

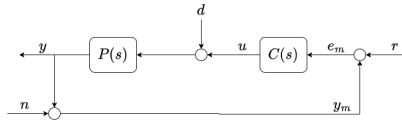
Control Theory (035188) - Tutorial 1

Recap on loop shaping tools

Faculty of Mechanical Engineering, Technion

March 31, 2025

Unity Feedback Closed-Loop



y : Regulated signal

u : Control signal

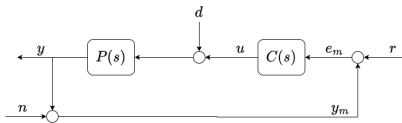
d : Load disturbance

r : Reference signal

n : Measurement noise

e : Tracking error $e = r - y = e_m + n$

Unity Feedback Closed-Loop



- The loop transfer function is $L(s) = C(s)P(s)$
- The Gang of Four is

$$\begin{bmatrix} S(s) & T_c(s) \\ T_d(s) & T(s) \end{bmatrix} = \frac{1}{1 + C(s)P(s)} \begin{bmatrix} 1 & C(s) \\ P(s) & C(s)P(s) \end{bmatrix}$$

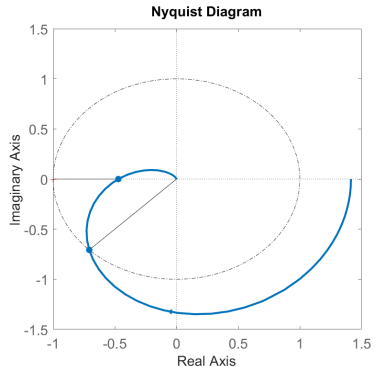
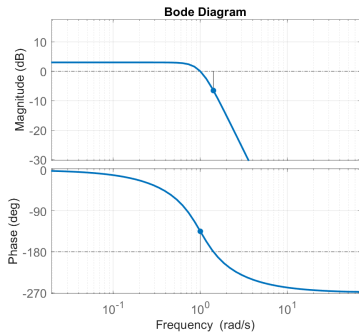
Stability Margins

For a stable Closed-loop system :

Gain Margin μ_g is the *minimal* factor by which the loop gain should be changed to render the closed-loop system unstable.

Phase Margin μ_{ph} is the *minimal* amount by which the loop phase should be changed to render the closed-loop system unstable.

Stability Margins on Frequency Response Plots



Loop Shaping Process

The typical course of action when designing a loop using loop shaping:

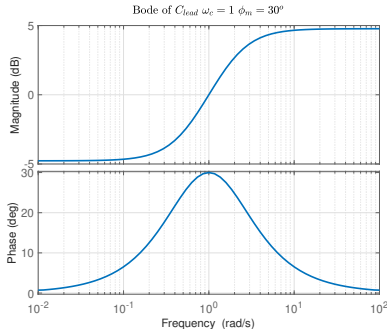
- choose magnitude crossover frequency ω_c according to requirements.
- shape high-frequencies roll-off, if required, using a low-pass filter.
- set required crossover ω_c using a proportional gain.
- shape phase around ω_c using a lead controller.
- shape low-frequency gain using a lag controller.

This standard order is a guideline that can be adjusted to fit the needs of a problem.

1st-order Lead

$$C_{\text{lead}}(s) = \frac{\sqrt{\alpha}s + \omega_m}{s + \sqrt{\alpha}\omega_m} \quad \alpha = \frac{1 + \sin \phi_m}{1 - \sin \phi_m}$$

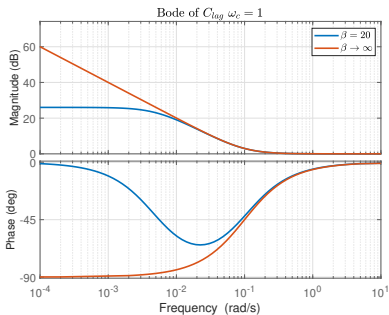
where $\phi_m \in (0, 90^\circ)$ is the maximal phase lead.



1st-order Lag

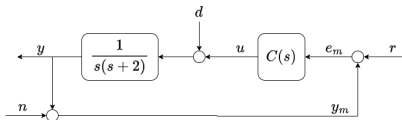
$$C_{\text{lag}}(s) = \frac{10s + \omega_m}{10s + \omega_m/\beta} \quad \beta > 1$$

where the phase lag at ω_m is at most 5.7° .



Design Example

Consider a DC motor controlled by the closed-loop controller $C(s)$

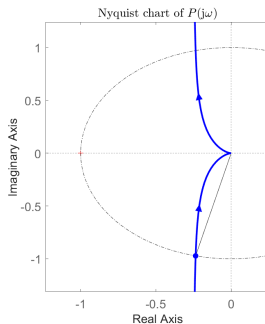
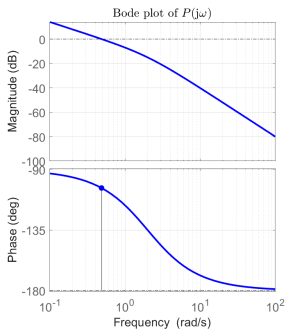


Design $C(s)$ according to the following requirements:

- stable closed-loop
- $\omega_c = 5 \left[\frac{\text{rad}}{\text{s}} \right]$
- zero steady-state error for a step in r
- zero steady-state error for a step in d
- $\mu_{ph} \in \{45, 60\}$

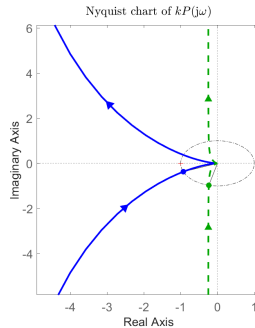
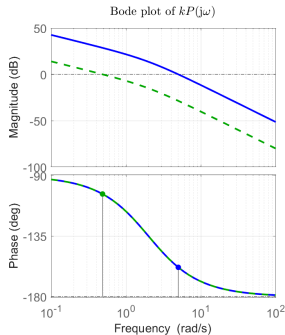
Plant Frequency Response

We will first look at the plant $P(s)$



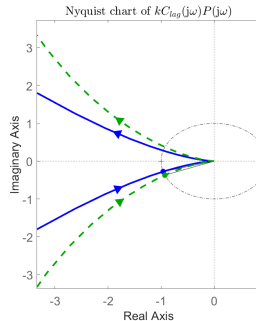
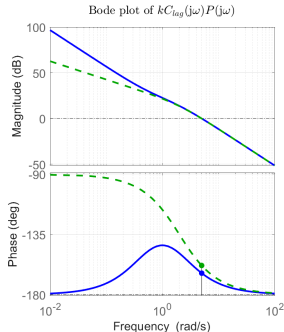
- The gain crossover $|P(j\omega_c)| = 1$ is at $\omega_c = 0.49 < \omega_{c,req}$.
- We choose the gain according to $k = \frac{1}{|P(j\omega_{c,req})|} = 26.9$

Setting Crossover Frequency



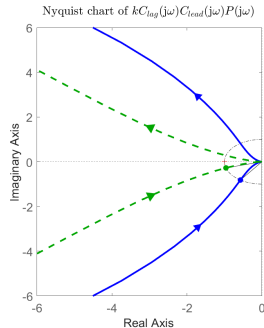
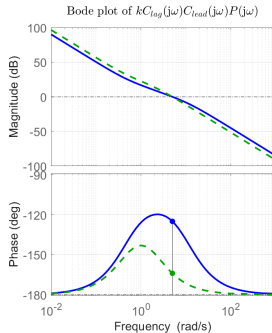
- For zero steady-state due to disturbance, a lag is added with $\beta \rightarrow \infty$.

Shaping Low-Frequency Gain



- The phase margin is $\mu_{ph} = 16.1^\circ$.
- added phase lead $\phi_m = 55^\circ - 16.1^\circ = 38.1^\circ$.
- one lead is enough.

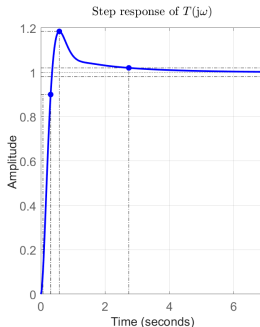
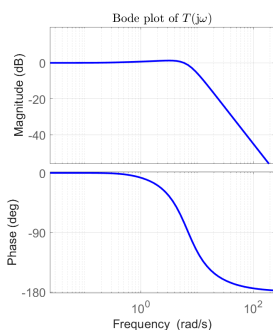
Shaping Crossover Phase



- The phase margin is now $\mu_{ph} = 54.9^\circ$ as required.
- The controller is

$$C(s) = 26.9 \cdot \frac{10s + 5}{10s} \cdot \frac{2.1s + 5}{s + 10.5}$$

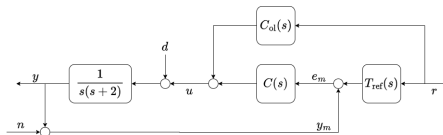
Closed-Loop Performance



- The closed-loop bandwidth is $\omega_b \approx 8.2 \left[\frac{\text{rad}}{\text{s}} \right]$
- Step response with $OS \approx 18\%$ $RT \approx 0.24[\text{s}]$

Improve Command Response

Since the OS is quite high, we will try to improve it using a 2DOF architecture.



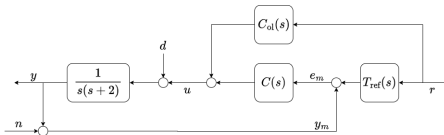
Where the resulting signals of interest are

$$y = T_{\text{ref}}r + T_d d - Tn \quad u = C_{\text{ol}}r - Td - T_c n$$

and the open-loop controller is

$$C_{\text{ol}} = \frac{T_{\text{ref}}}{P}$$

Reference Model Choice



Since $P(s)$ is a minimum-phase transfer function, we only need to choose $T_{\text{ref}}(s)$ with a pole excess of at least 2.

$$T_{\text{ref}}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow C_{\text{ol}}(s) = \frac{\omega_n^2 s(s+2)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where ζ, ω_n are design parameters.

Reference Model Tuning

To improve the overshoot of the command response, we choose $\zeta = 1$, and we would like to keep the same bandwidth.

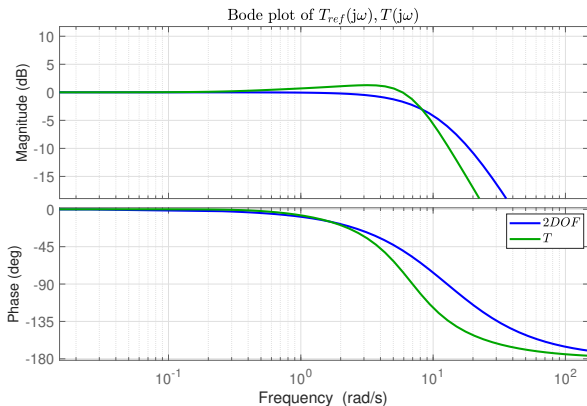
$$|T_{\text{ref}}(j\omega_b)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\omega_n\omega_b)^2}} = -3[\text{dB}] = \frac{1}{\sqrt{2}}$$
$$\omega_b^2 = (\sqrt{2} - 1) \omega_n^2$$
$$\omega_b \approx 0.644\omega_n$$

We choose

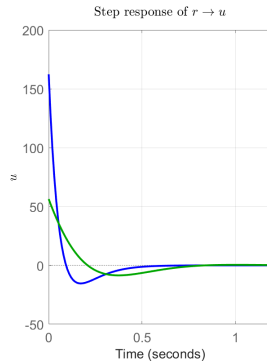
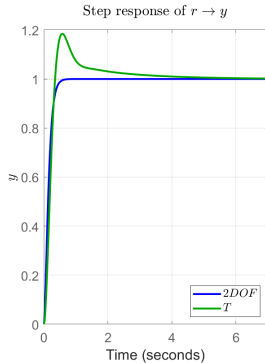
$$\omega_n = 12.74$$

Reference Model Tuning

For $\zeta = 1$ and $\omega_n = 12.74$ we get



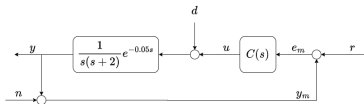
Closed-Loop Performance



- The response is now with $OS = 0\%$ and the same rise time $RT \approx 0.24[s]$

Class Exercise

Consider the same motor from the example, with an added delay of $h = 0.05[s]$



Design $C(s)$ according to the following requirements:

- stable closed-loop
- $\omega_c > 5 \left[\frac{\text{rad}}{\text{s}} \right]$
- zero steady-state error for a step in r
- zero steady-state error for a step in d
- $\mu_{ph} \in \{45, 60\}$

Try to increase ω_c
as much as you can!