

Introduction to Control (00340040)

lecture no. 9

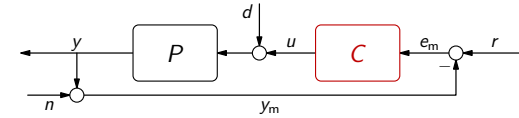
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Loop shaping



Philosophy:

- analyze, and affect, properties of the closed-loop system via properties of the frequency response $L(j\omega)$ of the loop transfer function $L = PC$,

i.e.

$$L(j\omega) \rightarrow \begin{cases} S : r \mapsto e \\ T : r \mapsto y, -n \mapsto y \\ T_d : d \mapsto y \\ T_c : r \mapsto u, -n \mapsto u \end{cases}$$

Today:

- internal stability via properties of the Nyquist (polar +) plot of $L(j\omega)$.

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Outline

Nyquist stability criterion: no open-loop $j\omega$ -poles

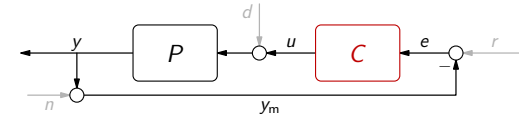
Nyquist stability criterion: including open-loop poles at the origin

Simplifications

Analysis of $kL(s)$ loops

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Preliminary: return difference



Let

$$L(s) := P(s)C(s) = \frac{N_L(s)}{D_L(s)}$$

(loop transfer function) and assume that there are **no unstable** pole / zero **cancellations** between $P(s)$ and $C(s)$. Consider the transfer function

$$1 + L(s) = \frac{N_L(s) + D_L(s)}{D_L(s)} = \frac{\chi_{cl}(s)}{D_P(s)D_C(s)},$$

called **return difference**. It is readily seen that

- **unstable poles** of $1 + L(s)$ are **open-loop unstable poles**
- **unstable zeros** of $1 + L(s)$ are **closed-loop unstable poles**.

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Assumptions

We consider the system



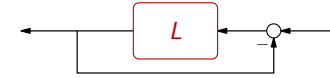
and assume that

- $L(s)$ is proper
- $L(s)$ has no $j\omega$ -axis poles

will be relaxed later on

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Idea



Technical steps:

1. define a simple closed contour Γ_s containing all singularities of $1 + L(s)$ in the ORHP $\mathbb{C}_0 := \{s \in \mathbb{C} \mid \text{Re } s > 0\}$;
2. determine the mapping Γ_{1+L} of Γ_s by the return difference;
3. count the number ν of clockwise encirclings the origin by Γ_{1+L} .

By the argument principle,

$$\nu = \pi_{cl}^+ - \pi_{ol}^+ \iff \pi_{cl}^+ = \nu + \pi_{ol}^+,$$

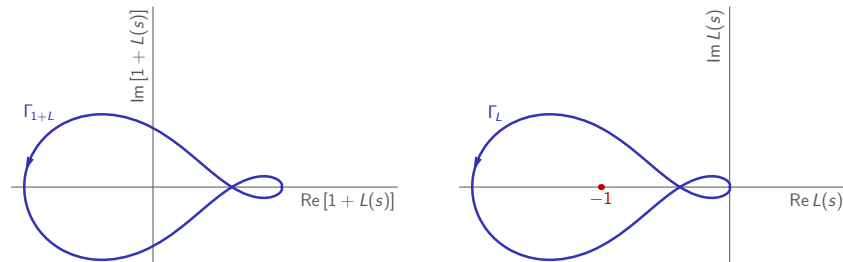
where

- π_{cl}^+ is the number of closed-loop poles in $\mathbb{C}_0$
- π_{ol}^+ is the number of open-loop poles (those of $L(s)$) in $\mathbb{C}_0$

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Idea: slight modification

The difference between $[1 + L(s)]$ -plane and $L(s)$ -plane is just the horizontal shift by 1:



Hence,

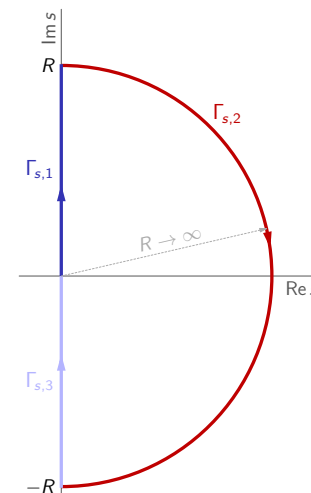
$$\# \text{encirclings the origin by } \Gamma_{1+L} = \# \text{encirclings the point } -1 + j0 \text{ by } \Gamma_L$$

and we may replace steps 2. and 3. above with

2. determine the mapping Γ_L of Γ_s by the loop transfer function;
3. count the number ν of clockwise encirclings the point $-1 + j0$ by Γ_L .

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Nyquist contour



The contour

$$\Gamma_s = \Gamma_{s,1} \cup \Gamma_{s,2} \cup \Gamma_{s,3},$$

where

$$\begin{aligned} \Gamma_{s,1} &= j\omega, & \omega &: 0 \rightarrow R, \\ \Gamma_{s,2} &= Re^{j\theta}, & \theta &: \frac{\pi}{2} \rightarrow -\frac{\pi}{2}, \\ \Gamma_{s,3} &= j\omega, & \omega &: -R \rightarrow 0, \end{aligned}$$

with $R \rightarrow \infty$ is called the **Nyquist contour**. It contains all ORHP poles of transfer functions having a finite number of such poles.

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Mapping the Nyquist contour by L

Consider each of the Nyquist contour segments separately:

$\Gamma_{s,1}$: In this case $s = j\omega$ ($\omega \geq 0$) mapped to $L(j\omega)$, so that

- $\Gamma_{L,1}$ is the **polar plot of the frequency response of L** .

$\Gamma_{s,2}$: As $L(s)$ is proper ($b_n = 0$ if $L(s)$ is strictly proper),

$$\begin{aligned} L(Re^{j\theta}) &= \frac{b_n R^n e^{j\theta n} + b_{n-1} R^{n-1} e^{j\theta(n-1)} + \dots + b_1 R e^{j\theta} + b_0}{R^n e^{j\theta n} + a_{n-1} R^{n-1} e^{j\theta(n-1)} + \dots + a_1 R e^{j\theta} + a_0} \\ &= \frac{b_n + b_{n-1} R^{-1} e^{-j\theta} + \dots + b_1 R^{1-n} e^{j\theta(1-n)} + b_0 R^{-n} e^{-j\theta n}}{1 + a_{1-n} R^{-1} e^{-j\theta} + \dots + a_1 R^{1-n} e^{-j\theta(n-1)} + a_0 R^{-n} e^{-j\theta n}} \end{aligned}$$

Thus, $\lim_{R \rightarrow \infty} L(Re^{j\theta}) = b_n$ for every θ , i.e.

- $\Gamma_{L,2}$ collapses to a **single point, b_n** .

Since this point already belongs to $\Gamma_{L,1}$, we may effectively **omit** $\Gamma_{L,2}$.

$\Gamma_{s,3}$: As $L(s)$ has real coefficients, $L(-j\omega) = \overline{L(j\omega)}$, so that

- $\Gamma_{L,3}$ is the **mirror of $\Gamma_{L,1}$ around the real axis**.

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Nyquist plot

The union of $\Gamma_{L,1}$ and $\Gamma_{L,3}$, which is the

- graph of $L(j\omega)$ in polar coordinates as ω runs from $-\infty$ to $+\infty$, is called the **Nyquist plot** of $L(j\omega)$.

The Nyquist plot can be constructed in two steps:

- construct the polar plot of $L(j\omega)$
(which is the graph of $L(j\omega)$ in polar coordinates as ω runs from 0 to $+\infty$)
- add the reflection of the polar plot about the real axis

The **critical point, $-1 + j0$** , on the $L(s)$ -plane is also presented, owing to its importance in the stability analysis.

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Nyquist stability criterion



Theorem (Nyquist)

The closed-loop system is stable iff the Nyquist plot of $L(j\omega)$

- does not intersect the critical point $-1 + j0$
- encircles the critical point π_{ol}^+ times in the counterclockwise direction as ω grows from $-\infty$ to $+\infty$, where π_{ol}^+ is the number of poles of $L(s)$ in $\mathbb{C}_0$.

Proof:

- If $\exists \omega_0$ s.t. $L(j\omega_0) = -1$, $\chi_{cl}(j\omega_0) = 0$ and closed-loop system unstable.
- We know that the number of clockwise encirclements of the critical point by the mapping of the Nyquist contour by $L(s)$ (i.e. the Nyquist plot) is $\nu = \pi_{cl}^+ - \pi_{ol}^+$. The stability of the closed-loop system is equivalent to $\pi_{cl}^+ = 0$, hence the system stable iff $\nu = -\pi_{ol}^+$. $\square$

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Nyquist stability criterion (contd)



Corollary 1: If the Nyquist plot of $L(j\omega)$ does not cross the critical point, then the number of closed-loop unstable poles

$$\pi_{cl}^+ = \pi_{ol}^+ + \nu,$$

where ν is the number of clockwise encirclements of the critical point $-1 + j0$ by the Nyquist plot.

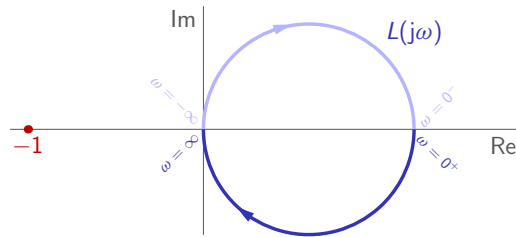
Corollary 2: If the Nyquist plot of $L(j\omega)$ intersects the critical point at some frequency, say $\omega = \omega_0$, then $\chi_{cl}(s)$ has at least one root at $s = j\omega_0$.

Corollary 3: If L is **stable** itself, then the closed-loop system is stable iff the Nyquist plot of $L(j\omega)$ neither intersects nor encircles the critical point as ω increases from $-\infty$ to $+\infty$.

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Example 1

Let $L(s) = \frac{1.2}{s+1}$. Its Nyquist plot is



We have:

- L is stable (i.e. $\pi_{ol}^+ = 0$)
- Nyquist plot does not encircle the critical point

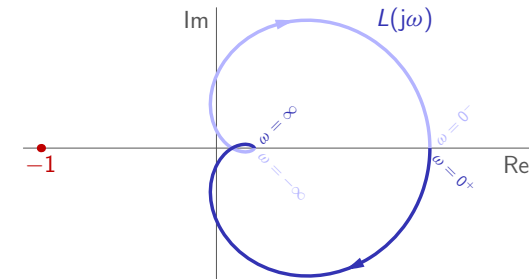
Hence,

- the closed-loop system is **stable**

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Example 2

Let $L(s) = \frac{(3s+8)(9s^2+3s+19)}{125(s+1)^3}$. Its Nyquist plot is



We have:

- L is stable (i.e. $\pi_{ol}^+ = 0$)
- Nyquist plot does not encircle the critical point

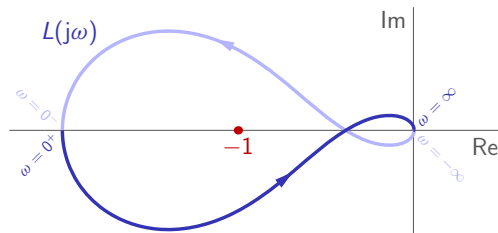
Hence,

- the closed-loop system is **stable**

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Example 3

Let $L(s) = \frac{2}{(s-1)(0.125s+1)^3}$. Its Nyquist plot is



We have:

- L is unstable with $L(s)$ having one unstable pole (i.e. $\pi_{ol}^+ = 1$)
- Nyquist plot encircles the critical point once in the counterclockwise direction

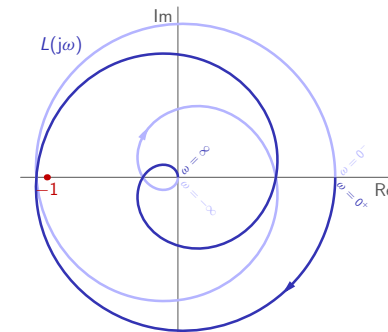
Hence,

- the closed-loop system is **stable**

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Example 4

Let $L(s) = \frac{(-s+1)^3}{(s+1)^4}$. Its Nyquist plot is



We have:

- L is stable (i.e. $\pi_{ol}^+ = 0$)
- Nyquist plot encircles the critical point twice in the clockwise direction

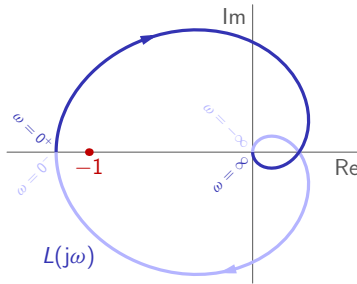
Hence,

- the closed-loop system is **unstable** (in fact, $\pi_{cl}^+ = 2$).

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Example 5

Let $L(s) = \frac{1.2(-0.5s + 1)^2}{(0.5s + 1)^2(s - 1)}$. Its Nyquist plot is



We have:

- L is unstable with $L(s)$ having one unstable pole (i.e. $\pi_{ol}^+ = 1$)
- Nyquist plot encircles the critical point once in the clockwise direction

Hence,

- the closed-loop system is **unstable** (in fact, $\pi_{cl}^+ = 2$).

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Outline

Nyquist stability criterion: no open-loop $j\omega$ -poles

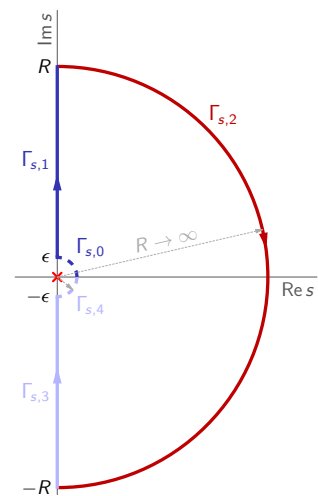
Nyquist stability criterion: including open-loop poles at the origin

Simplifications

Analysis of $kL(s)$ loops

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Nyquist contour if $L(s)$ has pole(s) at the origin



The modified contour is

$$\Gamma_s = \Gamma_{s,0} \cup \Gamma_{s,1} \cup \Gamma_{s,2} \cup \Gamma_{s,3} \cup \Gamma_{s,4},$$

where

$$\begin{aligned} \Gamma_{s,0} &= \epsilon e^{j\theta}, & \theta &: 0 \rightarrow \frac{\pi}{2}, \\ \Gamma_{s,1} &= j\omega, & \omega &: \epsilon \rightarrow R, \\ \Gamma_{s,2} &= R e^{j\theta}, & \theta &: \frac{\pi}{2} \rightarrow -\frac{\pi}{2}, \\ \Gamma_{s,3} &= j\omega, & \omega &: -R \rightarrow -\epsilon, \\ \Gamma_{s,4} &= \epsilon e^{j\theta}, & \theta &: -\frac{\pi}{2} \rightarrow 0, \end{aligned}$$

with $R \rightarrow \infty$ and $\epsilon \rightarrow 0$. It still contains all ORHP poles of transfer functions having a finite number of such poles.

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Mapping $\Gamma_{s,0}$ and $\Gamma_{s,4}$ by $L(s)$

The mapping of $\Gamma_{s,1} \cup \Gamma_{s,2} \cup \Gamma_{s,3}$ are practically unchanged. So to complete the picture we only need

$\Gamma_{s,0}$: Let $L(s) = \frac{1}{s^n} \tilde{L}(s)$, where $n \in \mathbb{N}$ and $\tilde{L}(0) \neq 0$ is **finite**. In this case

$$L(\epsilon e^{j\theta}) = \frac{1}{\epsilon^n e^{jn\theta}} \tilde{L}(\epsilon e^{j\theta}) \xrightarrow{\epsilon \rightarrow 0} \frac{\tilde{L}(0)}{\epsilon^n} e^{-jn\theta},$$

so that

- $\Gamma_{L,0}$ is an arc of ∞ radius, starting at either positive (if $\tilde{L}(0) > 0$) or negative (if $\tilde{L}(0) < 0$) real axis and going $n\pi/2$ [rad] clockwise until it intersect the beginning of $\Gamma_{s,1}$.

$\Gamma_{s,4}$: As $L(s)$ has real coefficients, $L(-j\omega) = \overline{L(j\omega)}$, hence

- $\Gamma_{L,4}$ is the mirror of $\Gamma_{L,0}$ around the real axis.

Remark: $\Gamma_{L,4} \cup \Gamma_{L,0}$ is an arc connecting the end of $\Gamma_{L,3}$ ($\omega \rightarrow 0^-$) and the beginning of $\Gamma_{L,1}$ ($\omega \rightarrow 0^+$) through the angle πn in the **clockwise** direction.

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Nyquist stability criterion



Since the modified Nyquist contour **excludes** the **poles of $L(s)$ at the origin**,

- these poles are not counted in π_{ol}^+ .

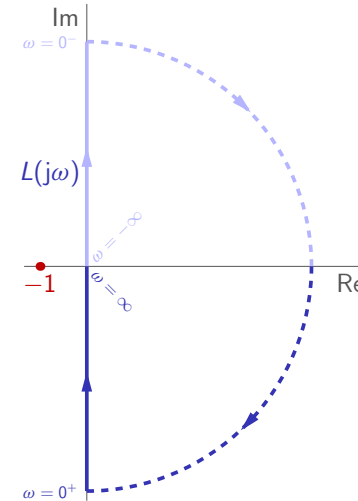
That may be done since $\chi_{cl}(s)$ cannot¹ have roots at $s = 0$, so all unstable closed-loop poles are separated from the origin and, therefore, lie inside the modified Nyquist contour if ϵ is sufficiently small.

The rest of the criterion remains unchanged...

¹Common roots of $D_L(s)$ and $\chi_{cl}(s) = N_L(s) + D_L(s)$ are common roots of $D_L(s)$ and $N_L(s)$, so cannot be unstable under our assumption.

Example 6

Let $L(s) = \frac{1}{s}$. Its Nyquist plot is



We have:

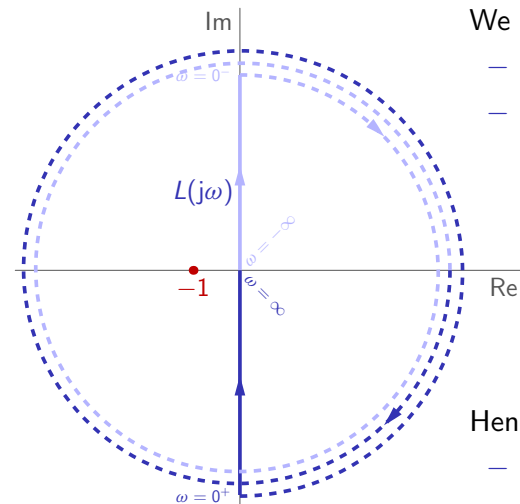
- L is “stable” (i.e. $\pi_{ol}^+ = 0$)
- Nyquist plot does not encircle the critical point

Hence,

- the closed-loop system is **stable**

Example 7

Let $L(s) = \frac{1}{s^5}$. Its Nyquist plot is



We have:

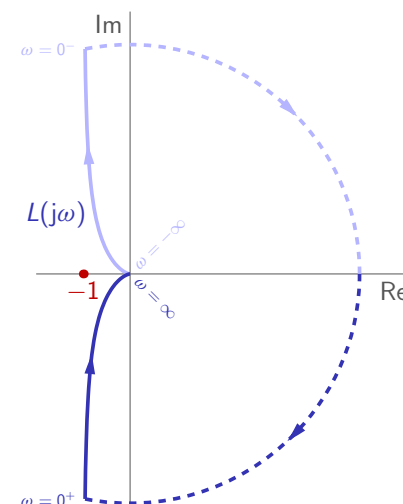
- L is “stable” (i.e. $\pi_{ol}^+ = 0$)
- Nyquist plot encircles the critical point twice in the clockwise dir.

Hence,

- the closed-loop system is **unstable** ($\pi_{cl}^+ = 2$)

Example 8

Let $L(s) = \frac{1}{s(s+1)}$. Its Nyquist plot is



We have:

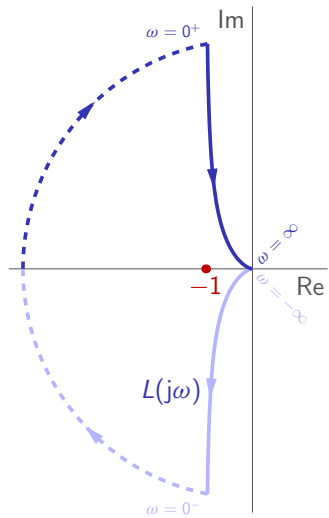
- L is “stable” (i.e. $\pi_{ol}^+ = 0$)
- Nyquist plot does not encircle the critical point

Hence,

- the closed-loop system is **stable**

Example 9

Let $L(s) = \frac{1}{s(s-1)}$. Its Nyquist plot is



We have:

- $L(s)$ has one unstable pole (i.e. $\pi_{ol}^+ = 1$)
- Nyquist plot encircles the critical point once in the clockwise dir.

Hence,

- the closed-loop system is **unstable** ($\pi_{cl}^+ = 2$)

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Outline

Nyquist stability criterion: no open-loop $j\omega$ -poles

Nyquist stability criterion: including open-loop poles at the origin

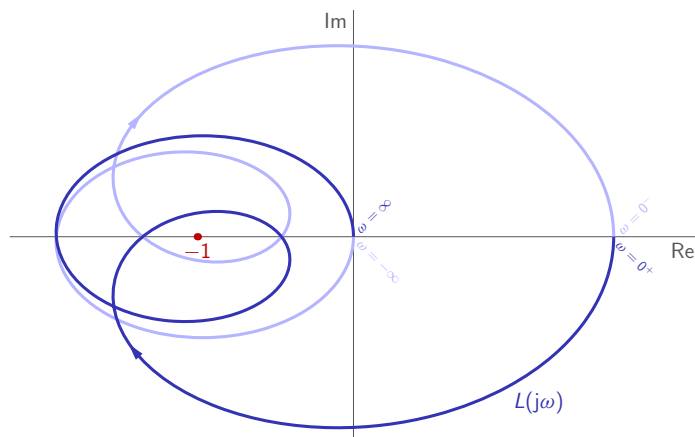
Simplifications

Analysis of $kL(s)$ loops

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Counting encirclements

Counting counterclockwise encirclements may be tedious (-4 in this case):

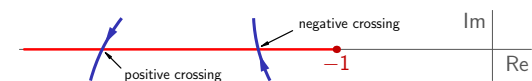


But there is a way to simplify this procedure...

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Positive and negative crossings

Consider the ray $(-\infty, -1]$ in the $L(j\omega)$ -plane:



A **crossing** occurs when the plot of $L(j\omega)$ intersects the ray. It is said to be

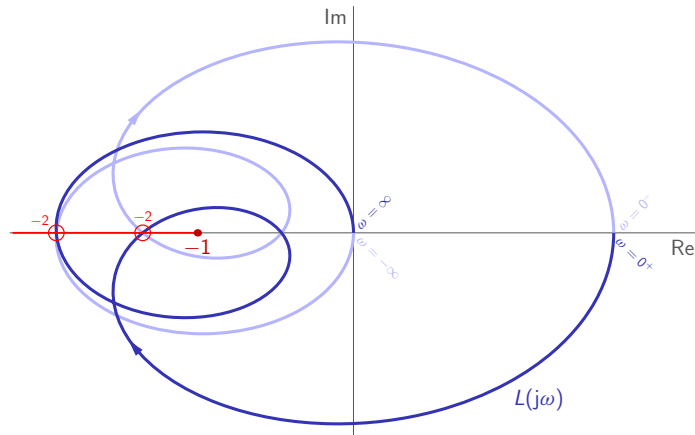
- *positive* if the direction of $L(j\omega)$ is downward
- *negative* if the direction of $L(j\omega)$ is upward

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Encirclements via the Nyquist plot

Lemma

The number of counterclockwise encirclements around the critical point by the Nyquist plot of $L(j\omega)$ equals the net sum of crossings the ray $(-\infty, -1]$ by the Nyquist plot of $L(j\omega)$.

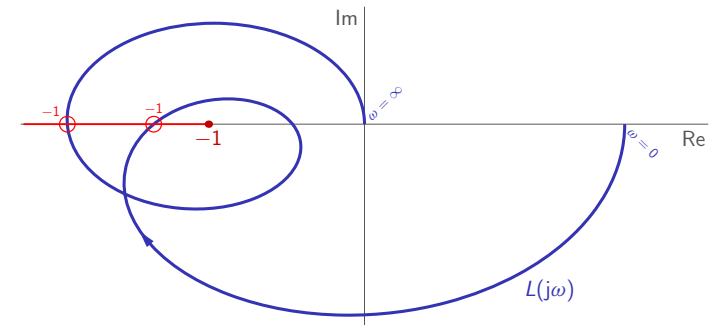


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Encirclements via the polar plot

Lemma

The number of counterclockwise encirclements around the critical point by the Nyquist plot of $L(j\omega)$ equals twice² the net sum of crossings the ray $(-\infty, -1]$ by the polar plot of $L(j\omega)$.

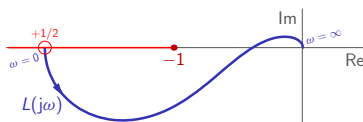


²One should be careful with the cases when the polar plot starts or / and ends at the ray. This situation, if happens, may be counted as "half crossings."

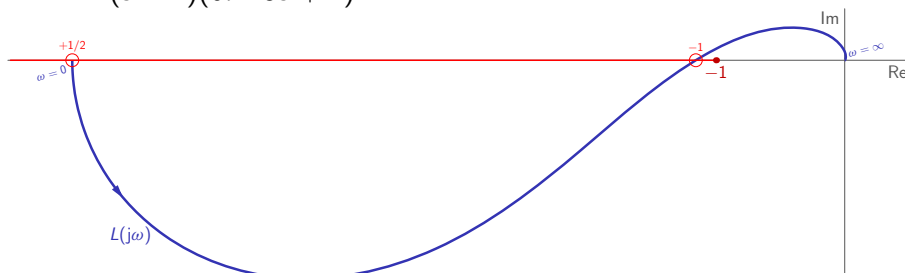
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Example 3 (contd)

If $L(s) = \frac{2}{(s-1)(0.125s+1)^3}$ (one counterclockwise encirclement):



If $L(s) = \frac{6}{(s-1)(0.125s+1)^3}$ (one clockwise encirclement):



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Outline

Nyquist stability criterion: no open-loop $j\omega$ -poles

Nyquist stability criterion: including open-loop poles at the origin

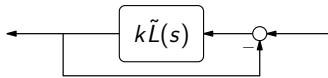
Simplifications

Analysis of $kL(s)$ loops

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Problem

Consider (in fact, $L(s) = k\tilde{L}(s)$)

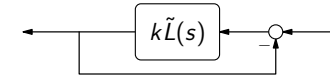


where $\tilde{L}(s)$ is a proper transfer function and k is the parameter we want to choose. The problem is to

- find all k stabilizing the system.

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Nyquist criterion revisited



The return difference transfer function rewrites as

$$1 + L(s) = 1 + k\tilde{L}(s) = k \left(\frac{1}{k} + \tilde{L}(s) \right)$$

Since the multiplication by k affects neither poles nor zeros of this transfer function, we may analyze stability by mapping the Nyquist contour by $\frac{1}{k} + \tilde{L}(s)$. This, in turn, implies that we should

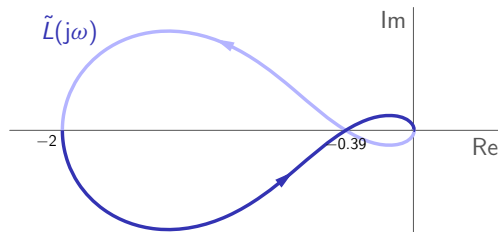
- analyze the Nyquist plot of $\tilde{L}(j\omega)$ with the **critical point** $-\frac{1}{k} + j0$.

In other words, different critical point is the only modification we need.

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Example 3 (contd)

Let $\tilde{L}(s) = \frac{2}{(s-1)(0.125s+1)^3}$. Its Nyquist plot³ is



In this case $\pi_{ol}^+ = 1$ and there are four intervals of k :

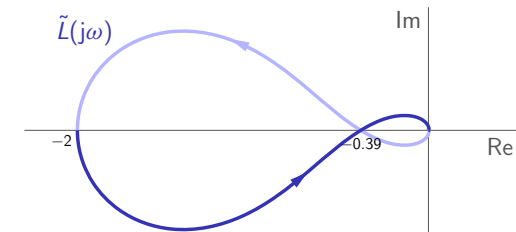
- $-\frac{1}{k} < -2$: no encirclings $\Rightarrow \pi_{cl}^+ = 1$;
- $-2 < -\frac{1}{k} < -\frac{529}{1372} \approx -0.39$: 1 counterclockwise encircling $\Rightarrow \pi_{cl}^+ = 0$;
- $-0.39 \approx -\frac{529}{1372} < -\frac{1}{k} < 0$: 1 clockwise encircling $\Rightarrow \pi_{cl}^+ = 2$;
- $-\frac{1}{k} > 0$: no encirclings $\Rightarrow \pi_{cl}^+ = 1$.

³Points of intersection with the real axis found by solving $\tilde{L}(j\omega) = \tilde{L}(-j\omega)$ in ω .

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Example 3 (contd)

Let $\tilde{L}(s) = \frac{2}{(s-1)(0.125s+1)^3}$. Its Nyquist plot is



In this case $\pi_{ol}^+ = 1$ and there are three intervals of k :

- $k < \frac{1}{2}$: no encirclings $\Rightarrow \pi_{cl}^+ = 1$;
- $\frac{1}{2} < k < \frac{1372}{529} \approx 2.59$: 1 counterclockwise encircling $\Rightarrow \pi_{cl}^+ = 0$;
- $k > \frac{1372}{529} \approx 2.59$: 1 clockwise encircling $\Rightarrow \pi_{cl}^+ = 2$.

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