx 1$$

so $b_b \approx a(R_0 - 1) = b(1 - 1/R_0)s_0$. Covid-19 might have $R_0 \in [1.4, 3.9]$

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Basic properties

- A system G is said to be **linear** if

$$G(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 (Gu_1) + \alpha_2 (Gu_2)$$

for all admissible signals u_1, u_2 and all scalars α_1, α_2 (superposition).

- A system $G : u \mapsto y$ is said to be **shift invariant** if any time shift of its input $u[t]$ results in the same time shift of its output $y[t]$, i.e. if

$$G(qu) = q(Gu) \implies G(q^i u) = q^i(Gu), \quad \forall i \in \mathbb{Z}.$$

- A system $G : u \mapsto y$ is said to be **causal** if $y[t]$ at every t can depend only on $u[s]$ for $s \leq t$ and **strictly causal** if only for $s < t$.
- A system $G : u \mapsto y$ is said to be **ℓ_p -stable** (mostly $p = 2$ or $p = \infty$) if

$$u \in \ell_p \implies y \in \ell_p \quad \text{and} \quad \|y\|_p \leq \gamma \|u\|_p$$

for all u and some $\gamma \geq 0$ independent of u (BIBO stability for $p = \infty$).

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Kernel representation

Note that

$$u[t] = \sum_{s \in \mathbb{Z}} \delta[t-s] u[s] = (\delta * u)[t].$$

If $G : u \mapsto y$ is linear, then (ignoring convergence issues)

$$\begin{aligned} y[t] &= (Gu)[t] = \left(G \sum_{s \in \mathbb{Z}} \delta[t-s] u[s] \right)[t] = \sum_{s \in \mathbb{Z}} (G\delta[t-s])[t] u[s] \\ &= \sum_{s \in \mathbb{Z}} g[t, s] u[s], \end{aligned}$$

where

$$g[t, s] := (G\delta[t-s])[t],$$

which is the response of G , at the time instance t , to the unit pulse $\delta[t-s]$ applied at the time s (the **impulse response** of G). This is known as

- the **kernel representation** of G .

It implies that the behavior of a linear system is

- completely determined by its responses to unit pulses.

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Causality in terms of the impulse response

If $G : u \mapsto y$ and $g[t, s]$ is its impulse response, then

$$y[t] = \sum_{s \in \mathbb{Z}} g[t, s] u[s] = \underbrace{\sum_{s=-\infty}^{t-1} g[t, s] u[s]}_{\text{past}} + \underbrace{g[t, t] u[t]}_{\text{present}} + \underbrace{\sum_{s=t+1}^{\infty} g[t, s] u[s]}_{\text{future}}$$

and G is

- causal $\iff g[t, s] = 0$ whenever $s > t$
- strictly causal $\iff g[t, s] = 0$ whenever $s \geq t$

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Shift invariance in terms of the impulse response

If $G : u \mapsto y$ and $g[t, s]$ is its impulse response, then

- $G(qu)$ at the time instance t reads

$$\sum_{s \in \mathbb{Z}} g[t, s] u[s+1] = \sum_{s \in \mathbb{Z}} g[t, s-1] u[s],$$

- $q(Gu)$ at the time instance t reads

$$\sum_{s \in \mathbb{Z}} g[t+1, s] u[s].$$

Hence, G is shift invariant iff $g[t, s-1] = g[t+1, s]$, $\forall t, s \in \mathbb{Z}$, or iff

$$g[t, s] = g[t+1, s+1] \implies g[t, s] = g[t+i, s+i], \quad \forall i \in \mathbb{Z}.$$

Thus, for linear shift-invariant (LSI) systems

- it is sufficient to consider $g[t] := g[t, 0]$,

which is the response of G at the time instance t to the pulse applied at 0.

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Kernel representation of LSI systems

If $G : u \mapsto y$ is LSI, then its impulse response $g[t, s] = g[t - s]$ and

$$y[t] = \sum_{s \in \mathbb{Z}} g[t - s] u[s] \iff y = g * u.$$

If G is causal, then $g[t] = 0$ for all $t < 0$ and

$$y[t] = \sum_{s=-\infty}^t g[t - s] u[s].$$

If, in addition, $u[t] = 0$ for all $t < 0$, then

$$y[t] = \sum_{s=0}^t g[t - s] u[s].$$

The convolution representation implies that the behavior of LSI systems is

- completely determined by its response to the unit pulse applied at 0.

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Examples of LSI systems

1. static gain has $g[t] = k\delta[t] = \text{---}\delta[t]\text{---}$ and acts as

$$y[t] = \sum_{s \in \mathbb{Z}} k\delta[t - s] u[s] = ku[t].$$

2. 2-step delay has $g[t] = \delta[t - 2] = \text{---}\delta[t - 2]\text{---}$ and acts as

$$y[t] = \sum_{s \in \mathbb{Z}} \delta[t - s - 2] u[s] = u[t - 2].$$

3. integrator has $g[t] = \mathbb{1}[t] = \text{---}\mathbb{1}[t]\text{---}$ and acts as

$$y[t] = \sum_{s \in \mathbb{Z}} \mathbb{1}[t - s] u[s] = \sum_{s=-\infty}^t u[s].$$

4. backward difference has $g[t] = \delta[t] - \delta[t - 1] = \text{---}\delta[t] - \delta[t - 1]\text{---}$ and acts as

$$y[t] = \sum_{s \in \mathbb{Z}} (\mathbb{1}[t - s] - \mathbb{1}[t - s - 1]) u[s] = u[t] - u[t - 1].$$

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ℓ_∞ -stability of LSI systems

If $G : u \mapsto y$ is LSI and $g[t]$ is its impulse response, then it is

- ℓ_∞ -stable iff $g \in \ell_1(\mathbb{Z})$, i.e. iff $\sum_{t \in \mathbb{Z}} |g[t]| < \infty$.

A consequence is that

- systems with decaying impulse responses are not necessarily stable.

Example

Consider the (causal) system G with

$$g[t] = \frac{1}{t+1} \mathbb{1}[t].$$

This function is not in ℓ_1 . To see instability, consider its step response:

$$y[t] = \sum_{s=0}^t g[t - s] \mathbb{1}[s] = \sum_{s=0}^t \frac{1}{t - s + 1} = \sum_{s=1}^{t+1} \frac{1}{s} > \ln[t + 2] \xrightarrow{t \rightarrow \infty} \infty.$$

▽

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The Frequency Response Theorem

If $G : u \mapsto y$ is LSI and stable, then the DTFT of its impulse response $g[t]$ exists and the response to the harmonic input $u[t] = \gamma e^{j\theta t}$ is

$$\begin{aligned} y[t] &= \sum_{s \in \mathbb{Z}} g[t - s] \gamma e^{j\theta s} = \sum_{s \in \mathbb{Z}} g[s] \gamma e^{j\theta(t-s)} = \left(\sum_{s \in \mathbb{Z}} g[s] e^{-j\theta s} \right) \gamma e^{j\theta t} \\ &= (G(e^{j\theta}) \gamma) e^{j\theta t}. \end{aligned}$$

This is also a harmonic signal, whose

- amplitude is $|\gamma| |G(e^{j\theta})|$, i.e. scaled by a factor of $|G(e^{j\theta})|$
- initial phase is $\arg \gamma + \arg G(e^{j\theta})$, i.e. shifted by $\arg G(e^{j\theta})$

The function

- $G(e^{j\theta}) = \mathfrak{F}\{g\}$ is dubbed the **frequency response** of the system G .

It is a periodic function of θ and thus may be considered for $\theta \in [-\pi, \pi]$.

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LSI systems in the frequency domain

If $G : u \mapsto y$ is LSI and stable and $\mathfrak{F}\{u\}$ exists, then

$$\mathfrak{F}\{y\} = \mathfrak{F}\{g * u\} = \mathfrak{F}\{g\}\mathfrak{F}\{u\} \iff Y(e^{j\theta}) = G(e^{j\theta})U(e^{j\theta}),$$

where $G(e^{j\theta})$ is the frequency response of G . Thus,

- spectra of signals passing through an LSI system **reshape**,

This property, aka filtering, is key in many control (and signal processing) applications. Like in the continuous-time case, we may define

- low-pass filters, with associated bandwidth
- high-pass filters, with associated cut-off frequency
- band-pass filters, with associated band

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LSI systems in the z-domain

If $G : u \mapsto y$ is LSI, then

$$\mathfrak{Z}\{y\} = \mathfrak{Z}\{g * u\} = \mathfrak{Z}\{g\}\mathfrak{Z}\{u\} \iff Y(z) = G(z)U(z)$$

for all $z \in \text{RoC}_u \cap \text{RoC}_g$. The function

- $G(z) = \mathfrak{Z}\{g\}$ is dubbed the **transfer function** of G .

The following are counterparts of continuous-time definitions / facts:

- if G is causal, then the RoC of $G(z)$ is normally $\mathbb{C} \setminus \bar{\mathbb{D}}_\alpha$ for some α ,
- if G is finite dimensional, then $G(z)$ is rational, i.e.

$$G(z) = \frac{N(z)}{D(z)} \quad \text{for some polynomials } N(z) \text{ and } D(z),$$

- a rational $G(z)$ is said to be proper if $\deg D(z) \geq \deg N(z)$,
- a rational $G(z)$ is said to be strictly proper if $\deg D(z) > \deg N(z)$.

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Examples of LSI systems (contd)

1. static gain has $g[t] = k\delta[t] = \text{---}\uparrow\text{---}$, so

$$G(z) = \mathfrak{Z}\{k\delta\} = k \quad \text{and} \quad G(e^{j\theta}) = \mathfrak{F}\{k\delta\} = k.$$

2. 2-step delay has $g[t] = \delta[t-2] = \text{---}\uparrow\text{---}$, so

$$G(z) = \mathfrak{Z}\{q^{-2}\delta\} = \frac{1}{z^2} \quad \text{and} \quad G(e^{j\theta}) = \mathfrak{F}\{q^{-2}\delta\} = e^{-2j\theta}.$$

3. integrator has $g[t] = \mathbb{1}[t] = \text{---}\uparrow\text{---}$, so

$$G(z) = \mathfrak{Z}\{\mathbb{1}\} = \frac{z}{z-1} \quad \text{and} \quad G(e^{j\theta}) = G(z)|_{z=e^{j\theta}} = \frac{e^{j\theta}}{e^{j\theta}-1}.$$

4. backward difference has $g[t] = \delta[t] - \delta[t-1] = \text{---}\uparrow\text{---}$, so

$$G(z) = \mathfrak{Z}\{\delta - q^{-1}\delta\} = \frac{z-1}{z} \quad \text{and} \quad G(e^{j\theta}) = \mathfrak{F}\{\delta - q^{-1}\delta\} = \frac{e^{j\theta}-1}{e^{j\theta}}.$$

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Transfer functions and causality

If $G : u \mapsto y$ is LSI and such that

- its transfer function $G(z)$ exists and its RoC is $\mathbb{C} \setminus \bar{\mathbb{D}}_\alpha$ for some $\alpha \geq 0$,
- then
- G is causal iff $G(z)$ is proper
 - G is strictly causal iff $G(z)$ is strictly proper
 - the pole excess of $G(z)$ equals the number of delays between u and y

Examples:

1. if $g[t] = \delta[t+1] = \text{---}\uparrow\text{---}$, then $G(z) = z$ is not proper

2. if $g[t] = \delta[t-2] = \text{---}\uparrow\text{---}$, then $G(z) = \frac{1}{z^2}$ is strictly proper

3. if $g[t] = \lambda^t \mathbb{1}[t] = \text{---}\uparrow\text{---}$, then $G(z) = \frac{z}{z-\lambda}$ is proper

4. if $g[t] = -\lambda^t \mathbb{1}[-1-t] = \text{---}\uparrow\text{---}$, then $G(z) = \frac{z}{z-\lambda}$ is proper
($\text{RoC}_1 = \text{RoC}_2 = \mathbb{C}$, $\text{RoC}_3 = \mathbb{C} \setminus \bar{\mathbb{D}}_{|\lambda|}$, and $\text{RoC}_4 = \mathbb{D}_{|\lambda|}$).

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Transfer functions and ℓ_2 -stability

If $G : u \mapsto y$ is LSI and such that

- its transfer function $G(z)$ exists and its RoC is $\mathbb{C} \setminus \bar{\mathbb{D}}_\alpha$ for some $\alpha \geq 0$,

then

- G is **causal and ℓ_2 -stable** iff $G(z)$ is **bounded and holomorphic** $\mathbb{C} \setminus \bar{\mathbb{D}}_1$.

The space of all such transfer functions is known as H_∞ . The RoC of every $G \in H_\infty$ must contain $\mathbb{C} \setminus \bar{\mathbb{D}}_1$.

Examples:

1. if $g[t] = \delta[t + 1]$, then $G(z) = z$ isn't in H_∞ (holomorphic, *un*bounded)
2. if $g[t] = \delta[t - 1]$, then $G(z) = \frac{1}{z}$ is in H_∞
3. if $g[t] = \lambda^t \mathbb{1}[t]$, then $G(z) = \frac{z}{z - \lambda}$ is in H_∞ iff $|\lambda| < 1$

If G is finite dimensional, then

- $G \in H_\infty \iff G(z)$ is proper and has no poles in $\mathbb{C} \setminus \mathbb{D}_1$
- G is ℓ_2 -stable $\iff G$ is ℓ_∞ -stable

Steady-state response to a step input

Let $G : u \mapsto y$ be LSI and ℓ_∞ -stable. The response to $u[t] = u_0 1[t]$ is

$$y[t] = \sum_{s \in \mathbb{Z}} g[t-s] u_0 \mathbb{1}[s] = \sum_{s=0}^{\infty} g[t-s] u_0 = \sum_{s=-\infty}^t g[s] u_0$$

Hence,

$$\lim_{t \rightarrow \infty} y[t] = \sum_{s \in \mathbb{Z}} g[s] u_0 = \sum_{s \in \mathbb{Z}} g[s] 1^{-s} u_0 = G(1) u_0,$$

where the series converges because $g \in \ell_1$. The quantity

- $G(1) = G(z)|_{z=1}$ is called the **static gain** of G .

In parallel to the continuous-time case, it equals the frequency response at the slowest frequency, i.e. $G(1) = G(e^{j\theta})|_{\theta=0}$.

Matrix representation

The kernel representation $y[t] = \sum_{s \in \mathbb{Z}} g[t, s]u[s]$ can also be written as

$$\begin{bmatrix} \vdots \\ y[t-1] \\ y[t] \\ y[t+1] \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots \\ \cdots & g[t-1, t-1] & g[t-1, t] & g[t-1, t+1] & \cdots \\ \cdots & g[t, t-1] & g[t, t] & g[t, t+1] & \cdots \\ \cdots & g[t+1, t-1] & g[t+1, t] & g[t+1, t+1] & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\text{system matrix of } G, [G]} \begin{bmatrix} \vdots \\ u[t-1] \\ u[t] \\ u[t+1] \\ \vdots \end{bmatrix}$$

i.e. $\mathbb{[y]} = \mathbb{[G]}\mathbb{[u]}$. Causal systems have lower-triangular system matrices. If G is LSI, then its system matrix is **Toeplitz**:

$$\mathbb{[G]} = \begin{bmatrix} \vdots & \vdots & \vdots & & \\ \cdots & g[0] & g[-1] & g[-2] & \cdots \\ \cdots & g[1] & g[0] & g[-1] & \cdots \\ \cdots & g[2] & g[1] & g[0] & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}.$$

Matrix representation: examples

1. integrator has $g[t] = \mathbb{1}[t] =$ , so

$$\llbracket G \rrbracket = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & 1 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & 1 & 1 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & 1 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & -1 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & 0 & -1 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}^{-1}$$

2. unit delay has $g[t] = \delta[t - 1] =$ , so

$$\llbracket G \rrbracket = \begin{bmatrix} \vdots & \vdots & \vdots & & \\ \dots & 0 & 0 & 0 & \dots \\ \dots & 1 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & & \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots & & \\ \dots & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 1 & \dots \\ \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & & \end{bmatrix}^{-1}$$