

Sampled-Data Control Systems (036709)

lecture no. 1

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Outline

Course info

Introduction

Review of analog signals

Review of analog systems

Review of control principles

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Course info

- Credit points: 3
- Prerequisite: **Control Theory** (035188)
- Grading policy: homework **100%** (5 best out of 6 assignments)
Homework solutions must be submitted **electronically** to c036709@technion.ac.il
- Course site: <http://leo.technion.ac.il/Courses/SDCS/>
- Literature:
 1. My slides.
 2. K. J. Åström and B. Wittenmark, *Computer-Controlled Systems: Theory and Design*. Englewood Cliffs, NJ: Prentice-Hall, 1997, 3rd ed.
 3. T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*. London: Springer-Verlag, 1995.

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Syllabus

1. Discrete signals and systems in time and transformed domains
 - time-domain properties and representations
 - discrete Fourier and z transforms
2. Connecting analog and digital (including the Sampling Theorem)
3. Discrete design methods in state space
 - state feedback, state observers, observer-based feedback
 - optimal control methods (LQR, state-feedback H_∞)
4. Digital redesign of analog controllers
5. Sampled-data design methods
6. Time-delay systems in discrete time
7. Multi-rate systems, lifting technique
8. Networked control systems
9. Quantization effects

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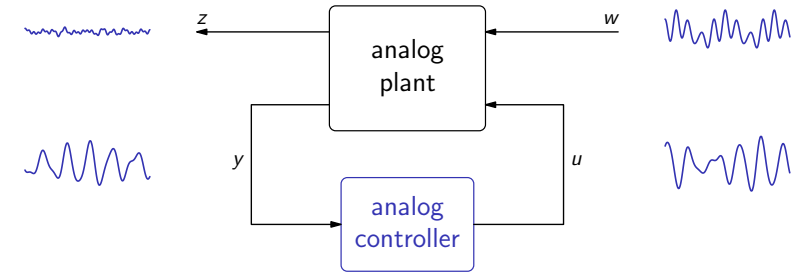
Review of analog signals

Review of analog systems

Review of control principles

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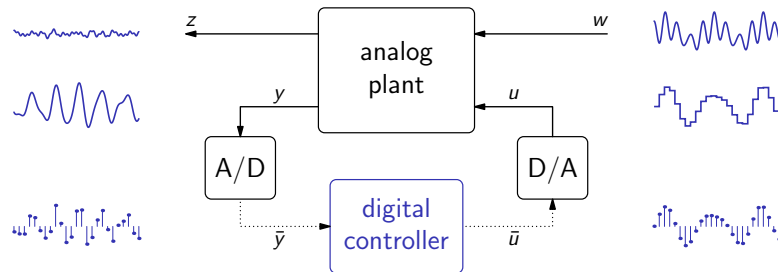
Analog control systems



- analog plant
- analog goals
 - w : contains exogenous signals (references, disturbances, measurement noise)
 - z : control goals, i.e. signals to be made small (tracking errors, control effort)
- analog control
 - (info about controlled processes, y ; its processing; control signal, u)

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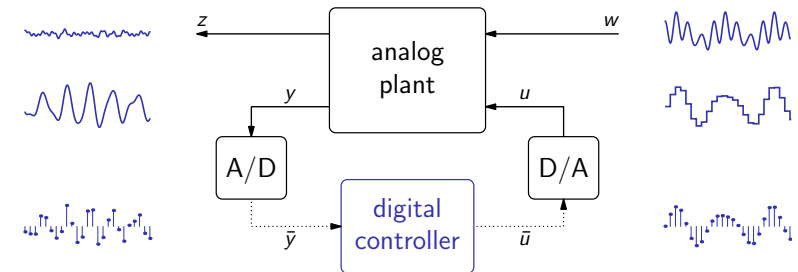
Sampled-data control systems



- analog plant
- analog goals
 - w : contains exogenous signals (references, disturbances, measurement noise)
 - z : control goals, i.e. signals to be made small (tracking errors, control effort)
- digital control
 - (info about controlled processes, $\bar{y}$; its processing; control signal, $\bar{u}$)

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Why sampled-data systems



Digital hardware

- more reliable
- more flexible
- cheaper

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Little history

1. '40s–'50s: initial period (theory before hardware)
 - transformed-domain methods
 - emphasis on analog behavior, but cumbersome theory
 - first implementations, from late '50s
2. '60s–'80s: discrete theory, mass implementations
 - state-space methods
 - emphasis on discrete analysis and design
 - struggles to keep sampling rate “fast enough”
3. '80s–'90s: golden age of sampled-data analysis
 - emphasis on analog performance, driven by overcoming “fast enough” sampling
 - great progress in theory (lifting, systems with jumps, multi-rate)
 - demotivated by silicon revolution
4. since '00s: networked control challenges
 - emphasis on intermittent and event-triggered sampling
 - great progress in theory
 - yet to see real implementations

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Nomenclature

$\mathbb{R}$	the set of real numbers, $\mathbb{R} = (-\infty, \infty)$
$\mathbb{R}_+$	the set of nonnegative real numbers, $\mathbb{R}_+ = [0, \infty)$
$\mathbb{R}_-$	the set of nonpositive real numbers, $\mathbb{R}_- = (-\infty, 0]$
$j\mathbb{R}$	the set of pure imaginary numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}_\alpha$	the half plane to the right of $\alpha \in \mathbb{R}$, i.e. $\mathbb{C}_\alpha = \{z \in \mathbb{C} \mid \operatorname{Re} z > \alpha\}$
$\bar{\mathbb{C}}_\alpha$	the closure of $\mathbb{C}_\alpha$, i.e. $\bar{\mathbb{C}}_\alpha = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \alpha\}$
$\mathbb{T}$	the unit circle, $\mathbb{T} := \{z \in \mathbb{C} \mid z = 1\}$
$\mathbb{D}_\alpha$	the open α -disk, $\mathbb{D} := \{z \in \mathbb{C} \mid z < \alpha\}$
$\bar{\mathbb{D}}_\alpha$	the closed α -disk, $\bar{\mathbb{D}} := \{z \in \mathbb{C} \mid z \leq \alpha\} = \mathbb{D}_\alpha \cup (\alpha \mathbb{T})$
$\mathbb{F}$	alias of either $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of positive integers (natural numbers)
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{Z}_-$	the set of nonpositive integers, $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$
$\mathbb{Z}_{i_1..i_2}$	the interval $\{i_1, i_1 + 1, \dots, i_2\}$

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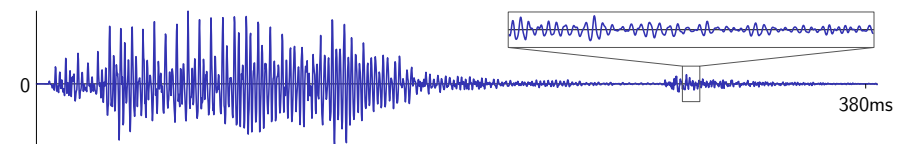
Review of analog systems

Review of control principles

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Signals

reflect evolving information:



Mathematically, signals are

- functions of independent variables, like time
- analog signals are functions $\mathbb{R} \supset \mathbb{I} \rightarrow \mathbb{F}^n$ ($\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$)
- denoted as $f(t)$

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Time-domain properties

Spaces (boundedness measures):

$$L_1(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{F} \mid \|f\|_1 := \int_{\mathbb{R}} |f(t)| dt < \infty \right\}$$

$$L_2(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{F} \mid \|f\|_2 := \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{1/2} < \infty \right\} \text{ (bounded energy)}$$

$$L_\infty(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{F} \mid \|f\|_\infty := \sup_{t \in \mathbb{R}} |f(t)| < \infty \right\} \text{ (bounded amplitude)}$$

Examples:

- $f(t) = e^{-t} \mathbb{1}(t)$ belongs to L_1 , L_2 and L_∞
- $f(t) = \frac{1}{t+1} \mathbb{1}(t)$ belongs to L_2 and L_∞ , but not to L_1
- $f(t) = \mathbb{1}(t)$ belongs to L_∞ , but not to L_1 or L_2

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Fourier transform

Given $f : \mathbb{R} \rightarrow \mathbb{F}^n$, its Fourier transform

$$\mathfrak{F}\{f\} = F(j\omega) := \int_{\mathbb{R}} f(t) e^{-j\omega t} dt,$$

where $\omega \in \mathbb{R}$ is frequency (in radians per time unit). Inverse transform

$$\mathfrak{F}^{-1}\{F\} = f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} F(j\omega) e^{j\omega t} d\omega.$$

Parseval:

$$\|f\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |F(j\omega)|^2 d\omega = \|F\|_2^2.$$

Remark: Strictly speaking, the Fourier transform is well defined (converges) for signals from $L_1(\mathbb{R})$ or, in a weaker sense, from $f \in L_2(\mathbb{R})$ only. A wider class of functions can be treated by allowing distributions (generalized functions) to be a part of signals in either time or frequency domains and taking some liberties with convergence. For example, $\mathfrak{F}\{\delta\} = 1$ and $\mathfrak{F}\{1\} = 1/(j\omega) + \pi\delta(\omega)$. But we proceed as if signals of interest are transformable.

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Fourier transform: interpretation

It follows from

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} F(j\omega) e^{j\omega t} d\omega$$

that $f(t)$ is a superposition of elementary harmonic signals $e^{j\omega t}$. The signal

- $F(j\omega)$ is the **frequency-domain** representation (or **spectrum**) of $f(t)$.

$F(j\omega_0)$ quantifies the contribution of $e^{j\omega_0 t}$ to $f(t)$.

Harmonic signals can be categorized on the “fast–slow” principle, namely

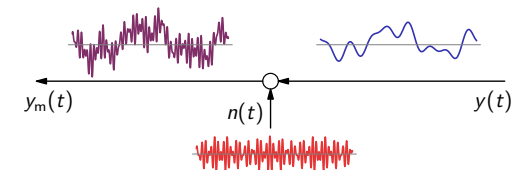
- $e^{j\omega_1 t}$ is faster (slower) than $e^{j\omega_2 t}$ if $|\omega_1| > |\omega_2|$ ($|\omega_1| < |\omega_2|$)

(just think of $\|\dot{f}\|_\bullet$ as a pace measure for f). Hence, spectrum offers a

- viewpoint on f , where fast and slow components are separated.

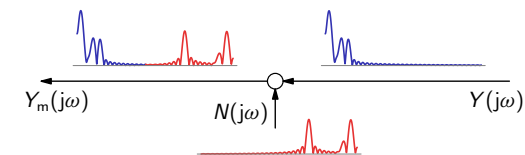
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Fourier transform: interpretation (contd)



Signal $y(t)$ and noise $n(t)$

- cannot be separated in the time domain,
- but can be in the frequency domain:



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Laplace transform

Given $f : \mathbb{R} \rightarrow \mathbb{F}^n$, its two-sided Laplace transform

$$\mathcal{L}\{f\} = F(s) := \int_{\mathbb{R}} f(t)e^{-st}dt$$

for all $s \in \mathbb{C}$ for which the integral converges. The set of all such $s \in \mathbb{C}$ is known as the **region of convergence** (RoC) of the transform. Technically,

- RoC requires that $f(t)e^{-st}$ belongs to $L_1(\mathbb{R})$.

If $f(t) = 0, \forall t < 0$, then a *typical* RoC is a right-half plane, $\mathbb{C}_\alpha$ or $\bar{\mathbb{C}}_\alpha$.

Examples:

- $f(t) = \mathbb{1}(t)$ has $F(s) = \frac{1}{s}$, with RoC in $\mathbb{C}_0$ (right-half plane)
- $f(t) = -\mathbb{1}(-t)$ has $F(s) = \frac{1}{s}$, with RoC in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$ (left-half plane)
- $f(t) = \mathbb{1}(t) - \mathbb{1}(t - \tau)$ has $F(s) = \frac{1 - e^{-\tau s}}{s}$, with RoC in $\mathbb{C}$
- $f(t) = e^{t^2}\mathbb{1}(t)$ has no Laplace transform

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Laplace transform: interpretation of s

For a differentiable $f(t)$,

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^+).$$

Thus, we may think of the variable “ s ” as

- the derivative in the transformed domain.

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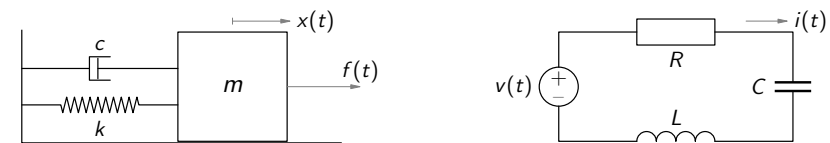
Review of analog systems

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Systems

are constraints imposed on signals:



- force $f(t)$ and mass position $x(t)$ are mutually dependent
- voltage $v(t)$ and current $i(t)$ are mutually dependent

I/O view on systems:

- some signals are considered as actions (**inputs**)
- some signals are considered as reactions (**outputs**)

Systems are then treated as mappings (operators) $u \mapsto y$ and their action is written as $y = Gu$ (not a product!).

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Basic properties

- A system G is said to be **linear** if

$$G(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 (Gu_1) + \alpha_2 (Gu_2)$$

for all admissible signals u_1, u_2 and all scalars α_1, α_2 (superposition).

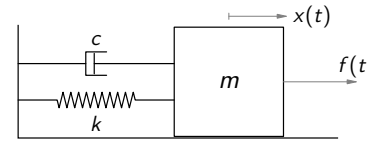
- A system $G : u \mapsto y$ is said to be **time invariant** if any time shift of its input $u(t)$ results in the same time shift of its output $y(t)$.
- A system $G : u \mapsto y$ is said to be **causal** if $y(t)$ at every t can depend only on $u(s)$ for $s \leq t$.
- A system $G : u \mapsto y$ is said to be **L_p -stable** (mostly $p = 2$ or $p = \infty$) if

$$u \in L_p \implies y \in L_p \quad \text{and} \quad \|y\|_p \leq \gamma \|u\|_p$$

for all u and some $\gamma \geq 0$ independent of u (BIBO stability for $p = \infty$).

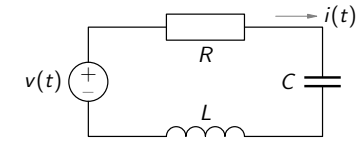
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Mathematical models



Position and force linked as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$



Charge ($\dot{q} = i$) and voltage linked as

$$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = v(t)$$

Both are of the form

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = k_{st}\omega_n^2 u(t)$$

and can be studied in a uniform fashion via their abstract **models**, which are

- descriptions of systems in a mathematical language.

Models are **always approximations** of physical process.

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Convolution representation

If $G : u \mapsto y$ is LTI (linear time-invariant), then

$$y(t) = \int_{-\infty}^{\infty} g(t-s)u(s)ds =: g * u,$$

where $g(t)$ is the impulse response of G . This system is causal iff $g(t) = 0$ whenever $t < 0$. Some more facts:

- if G is causal and $u(t) = 0$ for all $t < 0$ (a standard convention), then

$$y(t) = \int_0^t g(t-s)u(s)ds,$$

- if G is finite dimensional, then

$$g(t) = g_0\delta(t) + \sum_{i=1}^n \sum_{j=0}^{m_i} g_{ij} t^j e^{\lambda_i t}$$

($g_0 \in \mathbb{R}$ is known as the **feedthrough** term of G).

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Transfer function representation

If $G : u \mapsto y$ is LTI, then in the Laplace domain,

$$\mathcal{L}\{y\} = \mathcal{L}\{g * u\} = \mathcal{L}\{g\}\mathcal{L}\{u\}$$

in the intersection of their RoC's. In other words,

$$Y(s) = G(s)U(s),$$

where $G(s) = \mathcal{L}\{g\}$ is the **transfer function** of G . Some more facts:

- if G is causal, then the RoC of $G(s)$ is normally $\mathbb{C}_\alpha$ or $\bar{\mathbb{C}}_\alpha$ for some α ,
- if G is finite dimensional, then $G(s)$ is rational, i.e.

$$G(s) = \frac{N(s)}{D(s)} \quad \text{for some polynomials } N(s) \text{ and } D(s),$$

- a rational $G(s)$ is said to be proper if $\deg D(s) \geq \deg N(s)$,
- a finite-dimensional G is stable iff $G(s)$ is proper and $D(s)$ is Hurwitz.

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State-space representation

An LTI finite-dimensional $G : u \mapsto y$ can always be presented as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is a **state vector** of G . It is not unique, if $\tilde{x}(t) = Tx(t)$ for some nonsingular $T \in \mathbb{R}^{n \times n}$, then

$$\begin{aligned}\dot{\tilde{x}}(t) &= TAT^{-1}\tilde{x}(t) + TBu(t) \\ y(t) &= CT^{-1}\tilde{x}(t) + Du(t)\end{aligned}$$

is a realization of G too, *similar* to (A, B, C, D) . A realization is minimal iff it is controllable and observable. Every two minimal realizations are similar.

Connections:

- $g(t) = D\delta(t) + Ce^{At}B$, so $D = g_0$ is the feedthrough term
- $G(s) = D + C(sI - A)^{-1}B$, poles of $G(s)$ belong to $\text{spec}(A)$

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Steady-state responses

Let LTI $G : u \mapsto y$ be stable.

- If $u(t) = \gamma e^{j\omega t}$, then $y(t) = \gamma G(j\omega)e^{j\omega t}$, i.e. it remains harmonic, with
 - the amplitude of $u(t)$ scaled by a factor of $|G(j\omega)|$
 - the initial phase of $u(t)$ shifted by $\arg G(j\omega)$

where $G(j\omega) = G(s)|_{s=j\omega}$ is known as the frequency response of G .

- If $u(t) = \gamma e^{j\omega t}\mathbb{1}(t)$, then

$$\lim_{t \rightarrow \infty} y(t) = \gamma G(j\omega)e^{j\omega t}.$$

- If $u(t) = (u_0 + u_1 t)\mathbb{1}(t)$, then

$$\lim_{t \rightarrow \infty} y(t) = G(0)u_0 + G'(0)u_1 + G(0)u_1 t,$$

where $G(0)$ is known as the static (DC) gain of G .

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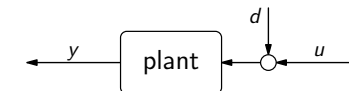
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A prototype control problem



Generate

- control signal u ,

for which

- regulated variable y behaves like a reference signal r

despite the presence of

- load disturbance d .

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Ultimate methodology: plant inversion

If we know

- plant model, i.e. how it responds to input signals, perfectly and
- measure the load disturbance d ,

we can attain the goal by

$$u = \frac{1}{\text{plant}} r - d$$

(it guarantees $y \equiv r$ for all r and d). This control strategy called

- **plant inversion**.

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Limitations of plant inversion

Even if we put implementability and the availability of d aside, control law

$$u = \frac{1}{\text{plant}} r - d$$

can seldom be feasible, as it might result in

- unstable response of u (if plant dynamics are nonminimum-phase)
- too large $|u|$ (control resources never unlimited)

For that reason, we may endeavor to attain

$$u \approx \frac{1}{\text{plant}} r - d$$

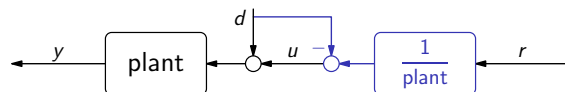
instead, where “ $\approx$ ” may refer to plant inversion

- over a limited frequency band
- with limited accuracy
- ...

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Ideal open-loop plant inversion

Natural way to implement by the open-loop scheme

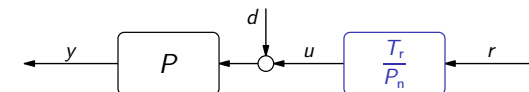


Limitations:

- disturbances are not always (well, seldom) measurable
- plant model is never perfectly known, while the scheme hinges upon modeling accuracy
- might require differentiations, which is sensitive to noise
- ...

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Realistic open-loop plant inversion



where

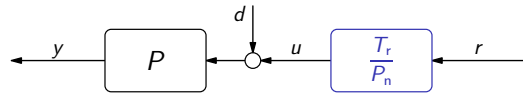
- P_n is a “nominal” model of the plant
- T_r is a reference model, i.e. a desired response of the system to r

Guidelines:

- T_r/P_n must be stable and implementable
 - all nonminimum-phase zeros of $P_n(s)$ are zeros of $T_r(s)$
 - pole excess of $T_r(s)$ is at least as large as that of $P_n(s)$
- T_r must be stable itself
- $T_r(j\omega) \approx 1$ at frequencies (normally, low) where good tracking needed
- T_r/P_n does not produce “large” control signal (bandwidth limitations)

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Realistic open-loop plant inversion: limitations



- Applies only to stable plants, otherwise is internally unstable.
- No way to handle disturbances.
- The actual response

$$y = T_{yr}r = P \frac{T_r}{P_n} r = T_r r - \left(1 - \frac{P}{P_n}\right) T_r r.$$

The relative mismatch between it and the expected response, $T_r r$, is

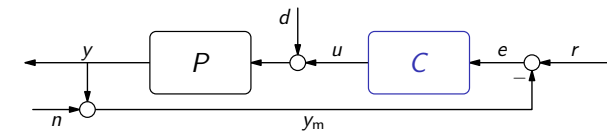
$$\left|1 - \frac{Y(j\omega)}{T_r(j\omega)R(j\omega)}\right| = \left|1 - \frac{P(j\omega)}{P_n(j\omega)}\right|.$$

This is the modeling uncertainty of P_n , no way we can reduce it.

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Closed-loop control

An alternative is closed-loop control, which uses feedback to generate u :



Closed-loop systems:

$$r \mapsto y: T = \frac{PC}{1 + PC} \quad \text{complementary sensitivity}$$

$$d \mapsto y: T_d = \frac{P}{1 + PC} \quad \text{disturbance sensitivity}$$

$$n \mapsto y: -T$$

$$r \mapsto u: T_c(s) = \frac{C}{1 + PC} \quad \text{control sensitivity}$$

$$d \mapsto u: -T$$

$$n \mapsto u: -T_c$$

plus sensitivity $S = 1 - T = 1/(1 + PC)$, which connects r with $r - y$.

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Closed-loop plant inversion

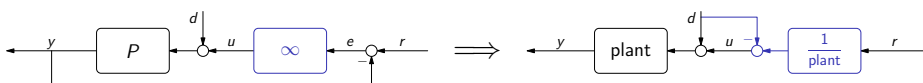
If $C \rightarrow \infty$ (practical meaning is that $|C| \gg |P|$, in whatever sense),

$$T_u = \frac{C}{1 + PC} \rightarrow \frac{1}{P} \quad \text{and} \quad T = \frac{PC}{1 + PC} \rightarrow 1$$

so that

$$u = \frac{1}{P} r - d \quad \Rightarrow \quad y = r.$$

In other words, in the absence of n



Important is that the

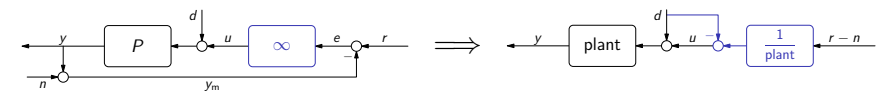
- choice $C \rightarrow \infty$ does not depend on the plant,
- so the ability to attain plant inversion in this case might be seen as **miracle**.

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Closed-loop plant inversion: limitations

It's not quite trivial though, we're limited by

- closed-loop stability
- sensitivity to measurement noise:



- limitations on $|u|$
- robustness
- ...

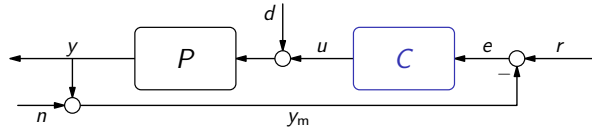
which effectively restrict high-gain feedback to a **finite frequency range** and thus leads to

- high-low loop gain game
- the need in crossover acrobatics

This, in turn, determines / limits the closed-loop bandwidth.

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Steady-state disturbance responses



Let C internally stabilize the system (i.e. T , S , T_d , and T_c are stable).

- If $d(t) = d_0 \mathbb{1}(t)$, then

$$\lim_{t \rightarrow \infty} y(t) = T_d(0)d_0 = 0 \iff P(0) = 0 \vee |C(0)| = \infty,$$

which is guaranteed by adding an **integral action** to the controller.

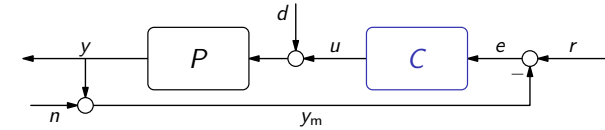
- If $d(t) = \gamma e^{j\omega_0 t} \mathbb{1}(t)$, then

$$\lim_{t \rightarrow \infty} y(t) = \gamma T_d(j\omega_0) e^{j\omega_0 t} = 0 \iff P(j\omega_0) = 0 \vee |C(j\omega_0)| = \infty,$$

which is guaranteed by adding a **harmonic oscillator** to the controller.

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Loop-shaping ideas



Assumptions (reasonable):

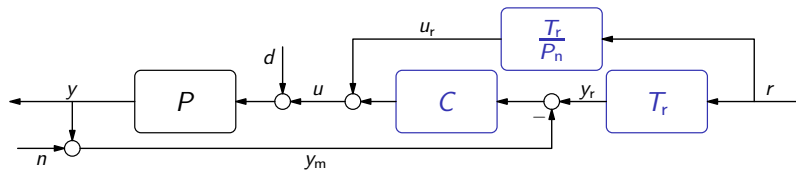
- references r are “slow”
- only “slow” parts of load disturbances d are handled by the controller
- measurement noise n is “fast”

Consequences for the loop transfer function $L(s) = P(s)C(s)$:

- gain $|L(j\omega)|$ should be high at low (i.e. sub-crossover) frequencies
- gain $|L(j\omega)|$ should be low at high (i.e. super-crossover) frequencies
- $L(j\omega)$ should be kept far from the critical point in the crossover range where the crossover frequency ω_c is a design parameter to choose.

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2-degrees-of-freedom control architecture



If $P_n = P$, then

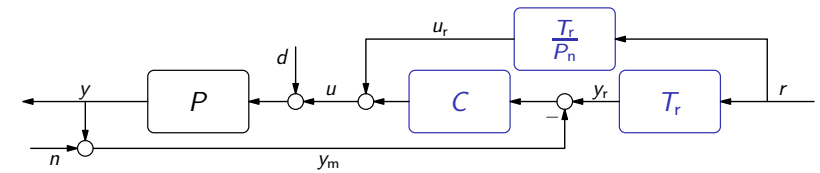
$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} T_r \\ T_r/P \end{bmatrix} r + \begin{bmatrix} T_d \\ -T \end{bmatrix} d - \begin{bmatrix} T \\ T_c \end{bmatrix} n.$$

Thus,

- the responses to d and n do not depend on the open-loop design,
- the response to r does not depend on the closed-loop design, viz. C ,
- open-loop design is applicable to unstable plants as well, provided T_r and T_r/P_n are both stable.

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2-degrees-of-freedom control architecture (contd)



If $P_n \neq P$, then the command response of this system is

$$y = T_r r - \frac{1}{1 + PC} \left(1 - \frac{P}{P_n} \right) T_r r.$$

The relative mismatch between it and the expected response, $T_r r$, is

$$\left| 1 - \frac{Y(j\omega)}{T_r(j\omega)R(j\omega)} \right| = |S(j\omega)| \left| 1 - \frac{P(j\omega)}{P_n(j\omega)} \right| = \left| 1 - \frac{T(j\omega)}{T_n(j\omega)} \right|.$$

Hence, the effect of plant modeling uncertainty can potentially be reduced by a **robust** feedback design, having a low sensitivity gain $|S(j\omega)|$.

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