

Introduction to Control (00340040)

lecture no. 7

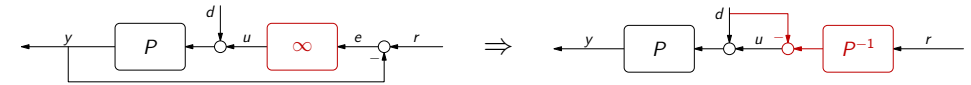
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The marvels of high-gain feedback (from Lecture 5)



and

$$T_c \rightarrow P^{-1} : r \mapsto u, \quad T \rightarrow 1 : r \mapsto y, \quad T_d \rightarrow 0 : d \mapsto y$$

i.e. high-gain feedback

- can invert the plant w/o knowing its model
- can compensate disturbance w/o measuring it directly

But

- is it feasible (stabilizing)?

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Outline

High-gain feedback limitations: root locus insight

Back to reality

Steady-state performance of closed-loop systems: conditions for $e_{ss} = 0$

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Root-locus behavior as $k \rightarrow \infty$

The root locus form

$$-\frac{1}{k} = G_k(s),$$

where $G_k(s)$ has n poles and $m \leq n$ (finite) zeros. In this case

- m loci end up at (finite) zeros of $G_k(s)$
- $n - m$ loci end up at infinity along asymptotes centered at

$$\sigma_c = \frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n - m}$$

and directed with angles

$$\phi_i = \frac{\arg b_m - \pi + 2\pi i}{n - m}, \quad i = 0, 1, \dots, n - m - 1$$

where b_m is known as the **high-frequency gain**¹ of G_k .

¹As high frequencies $|G_k(j\omega)|$ behaves like the gain frequency response of b_m/s^{n-m} .

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Strictly proper $G_k(s)$ with negative high-frequency gain

Let

- $n > m$
- $b_m < 0$

Then there always is an asymptote with $\phi_i = 0$, i.e.

- at least one closed-loop pole is in the RHP as $k \rightarrow \infty$,

implying that

- high-gain feedback is impossible in this case.

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Nonminimum-phase $G_k(s)$

Let

- at least one (finite) zero of $G_k(s)$ be in $\bar{\mathbb{C}}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$

In this case

- at least one locus ends up² in the RHP,

implying that

- high-gain feedback is impossible in this case.

²Even if all NMP zeros of $G_k(s)$ are on the $j\omega$ -axis, closed-loop poles approach the RHP as the gain grows, which is also unacceptable.

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$G_k(s)$ with pole excess more than 2

Let

- $n - m > 2$
- $b_m > 0$

Then one of the asymptotes has $\phi_i = \frac{\pi}{n-m} < \frac{\pi}{2}$, i.e.

- at least two (because of symmetry) closed-loop poles are in the RHP as $k \rightarrow \infty$,

implying that

- high-gain feedback is impossible in this case.

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Minimum-phase $G_k(s)$ with pole excess 2

Let

- $n - m = 2$
- $b_m > 0$
- all (finite) zeros of $G_k(s)$ be in $\operatorname{Re} s < 0$

Then $n - 2$ loci end up in the LHP (at stable zeros) and the others go to

$$\frac{\sum_{i=1}^n p_i - \sum_{i=1}^{n-2} z_i}{2} \pm j\infty,$$

implying that

- high-gain feedback is possible in this case iff

$$\sum_{i=1}^n p_i < \sum_{i=1}^{n-2} z_i \iff \frac{a_{n-1}}{a_n} > \frac{b_{n-3}}{b_{n-2}}.$$

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Minimum-phase $G_k(s)$ with pole excess 1

Let

- $n - m = 1$
- $b_m > 0$
- all (finite) zeros of $G_k(s)$ be in $\text{Re } s < 0$

Then $n - 1$ loci end up in the LHP (at stable zeros) and the last one goes to $-\infty$ along the real axis, implying that

- high-gain feedback is always possible in this case.

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Minimum-phase $G_k(s)$ with pole excess 0

Let

- $n - m = 0$
- all (finite) zeros of $G_k(s)$ be in $\text{Re } s < 0$

Then all loci end up in the LHP (at stable zeros), implying that

- high-gain feedback is always possible in this case.

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Summary

Systems, for which high-gain feedback can be applied:

1. minimum-phase and $m = n$
2. minimum-phase, $m = n - 1$, and $b_{n-1} > 0$
3. minimum-phase, $m = n - 2$, $b_{n-2} > 0$, and $\frac{a_{n-1}}{a_n} > \frac{b_{n-3}}{b_{n-2}}$
(although in this case we'll have a pair of lightly damped poles)

MP systems with a pole excess of at most 1 are the classes of systems

- easiest to control by feedback.

The problem is that such systems

- virtually do not exist in real world applications...

We may have such loops if we measure enough derivatives of the output y (equivalent to using non-proper controller). Yet measuring derivatives is

- prone to severe high-frequency noise,

which, as we know from Lecture 5, makes high-gain feedback unaffordable.

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Outline

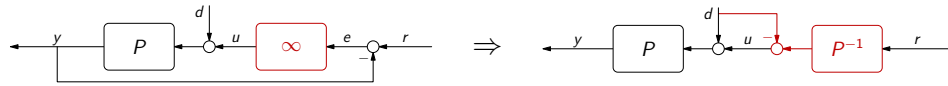
High-gain feedback limitations: root locus insight

Back to reality

Steady-state performance of closed-loop systems: conditions for $e_{ss} = 0$

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The marvels of high-gain feedback (contd)



This control law, which renders

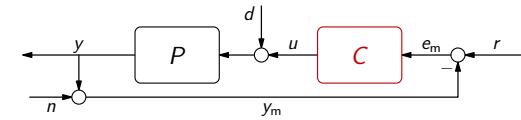
$$T_c \rightarrow P^{-1}, \quad T \rightarrow 1, \quad T_d \rightarrow 0$$

w/o the need to know the plant model, is **hardly ever feasible** because of

- stability constraints
- measurement imperfections
- implementation limitations
normally, $C(s)$ must be proper; too large coefficients $\Rightarrow$ numerical errors
- ...

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What can we do then ?



To understand what can be done, let us try to see how feedback affects

- steady-state errors
- transient behavior

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Outline

High-gain feedback limitations: root locus insight

Back to reality

Steady-state performance of closed-loop systems: conditions for $e_{ss} = 0$

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Steady-state error (from Lecture 3)

If

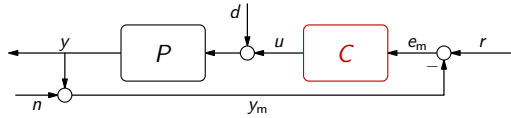
$$e = T_{ew} w$$

for some **stable** T_{ew} , then

$$e_{ss} = \begin{cases} |T_{ew}(j\omega)| & \text{if } w(t) = \sin(\omega t + \phi) \mathbb{1}(t) \\ |T'_{ew}(0)| & \text{if } w(t) = \text{ramp}(t) \text{ and } T_{ew}(0) = 0 \end{cases}$$

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Closed-loop relations (from Lecture 5)



We know that

$$y = Tr + T_d d - Tn$$

$\Downarrow$

$$e = r - y = Sr - T_d d + Tn,$$

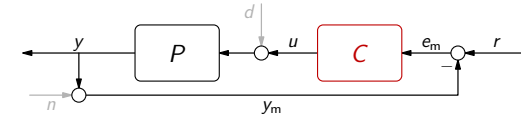
where

$$T(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}, \quad T_d(s) = \frac{P(s)}{1 + P(s)C(s)}, \quad S(s) = \frac{1}{1 + P(s)C(s)}$$

(mind that $S + T = 1$).

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e_{ss} for $r(t) = \sin(\omega t + \phi)\mathbb{1}(t)$



In this case

$$e_{ss} = |S(j\omega)| = \frac{1}{|1 + P(j\omega)C(j\omega)|}$$

provided the system is (internally) stable, of course. Thus

$$e_{ss} = 0 \iff |P(j\omega)C(j\omega)| = \infty \iff |P(j\omega)| = \infty \vee |C(j\omega)| = \infty.$$

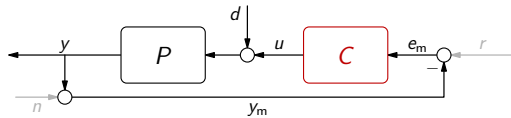
Note that this is a requirement

$$\text{— only on the gain of } PC \text{ at one frequency, } \omega,$$

and does not impose any requirements on the gain at other frequencies.

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e_{ss} for $d(t) = \sin(\omega t + \phi)\mathbb{1}(t)$



In this case

$$e_{ss} = |-T_d(j\omega)| = \left| \frac{P(j\omega)}{1 + P(j\omega)C(j\omega)} \right| = \frac{1}{|1/P(j\omega) + C(j\omega)|}$$

provided the system is (internally) stable, of course. Thus

$$e_{ss} = 0 \iff \text{either}^3 P(j\omega) = 0 \text{ or } |C(j\omega)| = \infty.$$

This is also a requirement

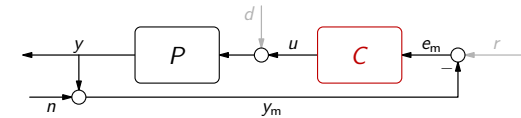
$$\text{— only on the gains of } P \text{ and } C \text{ at one frequency, } \omega,$$

and does not impose any requirements on the gains at other frequencies.

³In that case even open-loop control does the job, since the plant filters out harmonic component of d with the frequency ω irrespective of the controller.

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e_{ss} for $n(t) = \sin(\omega t + \phi)\mathbb{1}(t)$



In this case

$$e_{ss} = |T(j\omega)| = \left| \frac{P(j\omega)C(j\omega)}{1 + P(j\omega)C(j\omega)} \right|$$

provided the system is (internally) stable, of course. Thus

$$e_{ss} = 0 \iff P(j\omega)C(j\omega) = 0 \iff |P(j\omega)| = 0 \vee |C(j\omega)| = 0.$$

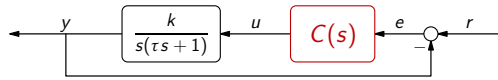
And yet again, this is a requirement

$$\text{— only on the gain of } PC \text{ at one frequency, } \omega,$$

and does not impose any requirements on the gain at other frequencies.

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DC motor: e_{ss} for $r(t) = \mathbb{1}(t)$



In this case the plant has a pole at the origin (an integrator), so static loop gain

$$|P(0)C(0)| = \lim_{s \rightarrow 0} \left| \frac{kC(s)}{s(\tau s + 1)} \right| = \infty$$

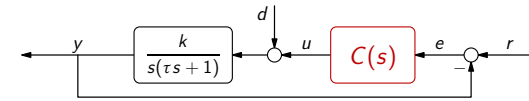
provided $C(0) \neq 0$ (true whenever C stabilizes the system, right?). Thus

- $e_{ss} = 0$ iff C is stabilizing,

which, for example, true with $C(s) = k_p$ (P controller) for all $k_p > 0$.

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DC motor: e_{ss} for $d(t) = \mathbb{1}(t)$



In this case

$$e_{ss} = \frac{1}{|1/P(0) + C(0)|} = \frac{1}{|C(0)|},$$

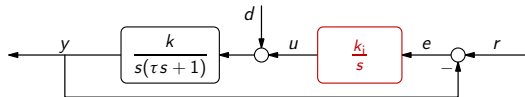
so that the integral action in P does not help. What could help is an

- integrator in C ,

which indeed guarantees that $|C(0)| = \infty$.

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DC motor: e_{ss} for $d(t) = \mathbb{1}(t)$ with I controller



This control law acting as

$$u(t) = k_i \int_0^t e(\theta) d\theta$$

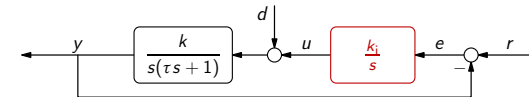
is called the **integral controller** (I controller). This **unstable** controller allows

- $u(t) \not\rightarrow 0$ even if $e(t) \rightarrow 0$,

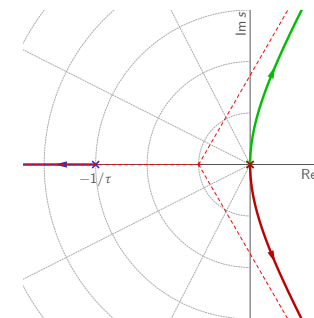
which is necessary for counteracting constant disturbances.

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DC motor: e_{ss} for $d(t) = \mathbb{1}(t)$ with I controller (contd)



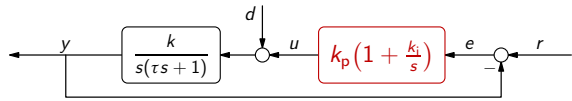
The root-locus form with respect to k_i is $-\frac{1}{k_i} = \frac{k}{s^2(\tau s + 1)}$, so that we have:



which is **never stable**.

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DC motor: e_{ss} for $d(t) = \mathbb{1}(t)$ with PI controller



This control law acting as

$$u(t) = k_p \left(e(t) + k_i \int_0^t e(\theta) d\theta \right)$$

is called the **proportional-integral controller** (PI controller). It also allows

— $u(t) \not\rightarrow 0$ even if $e(t) \rightarrow 0$,

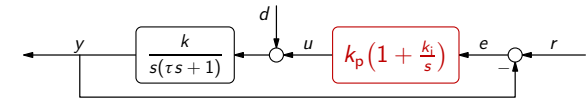
but has an additional degree of freedom, a zero at $s = -k_i$:

$$C(s) = k_p \left(1 + \frac{k_i}{s} \right) = \frac{k_p(s + k_i)}{s},$$

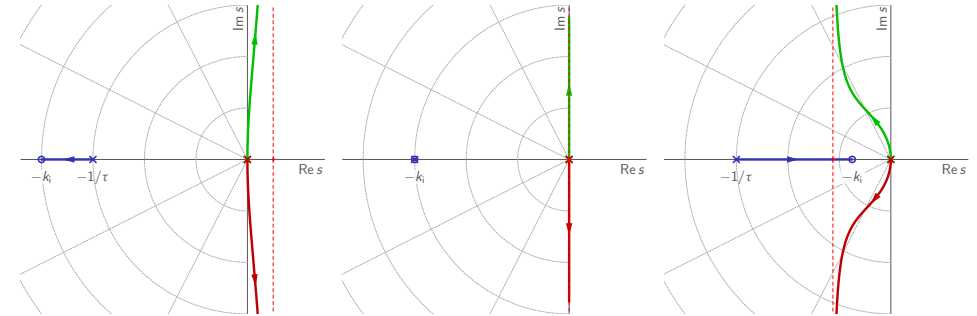
which can be used to stabilize the system.

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DC motor: e_{ss} for $d(t) = \mathbb{1}(t)$ with PI controller (contd)



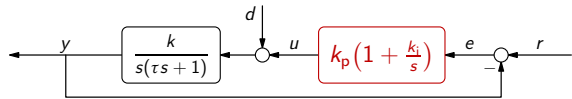
The root locus form with respect to k_p is $-\frac{1}{k_p} = \frac{k(s+k_i)}{s^2(\tau s+1)}$, so that we have:



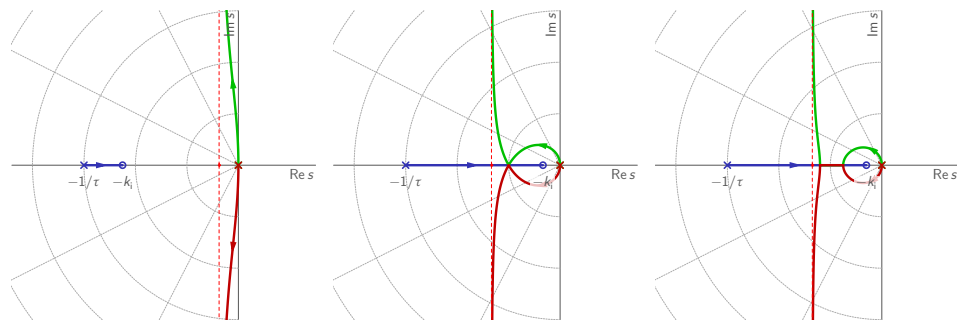
which may be **stabilized** iff $0 < \tau k_i < 1$ (as in this case $\sigma_c = \frac{\tau k_i - 1}{2\tau}$).

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DC motor: e_{ss} for $d(t) = \mathbb{1}(t)$ with PI controller (contd)

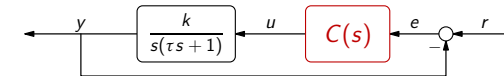


Root loci for various $k_i \in (0, 1/\tau)$ may take exotic forms:



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DC motor: e_{ss} for $r(t) = \text{ramp}(t)$



In this case

$$e_{ss} = \left| \lim_{s \rightarrow 0} s S(s) \frac{1}{s^2} \right| = \lim_{s \rightarrow 0} \frac{1}{|s(1 + P(s)C(s))|} = \lim_{s \rightarrow 0} \frac{1}{|s + kC(s)/(\tau s + 1)|}$$

As $C(0) \neq 0$ for any stabilizing C , we have that

$$e_{ss} = \frac{1}{k|C(0)|}.$$

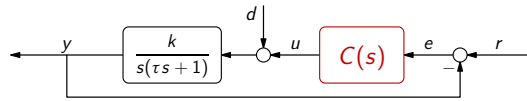
Thus, the larger $|C(0)|$ is, the smaller e_{ss} is and then

— $e_{ss} = 0$ only if $|C(0)| = \infty$,

which again requires an integral action in C (e.g. PI).

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DC motor: e_{ss} for $d(t) = \text{ramp}(t)$



Now,

$$e_{ss} = \left| \lim_{s \rightarrow 0} s T_d(s) \frac{1}{s^2} \right| = \lim_{s \rightarrow 0} \frac{|P(s)|}{|s(1 + P(s)C(s))|} = \lim_{s \rightarrow 0} \frac{k}{|s^2(\tau s + 1) + ksC(s)|}$$

so that

- if $C(0)$ is finite, then $e_{ss} = \infty$.

Thus we need an integrator in C just to keep e_{ss} bounded, in which case

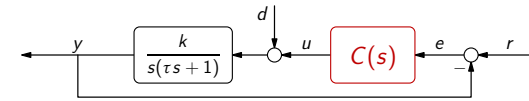
$$e_{ss} = \lim_{s \rightarrow 0} \frac{1}{|sC(s)|}$$

and

- $e_{ss} = 0$ only if $C(s)$ has at least 2 poles at the origin (double integrator).

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DC motor: e_{ss} for $r(t)/d(t) = \sin(\omega t + \phi)\mathbb{1}(t)$, $\omega > 0$



Because

$$|P(j\omega)| = \left| \frac{k}{j\omega(j\omega\tau + 1)} \right| = \frac{k}{\omega\sqrt{1 + \tau^2\omega^2}} \neq \infty$$

we have that

- $e_{ss} = 0 \iff |C(j\omega)| = \infty$.

The latter requires

- poles of $C(s)$ at $\pm j\omega$,

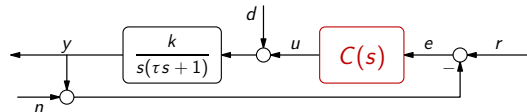
like

$$C(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + \omega^2}.$$

Try to find stabilizing b_i and show that $b_2 \neq 0$ is necessary for stability.

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DC motor: e_{ss} for $n(t) = \sin(\omega t + \phi)\mathbb{1}(t)$, $\omega > 0$



Because

$$P(j\omega) = \frac{k}{j\omega(j\omega\tau + 1)} = \frac{k}{\omega\sqrt{1 + \tau^2\omega^2}} \neq 0$$

we have that

- $e_{ss} = 0 \iff C(j\omega) = 0$.

The latter requires

- zeros of $C(s)$ at $\pm j\omega$,

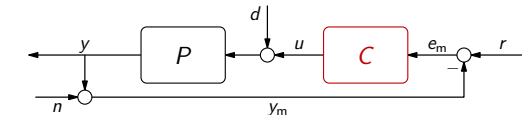
like

$$C(s) = \frac{s^2 + \omega^2}{a_2 s^2 + a_1 s + a_0}$$

(known as **notch**). Try to find stabilizing a_i .

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Summary



Zero steady-state errors to

- $r(t) = \sin(\omega t + \phi)\mathbb{1}(t)$ requires poles at $\pm j\omega$ in $P(s)C(s)$
- $d(t) = \sin(\omega t + \phi)\mathbb{1}(t)$ requires poles at $\pm j\omega$ in $C(s)$
- $r(t) = \mathbb{1}(t)$ requires an integrator in PC
- $d(t) = \mathbb{1}(t)$ requires an integrator in C
- $r(t) = \text{ramp}(t)$ requires a double integrator in PC
- $d(t) = \text{ramp}(t)$ requires a double integrator in C

which are infinite gains at isolated frequencies. Zero steady-state error to

- $n(t) = \sin(\omega t + \phi)\mathbb{1}(t)$ requires zeros at $\pm j\omega$ in $P(s)C(s)$

which is zero gain at isolated frequencies.

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