



Sampled-Data Control Systems

Spring 2019/2020

Homework no. 3

(submission deadline: 19.05.2020, 11:59pm; please make an effort to be concise, clear, accurate, and rigorous)

Problem 1 (20%). Let P be a causal LTI system, whose transfer function $P(s)$ is strictly proper, and $\bar{P}_h = \mathcal{S}_{\text{idl}} P \mathcal{H}_{\text{ZOH}}$ be its discretization for some sampling period $h > 0$. Find conditions, in terms of the impulse response $p(t)$ of P , under which the pulse-transfer function $\bar{P}_h(z)$ has a pole excess of at least 2. Give an example of a second-order system with a strictly proper transfer function of this kind (assume $h = 1$), namely $P(s)$ and the corresponding $\bar{P}_h(z)$.

Hint: Read carefully Slide 44 of Lecture 2.

Problem 2 (30%). The continuous-time delay element $\bar{D}_\tau : u \mapsto y$ acts as $y(t) = u(t - \tau)$. It is an LTI system, whose impulse response is $\delta(t - \tau)$ and transfer function is $e^{-\tau s}$. Consider the system $P = P_0 \bar{D}_\tau$ for some causal and finite-dimensional P_0 .

1. How are the impulse responses $p(t)$ and $p_0(t)$ of P and P_0 , respectively, related? In what time interval does $p(t)$ have its support?
2. Assume that $\tau/h = l$ for some $l \in \mathbb{N}$. Find the impulse response $\bar{p}[i]$ of the discretized system $\bar{P}_h = \mathcal{S}_{\text{idl}} P \mathcal{H}_{\text{ZOH}}$ in terms of $p_0(t)$. How is it related to the impulse response $\bar{p}_{0,h}[i]$ of $\bar{P}_{0,h} = \mathcal{S}_{\text{idl}} P_0 \mathcal{H}_{\text{ZOH}}$? How are the transfer functions $\bar{P}_h(z)$ and $\bar{P}_{0,h}(z)$ related? Write explicitly the first *potentially nonzero* element of $\bar{p}_h[i]$.
3. Now consider a general $\tau > 0$, assuming that $\tau = lh - \theta$ for some $l \in \mathbb{N}$ and $\theta \in (0, h)$. Find the impulse response $\bar{p}[i]$ of the discretized system $\bar{P}_h = \mathcal{S}_{\text{idl}} P \mathcal{H}_{\text{ZOH}}$ in terms of $p_0(t)$. Write explicitly the first *potentially nonzero* element of $\bar{p}_h[i]$.

Problem 3 (30%). Consider discretization under the (non-causal) first-order hold $\mathcal{H}_{\text{FOH}}$, whose hold function

$$\phi_{\text{FOH}}(t) = \frac{1}{h} \begin{cases} h+t & \text{if } t \in [-h, 0] \\ h-t & \text{if } t \in [0, h] \\ 0 & \text{otherwise} \end{cases} = \begin{array}{c} 1 \\ \diagup \quad \diagdown \\ -h \quad 0 \quad h \\ t \end{array}.$$

1. Find the impulse response of $\bar{P}_h = \mathcal{S}_{\text{idl}} P \mathcal{H}_{\text{FOH}}$ for a SISO LTI system with a strictly proper transfer function. When is it causal? Strictly causal?
2. Assuming $P(s) = C(sI - A)^{-1}B$ with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, and $C \in \mathbb{R}^n$, construct a state-space realization of $\bar{P}_h(z)$.
3. (bonus, +8pt) Express the parameters of the state-space realization of $\bar{P}_h$ above using *one* $(n+2) \times (n+2)$ -dimensional matrix exponential. Check the formula by comparing its outcome with the result of MATLAB's command `c2d(tf(1, [1 1]), h, 'foh')`, i.e. for $P(s) = 1/(s+1)$, for $h \in \{0.1, 1\}$.

Problem 4 (20%). Consider an LTI system P given in terms of its realization $(A, B, C, 0)$, where

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} 0 & \omega_n & 0 \\ -\omega_n & 0 & \omega_n \end{bmatrix}$$

(C doesn't matter).

1. Construct $(\bar{A}, \bar{B})$ of its stroboscopic model $\mathcal{S}_{\text{idl}} P \mathcal{H}_{\text{imp}}$ under the impulsive hold and $h = \pi/\omega_n$, assuming that C remains unchanged.
2. Characterize all reachable states of $x[t+1] = \bar{A}x[t] + \bar{B}u[t]$.