

Control Theory (00350188)

lecture no. 11

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Outline

Linear-quadratic state observer (Kalman filter)

Linear Quadratic Gaussian problem

LQG controller design (optional self-study)

What is sampled-data control

Background: A/D and D/A conversions in the frequency domain (from LS)

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Setup

Consider system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Bd(t), & x(0) = 0, \\ y(t) = Cx(t) + n(t), \end{cases}$$

where u is known input, d is (unknown) disturbance, and n is measurement noise.

Our purpose here is to reconstruct x so that reconstruction error,

$$\epsilon = x - \hat{x},$$

is “small.”

Clearly, solution of such state estimation problem depends on

1. what our assumptions about d and n are,
2. what measure of the estimation error’s “smallness” is taken.

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(Sloppy) introduction to white noise signals

By **white noise signal** we understand

- signal f , whose frequency spectrum satisfies $|F(j\omega)|^2 \equiv \sigma_f = \text{const}$ with $\sigma_f \geq 0$ called the **intensity** of f . In other words, a white noise signal
- has all harmonics equally represented in it.

We may interpret white noise signal as

- impulse response of any all-pass system—deterministic interpretation¹.

It may be safe to say that white noise doesn't exist in nature. It, however, is a convenient abstraction used as a building block for many realistic signals.

¹Conventionally, white noise is defined and interpreted from a stochastic point of view (white random process). Yet such approach goes far beyond the scope of this course.

Linear-quadratic state estimation problem

In this formulation we assume that in the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Bd(t), & x(0) = 0, \\ y(t) = Cx(t) + n(t), \end{cases}$$

inputs d and n are white noise signals with intensities σ_d and σ_n . We also assume that

$\mathcal{A}_4$: (C, A) is detectable,

$\mathcal{A}_5$: $\sigma_n > 0$,

$\mathcal{A}_6$: (A, B) has no uncontrollable $j\omega$ -axis modes.

Linear-quadratic estimation is formulated as reconstruction of x minimizing

$$\mathcal{J} := \int_0^\infty (x(t) - \hat{x}(t))' (x(t) - \hat{x}(t)) dt,$$

where $\hat{x}$ is the reconstruction of x .

Interpretation of d and n

Plant disturbance

- d reflects inaccuracies of our model of x .

Hence, a high intensity of d means that we should correct our model-based estimation more aggressively.

Measurement noise

- n reflects inaccuracies of our measurements.

As y is the only available information on deviations of x from its “predicted” values, high intensity of n means that we should rely more upon model.

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Solution (steady-state Kalman filter)

Theorem

If $\mathcal{A}_{4-6}$ hold, then the unique optimal linear-quadratic estimator is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \sigma_n^{-1} \bar{Y} C' (y(t) - C\hat{x}(t)), \quad \hat{x}(0) = 0,$$

where $\bar{Y} = \bar{Y}' \geq 0$ is (unique) stabilizing solution of CARE

$$\bar{Y} A' + A \bar{Y} + \sigma_d B B' - \sigma_n^{-1} \bar{Y} C' C \bar{Y} = 0$$

such that $A_L = A + LC$ is Hurwitz, where $L = -\sigma_n^{-1} \bar{Y} C'$.

Filtering CARE

The CARE

$$\bar{Y}A' + A\bar{Y} + \sigma_d BB' - \sigma_n^{-1} \bar{Y}C'C\bar{Y} = 0$$

is a special case of

$$\bar{A}'\bar{X} + \bar{X}\bar{A} + \bar{Q} - (\bar{S} + \bar{X}\bar{B})\bar{R}^{-1}(\bar{S}' + \bar{B}'\bar{X}) = 0$$

with $\bar{A} = A'$, $\bar{B} = C'$, $\bar{Q} = \sigma_d BB'$, $\bar{R} = \sigma_u$, and $\bar{S} = 0$. Here

- (C, A) is detectable $\implies (\bar{A}, \bar{B}) = (A', C')$ is stabilizable
- (A, B) has no uncontrollable $j\omega$ -axis modes and $\sigma_u > 0 \implies$

$$\begin{bmatrix} \bar{A} - j\omega I & \bar{B} \\ \bar{Q} & \bar{S} \\ \bar{S}' & \bar{R} \end{bmatrix} = \begin{bmatrix} A' - j\omega I & C' \\ \sigma_d BB' & 0 \\ 0 & \sigma_u \end{bmatrix}$$

has full column rank $\forall \omega \in \mathbb{R}$.

As such, it can be solved by `icare(A',C',sigma_d*B*B',sigma_u)`.

Kalman filter transfer functions

The Kalman filter has two inputs, u and y , and one vector output, $\hat{x}$. The transfer function from u to $\hat{x}$ and from y to $\hat{x}$ are

$$G_{\hat{x}u}(s) = (sI - A_L)^{-1}B \quad \text{and} \quad G_{\hat{x}y}(s) = -(sI - A_L)^{-1}L,$$

respectively, where $L = -\sigma_n^{-1}\bar{Y}C'$ and $A_L = A + LC$. Both result from the following form of the estimator equation:

$$\dot{\hat{x}}(t) = (A + LC)\hat{x}(t) + Bu(t) - Ly(t), \quad \hat{x}(0) = 0$$

which is a Luenberger observer under an optimization-based choice of L .

Estimating incomplete state

In some situations we do not need to reconstruct all state vector but rather only its part $v = C_v x$. Performance measure is then

$$\mathcal{J} := \int_0^{\infty} (C_v x(t) - \hat{v}(t))' (C_v x(t) - \hat{v}(t)) dt.$$

Remarkable property of Kalman filter is that in this case

$$\hat{v} = C_v \hat{x},$$

where $\hat{x}$ is the optimal estimate of x . In other words, the whole x should be estimated anyway. The transfer functions of the filter are then

$$G_{\hat{v}u}(s) = C_v(sI - A_L)^{-1}B \quad \text{and} \quad G_{\hat{v}y}(s) = -C_v(sI - A_L)^{-1}L$$

Example

Consider the process

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + n(t), \end{cases}$$

with $x = \begin{bmatrix} y \\ v \end{bmatrix}$, which is double integrator with noisy output measurements. Our purpose is

- to reconstruct $v(t)$ from $y(t)$ (i.e. $C_v = \begin{bmatrix} 0 & 1 \end{bmatrix}$).

We assume that

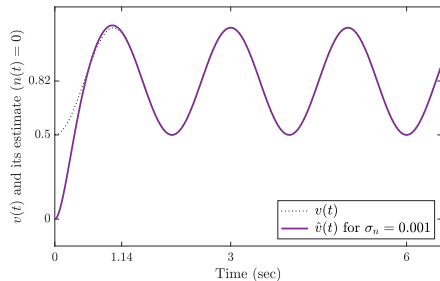
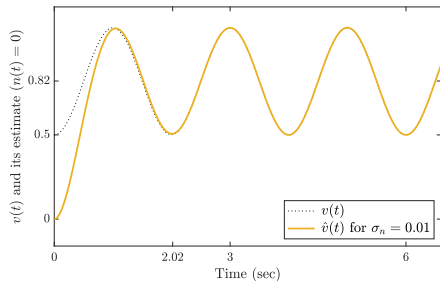
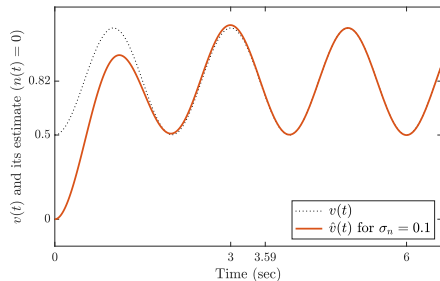
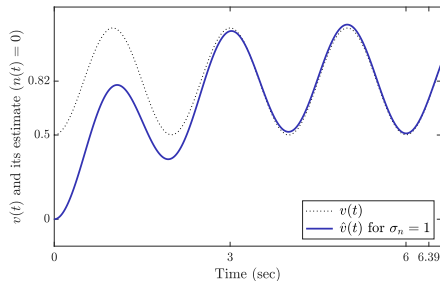
- $d(t) = \delta(t)$ (i.e. that $y(0) = 0$ and $v(0) = 1$)
- $n(t) = 0.2 \sin(20\pi t)$
- $u(t) = \sin(\pi t)$.

$$\sigma_d = 1$$

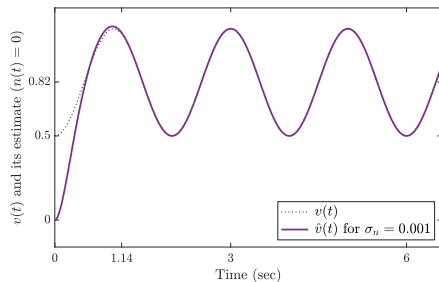
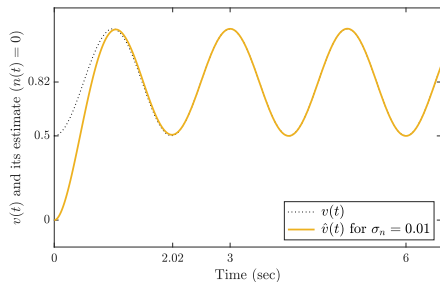
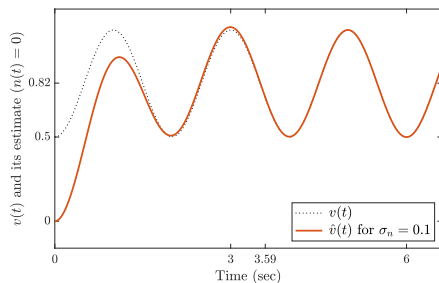
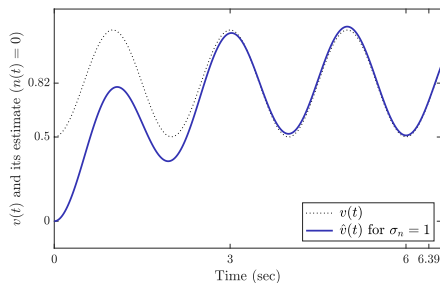
σ_n is tuned

We'll be playing with σ_n , which is fictitious parameter.

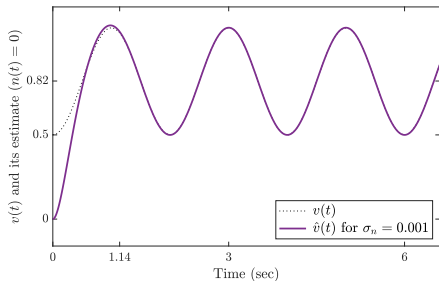
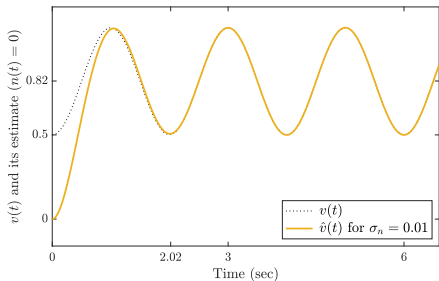
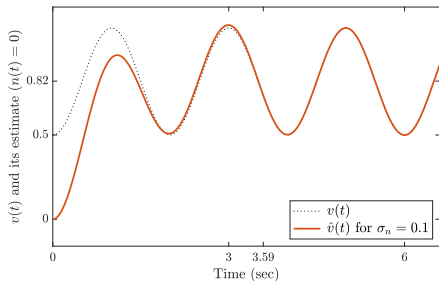
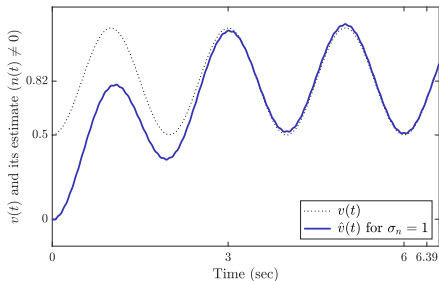
Example: results with $d(t) = \delta(t)$ & $n(t) = 0$



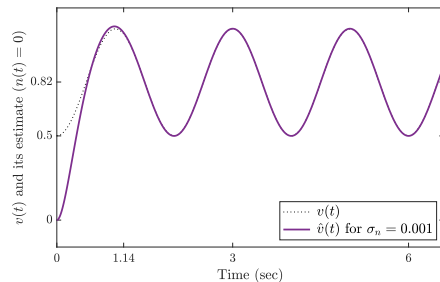
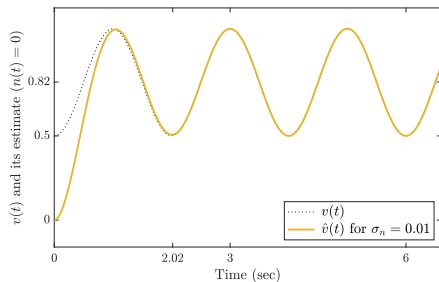
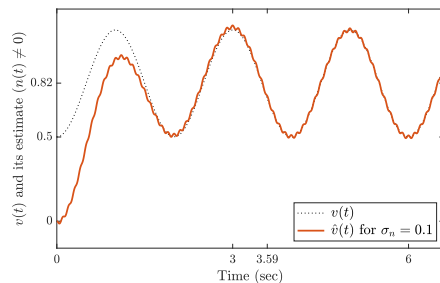
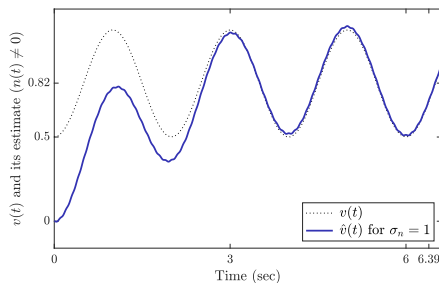
Example: results with $d(t) = \delta(t)$ & $n(t) = 0.2 \sin(20\pi t)$



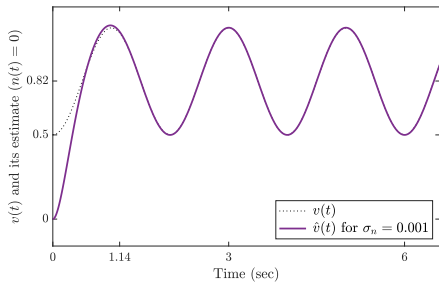
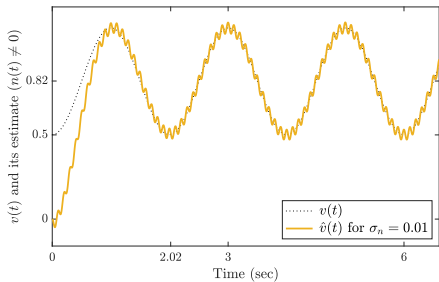
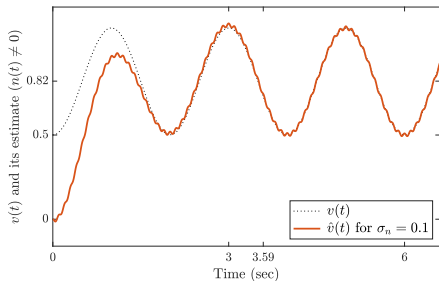
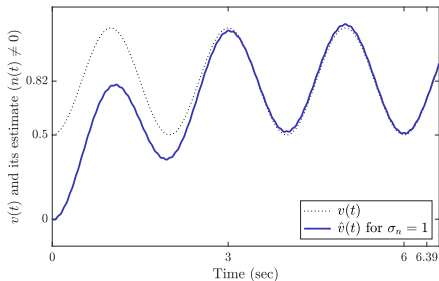
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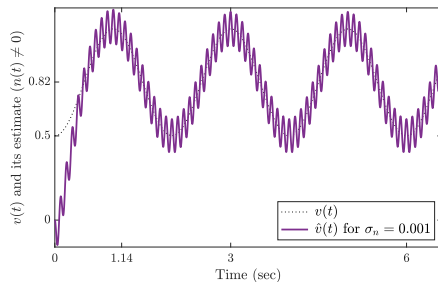
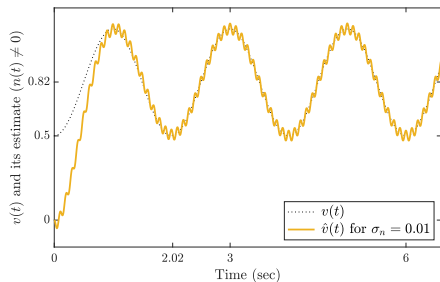
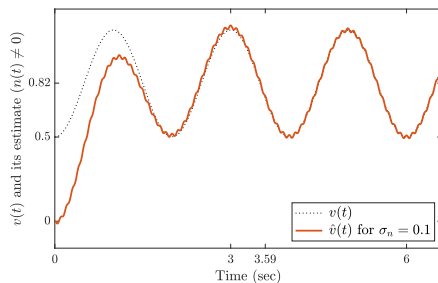
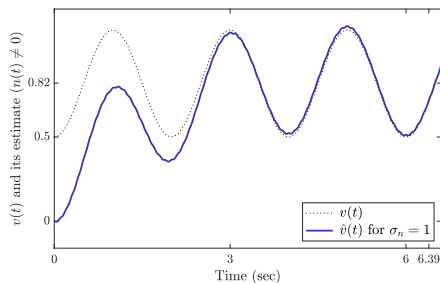
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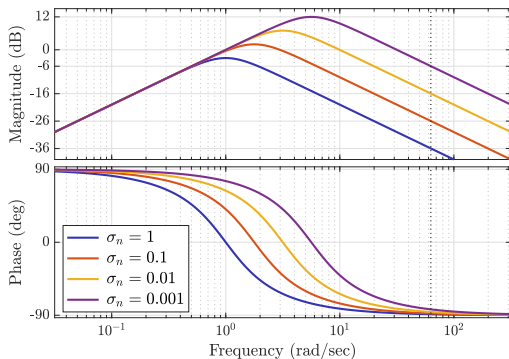


Example: frequency responses

Bode plots of the filter

$$G_{\hat{v}_y}(s) = -C_v(sI - A_L)^{-1}L$$

for different choices of σ_n :



It starts as a differentiator (logical), but then cuts it off.

Example: insight

Our *assumed* measurement signal,

$$y = \frac{1}{s^2}d + n =: y_s + y_n,$$

can be thought of as comprised

- the “signal” y_s , which contains information about v (to be estimated),
- the “noise” y_n , which contains no information about x .

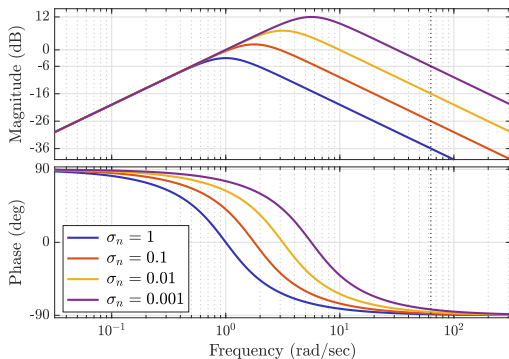
Convenient measure of the “usefulness” of y is the **signal to noise ratio**:

$$\text{SNR}(\omega) := \frac{|Y_s(j\omega)|}{|Y_n(j\omega)|} = \frac{\sqrt{\sigma_d}}{\sqrt{\sigma_n} \omega^2}$$

The estimator then

- imitates the differentiator at frequencies where $\text{SNR}(\omega) \gg 1$
- does nothing at frequencies where $\text{SNR}(\omega) \ll 1$

Example: insight (contd)



In our case, with $\sigma_d = 1$,

$$\text{SNR}(\omega) = 1 \iff \omega = \frac{1}{\sqrt[4]{\sigma_n}} = \{1, 1.7783, 3.1623, 5.6234\}$$

These frequencies agree with the breaks of tendency of $|G_{\hat{v}_Y}(j\omega)|$.

More general formulation

Let

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + B_w w(t), & x(0) = 0, \\ y(t) = Cx(t) + D_w w(t), \end{cases}$$

for an unmeasured w , each element of which is a white unit-intensity signal independent of other elements of w , and under

$\mathcal{A}_4$: (C, A) is detectable,

$\mathcal{A}_5$: D_w has full row rank,

$\mathcal{A}_6$: $\begin{bmatrix} A - j\omega I & B_w \\ C & D_w \end{bmatrix}$ has full row rank $\forall \omega \in \mathbb{R}$

The previous formulation is its special case with

$$w = \begin{bmatrix} d \\ n \end{bmatrix}, B_w = \begin{bmatrix} B\sqrt{\sigma_d} & 0 \end{bmatrix}, \text{ and } D_w = \begin{bmatrix} 0 & \sqrt{\sigma_n} \end{bmatrix}.$$

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$$- \quad w = \begin{bmatrix} d \\ n \end{bmatrix}, \quad B_w = \begin{bmatrix} B\sqrt{\sigma_d} & 0 \end{bmatrix}, \quad \text{and} \quad D_w = \begin{bmatrix} 0 & \sqrt{\sigma_n} \end{bmatrix}.$$

More general formulation: solution

Theorem

If $\mathcal{A}_{4-6}$ hold, then the unique optimal linear-quadratic estimator is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + (D_w D'_w)^{-1}(D_w B'_w + \bar{Y} C')(y(t) - C\hat{x}(t))$$

with $\hat{x}(0) = 0$, where $\bar{Y} = \bar{Y}' \geq 0$ is (unique) stabilizing solution of CARE

$$\bar{Y} A' + A \bar{Y} + B_w B'_w - (B_w D'_w + \bar{Y} C')(D_w D'_w)^{-1}(D_w B'_w + C \bar{Y}) = 0$$

such that $A_L = A - (D_w D'_w)^{-1}(D_w B'_w + \bar{Y} C')C$ is Hurwitz.

This corresponds to

- Luenberger observer with $L = -(D_w D'_w)^{-1}(D_w B'_w + \bar{Y} C')$.
- CARE solution² with $\bar{A} = A'$, $\bar{B} = C'$, $\begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}' & \bar{R} \end{bmatrix} = \begin{bmatrix} B_w \\ D_w \end{bmatrix} \begin{bmatrix} B'_w & D'_w \end{bmatrix}$

²So solved by `icare(A',C',Bw*Bw',Dw*Dw',Bw*Dw')`.

Is white noise assumption realistic?

No ! But many realistic signals can be generated by white noise (modeled).
For example, if $d = W_d w_w$ for a white w_d , then

$$\|D(j\omega)\| = \|W_d(j\omega)\|,$$

so if we know spectral properties of d , we can account for them.

Example (Dryden gusts): Spectral model of the lateral wind turbulence for aircraft flying at a (constant) speed V through a frozen turbulence field is

$$\Phi_v(\omega) = \frac{\sigma_v^2 L_v V}{\pi} \frac{3L_v^2 \omega^2 + V^2}{(L_v^2 \omega^2 + V^2)^2},$$

where σ_v and L_v are turbulence intensity and scale length, respectively. If

$$W_v(s) = \sigma_v \sqrt{\frac{L_v V}{\pi}} \frac{\sqrt{3L_v s + V}}{(L_v s + V)^2},$$

check that $|W_v(j\omega)|^2 = \Phi_v(\omega)$ for all ω . Then $v = W_v w_{dg}$ for a white w_{dg} .

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$$\Phi_v(\omega) = \frac{\sigma_v^2 L_v V}{\pi} \frac{3L_v^2 \omega^2 + V^2}{(L_v^2 \omega^2 + V^2)^2} = \left[\text{plot of a curve that is flat and then drops off} \right],$$

where σ_v and L_v are turbulence intensity and scale length, respectively. If

$$W_v(s) = \sigma_v \sqrt{\frac{L_v V}{\pi}} \frac{\sqrt{3}L_v s + V}{(L_v s + V)^2},$$

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Colorifying disturbances and noises

If disturbance and measurement noise are modeled as

$$\begin{cases} \dot{x}_d(t) = A_d x_d(t) + B_d w_d(t) \\ d(t) = C_d x_d(t) + D_d w_d(t) \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_n(t) = A_n x_n(t) + B_n w_n(t) \\ n(t) = C_n x_n(t) + D_n w_n(t) \end{cases}$$

for white w_d and w_n , then the augmented system (check that)

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_d(t) \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} A & BC_d & 0 \\ 0 & A_d & 0 \\ 0 & 0 & A_n \end{bmatrix} \begin{bmatrix} x(t) \\ x_d(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} BD_d & 0 \\ B_d & 0 \\ 0 & B_n \end{bmatrix} \begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 & C_n \end{bmatrix} \begin{bmatrix} x(t) \\ x_d(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & D_n \end{bmatrix} \begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix} \end{cases}$$

This fits the general Kalman filtering formulation under white $w = \begin{bmatrix} w_d \\ w_n \end{bmatrix}$.

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Linear Quadratic Gaussian problem

LQG controller design (optional self-study)

What is sampled-data control

Background: A/D and D/A conversions in the frequency domain (from LS)

Problem formulation

Given system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Bd(t), & x(0) = 0, \\ y(t) = Cx(t) + n(t), \end{cases}$$

where

- $u(t) \in \mathbb{R}$ is the control input
- $d(t) \in \mathbb{R}$ is the load disturbance (white, with intensity $\sigma_d \geq 0$)
- $n(t) \in \mathbb{R}$ is the measurement noise (white, with intensity $\sigma_n > 0$)

The **LQG** (Linear Quadratic Gaussian) problem is

- design a stabilizing $u(t)$, which is a function of $y(t)$, minimizing

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

for some $R > 0$ and $Q - SR^{-1}S' \geq 0$.

Solution structure

Because

- LQR optimal control is state feedback,
- Kalman filter optimal state estimator is a Luenberger observer,

it is obvious that

- LQR + Kalman results in a stabilizing observer-based controller, with

$$\chi_{cl}(s) = \det(sI - A_K) \det(sI - A_L)$$

Less obvious is that

- LQR + Kalman results in the LQG-optimal stabilizing controller (the separation principle).

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Problem solution

Theorem

If $\mathcal{A}_{1-6}$ hold, then the unique LQG controller is given by controller

$$\begin{cases} \dot{\hat{x}}(t) = (A + BK + LC)\hat{x}(t) - Ly(t), & \hat{x}(0) = 0, \\ u(t) = K\hat{x}(t), \end{cases}$$

where K and L are the LQR and Kalman filter gains. The optimal cost

$$\mathcal{J}_{opt} = \text{tr}(\sigma_d \bar{X} B B' + \bar{Y} Q + \bar{X} A \bar{Y} + \bar{Y} A' \bar{X}),$$

where $\bar{X} = \bar{X}' \geq 0$ and $\bar{Y} = \bar{Y}' \geq 0$ are the stabilizing solutions of the LQR and filtering CAREs.

Properties:

- LQG controller is stabilizing, but not necessarily stable itself
- LQG controller always has a strictly proper transfer function
- stability margins of LQR are not achieved by LQG

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$$\begin{cases} \dot{\hat{x}}(t) = (A + BK + LC)\hat{x}(t) - Ly(t), & \hat{x}(0) = 0, \\ u(t) = K\hat{x}(t), \end{cases}$$

where K and L are the LQR and Kalman filter gains. The optimal cost

$$\mathcal{J}_{opt} = \text{tr}(\sigma_d \bar{X} B B' + \bar{Y} Q + \bar{X} A \bar{Y} + \bar{Y} A' \bar{X}),$$

where $\bar{X} = \bar{X}' \geq 0$ and $\bar{Y} = \bar{Y}' \geq 0$ are the stabilizing solutions of the LQR and filtering CAREs.

Properties:

- LQG controller is stabilizing, but not necessarily stable itself
- LQG controller always has a strictly proper transfer function
- stability margins of LQR are not achieved by LQG

Outline

Linear-quadratic state observer (Kalman filter)

Linear Quadratic Gaussian problem

LQG controller design (optional self-study)

What is sampled-data control

Background: A/D and D/A conversions in the frequency domain (from LS)

Think loop shaping, act LQG

Loop shaping:

- easy on the magnitude shaping
- might be intricate in the phase shaping around crossover

LQG:

- guarantees closed-loop stability
- not directly interpretable in terms of frequency-domain properties

Combine:

- shape loop magnitude (via frequency-dependent weights)
- solve some LQG to stabilize the shaped loop

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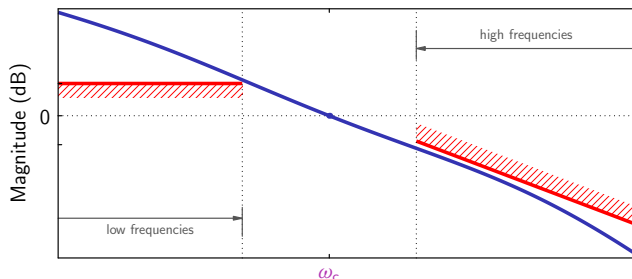
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Combine:

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- solve some LQG to stabilize the shaped loop

Magnitude shaping



Conceptually simple,

- pick W such that $|P_a(j\omega)|$, where $P_a = PW$, is shaped appropriately
 - possible choices
 - $W(s) = k_p$ to fix the crossover
 - $W(s) = k_p \omega_c / s$ (I) or $W(s) = k_p(1 + k_i/s)$ (PI) to add integral action
 - $W(s) = k_p/(\tau s + 1)$ to add low-pass filter
- whatever C_a designed for P_a is then implemented as $C = WC_a$ for P

LQR

Bring in

$$P_a : \begin{cases} \dot{x}_a(t) = A_a x_a(t) + B_a u_a(t) + B_a d_a(t), & x_a(0) = 0, \\ y(t) = C_a x_a(t) + n(t), \end{cases}$$

and consider a **balanced** LQG for it with the cost

$$\mathcal{J} = \int_0^\infty (y^2(t) + u_a^2(t)) dt = \int_0^\infty \begin{bmatrix} x_a'(t) & u_a'(t) \end{bmatrix} \begin{bmatrix} C_a' C_a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_a(t) \\ u_a(t) \end{bmatrix} dt$$

and unit intensities of d_a and n . In other words,

- cost puts equal importance to y and $u_a = (1/W)u$,
- load disturbance $d_a = (1/W)d$ and noise n are equally important.

Motivation:

- LQR shall just ensure stability $\implies$ unsophisticated and balanced.

LQR

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Controller

The resulted controller is

$$C(s) = W(s)C_a(s).$$

Some of its properties:

- the degree of $C(s)$, $\deg(C(s)) = \deg(P(s)) + 2 \deg(W(s))$
unless some poles or zeros of $W(s)$ are canceled by $C_a(s)$
- integrators of $W(s)$ become integrators of $C(s)$
- any nonzero roll-off in $W(s)$ increases that of $C(s)$

Success indicator:

$$\epsilon = \frac{1}{\sqrt{1 + \lambda_{\max}(\bar{X}_s \bar{Y}_s)}} < 1,$$

where $\bar{X}_s$ and $\bar{Y}_s$ are solutions to CAREs for P_s . The level $\epsilon \geq 0.25$ may be regarded as adequate, $\epsilon \geq 0.5$ is very good

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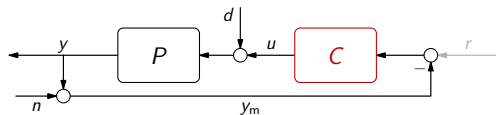
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Example: DC motor

A DC motor from Lecture 1, controlled in closed loop



Requirements:

- closed-loop stability (of course)
- zero steady-state error for a step in r
- zero steady-state error for a step in d
- good stability margins
- ω_c is treated as a tuning parameter

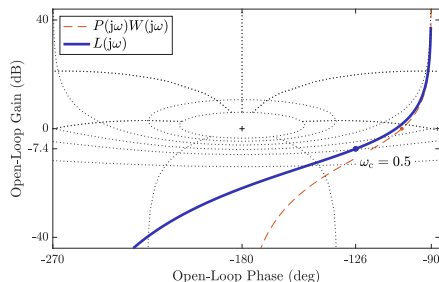
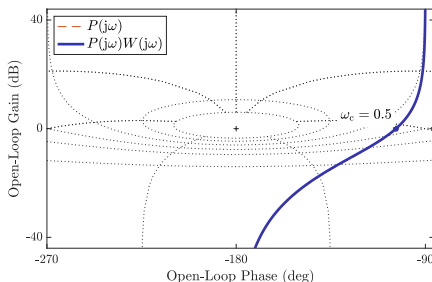
always holds
integrator in $C(s)$

Example: $\omega_c = 0.5$, P weight

Choose

$$W(s) = k_p$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.6432$, loops are



and the controller

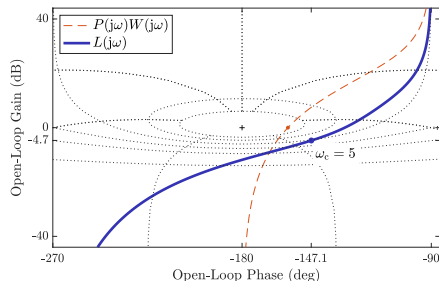
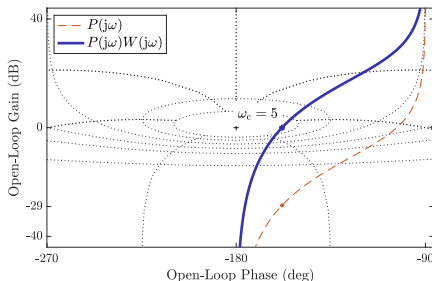
$$C(s) = \frac{0.52555(s + 2.022)}{s^2 + 2.924s + 2.275}$$

Example: $\omega_c = 5$, P weight

Choose

$$W(s) = k_p$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.4673$, loops are



and the controller

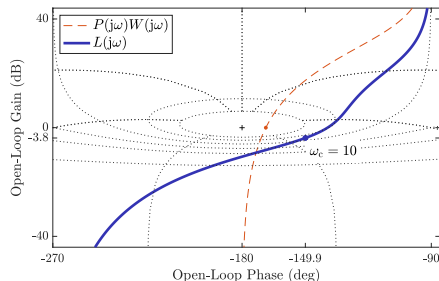
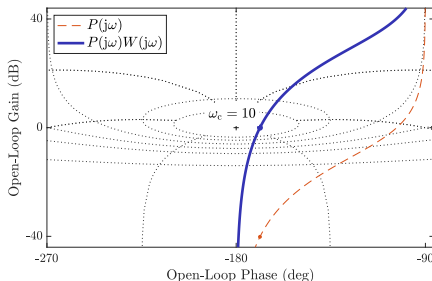
$$C(s) = \frac{239.04(s + 3.033)}{s^2 + 13.21s + 85.28}$$

Example: $\omega_c = 10$, P weight

Choose

$$W(s) = k_p$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.4276$, loops are



and the controller

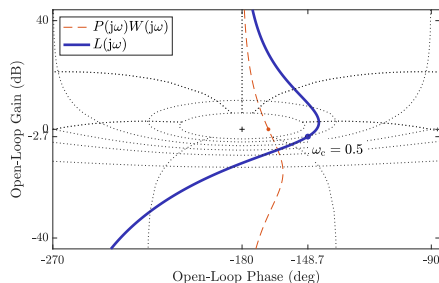
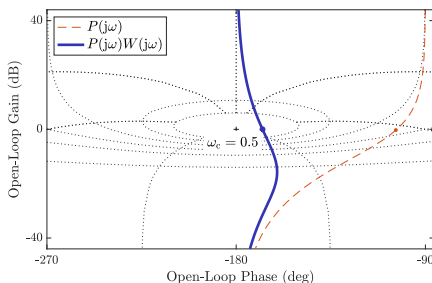
$$C(s) = \frac{2224.8(s + 4.675)}{s^2 + 26.84s + 358.2}$$

Example: $\omega_c = 0.5$, PI weight

Choose

$$W(s) = k_p \left(1 + \frac{1}{s} \right)$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.4223$, loops are



and the controller

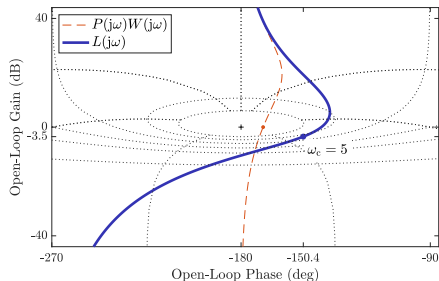
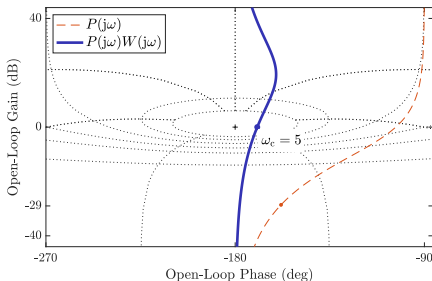
$$C(s) = \frac{0.59807(s + 2.001)(s + 1)(s + 0.1776)}{s(s + 1.821)(s^2 + 1.578s + 0.903)}$$

Example: $\omega_c = 5$, PI weight

Choose

$$W(s) = k_p \left(1 + \frac{1}{s} \right)$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting loops of $\epsilon = 0.4173$, loops are



and the controller (zero of $W(s)$ at $s = -1$ is canceled by a pole of $C_a(s)$)

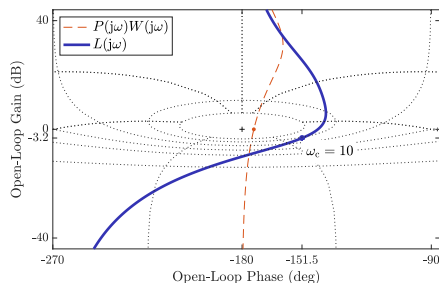
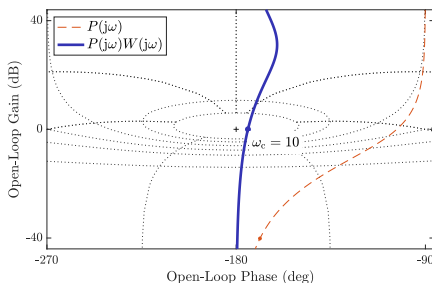
$$C(s) = \frac{311.03(s + 2.658)(s + 0.8432)}{s(s^2 + 13.93s + 95.51)}$$

Example: $\omega_c = 10$, PI weight

Choose

$$W(s) = k_p \left(1 + \frac{1}{s} \right)$$

where k_p renders $|P_a(j\omega_c)| = 1$. Resulting $\epsilon = 0.403$, loops are



and the controller (zero of $W(s)$ at $s = -1$ is canceled by a pole of $C_a(s)$)

$$C(s) = \frac{2556.5(s + 4.209)(s + 0.9569)}{s(s^2 + 27.7s + 382.1)}$$

Outline

Linear-quadratic state observer (Kalman filter)

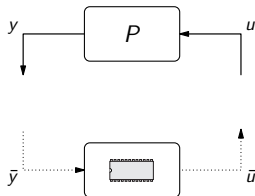
Linear Quadratic Gaussian problem

LQG controller design (optional self-study)

What is sampled-data control

Background: A/D and D/A conversions in the frequency domain (from LS)

Computer-controller systems



Control laws are often implemented on digital computers, mainly because of

- ‘spectacular success’ of digital technologies.

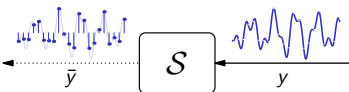
Feedback loops in this case contain

- analog plant, having analog i/o signals
- digital controller, having digital i/o signals (digital sequences)

We, therefore, need to provide analog/digital and digital/analog interfaces.

A/D conversion in the time domain

A conversion of a continuous-time (analog) signal, say y , to a discrete-time (digital) signal, say $\bar{y}$, is known as **sampling**. If for all $i \in \mathbb{Z}$

$$\bar{y}[i] = y(s_i), \quad s_i < s_{i+1}$$


then the term **ideal sampling** is used.

Terminology:

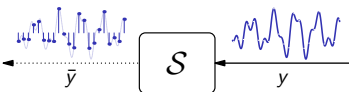
- time instances s_i are called **sampling instances**
- if $s_i = ih$ for some $h > 0$, we say that the sampling is periodic and call h the **sampling period**
- $\omega_s := 2\pi/h$ is known as the **sampling frequency**

This ideal sampling operation is

- well defined only if y is continuous at each sampling instance s_i .

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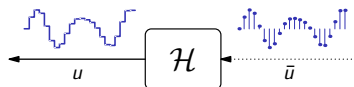
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D/A conversion in the time domain

A conversion of a discrete-time (digital) signal, say $\bar{u}$, to a continuous-time (analog) signal, say u , is known as **hold** (interpolation). Examples:

Zero-order hold (ZOH)

$$u(t) = \bar{u}[i], \quad \forall t \in (s_i, s_{i+1})$$



for given sampling instances s_i

First-order hold (FOH or linear interpolator)

$$u(t) = \frac{s_{i+1} - t}{s_{i+1} - s_i} \bar{u}[i] + \frac{t - s_i}{s_{i+1} - s_i} \bar{u}[i+1]$$

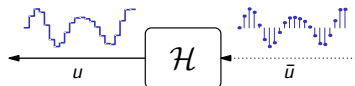
for given sampling instances s_i (not used in this presentation)

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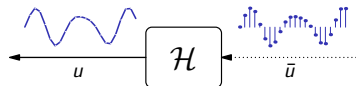
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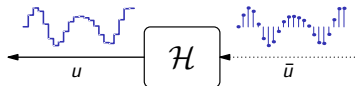
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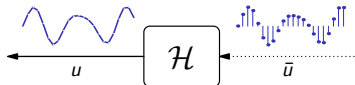
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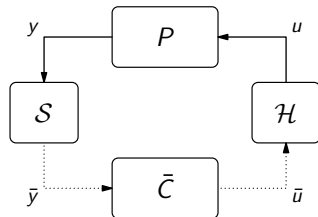
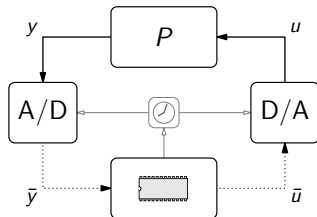
First-order hold (FOH or linear interpolator)

$$u(t) = \frac{s_{i+1} - t}{s_{i+1} - s_i} \bar{u}[i] + \frac{t - s_i}{s_{i+1} - s_i} \bar{u}[i + 1]$$



for given sampling instances s_i (not used in control, because it's non-causal)

Sampled-data systems



Thus, general sampled-data control system consists of

1. continuous-time plant P ,
2. digital part of the controller $\bar{C}$,
3. A/D converter (sampler) $\mathcal{S}$,
4. D/A converter (hold) $\mathcal{H}$.

We assume that the

- last three devices are **synchronized**, with sampling period h .

Pros and cons of sampled-data control

Digital equipment is

- 😊 more flexible
- 😊 more reliable
- 😊 cheaper

than its analog counterpart. Consequently,

- 😊 more sophisticated control algorithms can be implemented.

On the other hand,

- in intersample time control system is open loop;
- control signals are waveform limited (e.g. piecewise constant);
- rigorous analysis of sampled-data systems is more complicated comparing with their continuous-time counterparts.

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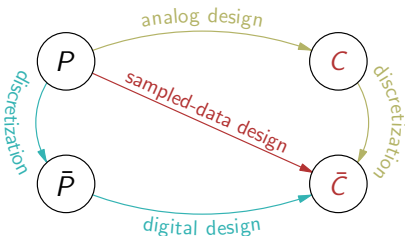
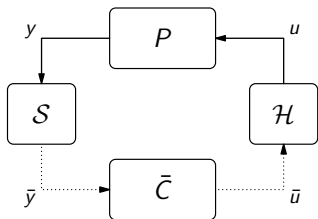
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Three approaches to sampled-data control design



1. Digital redesign of analog controllers

(do your favorite analog design first, then discretize the resulting controller)

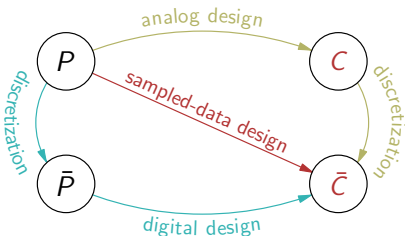
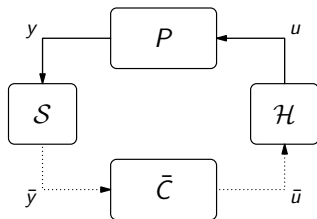
2. Discrete-time design

(discretize the problem first, then do your favorite discrete design)

3. Direct digital (sampled-data) design

(design discrete-time controller $\bar{C}(z)$ directly for analog specs)

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In this course we address mainly the first approach...

Outline

Linear-quadratic state observer (Kalman filter)

Linear Quadratic Gaussian problem

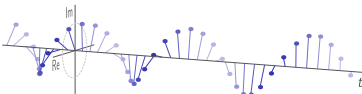
LQG controller design (optional self-study)

What is sampled-data control

Background: A/D and D/A conversions in the frequency domain (from LS)

Discrete harmonic signal

Signal $\alpha = a \exp_{e^{j\theta}} : \mathbb{Z} \rightarrow \mathbb{C}$,

$$\alpha[t] = ae^{j\theta t} = |a|e^{j(\theta t + \arg(a))} =$$


for $a \in \mathbb{C}$ and $\theta \in \mathbb{R}$ is called **harmonic signal** with frequency θ , amplitude $|a|$, and initial phase $\arg(a)$. Given $T \in \mathbb{N}$,

$$\alpha[t + T] = ae^{j\theta(t+T)} = ae^{j\theta t + j\theta T} = ae^{j\theta t} e^{j\theta T} = \alpha[t] e^{j\theta T}$$

so that

$$\alpha[t + T] = \alpha[t] \iff e^{j\theta T} = 1 \iff |\theta| T = 2\pi k \quad \text{for } k \in \mathbb{N}$$

and this condition is independent of t . Therefore,

- α is T -periodic iff $\exists k \in \mathbb{N}$ such that $|\theta| = \frac{2\pi k}{T}$,

which happens iff $2\pi/|\theta| \in \mathbb{Q}$.

Discrete harmonic signal (contd)

Because

$$e^{j(\theta+2\pi)t} = e^{j\theta t} e^{j2\pi t} = e^{j\theta t},$$

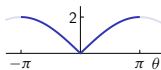
shifting θ by a multiple of 2π does *not* change the harmonic signal. Hence,

- discrete frequencies only make sense on intervals of the length 2π .

Thereafter we use the convention that $\theta \in [-\pi, \pi]$, which is a choice.

The normalized change at each step of the harmonic signal,

$$\frac{|a[t+1] - a[t]|}{|a[t]|} = \left| \frac{ae^{j\theta(t+1)}}{ae^{j\theta t}} - 1 \right| = |e^{j\theta} - 1| = 2 \left| \sin \frac{\theta}{2} \right| =$$



is independent of t and increases with $|\theta|$ (provided $\theta \in [-\pi, \pi]$, of course).

It is thus still justifiable to say that

- $a_1 \exp_{\omega_1}$ is faster (slower) than $a_2 \exp_{\omega_2}$ if $|\theta_1| > |\theta_2|$ ($|\theta_1| < |\theta_2|$).

The *fastest* harmonic signal is that for $\theta = \pm\pi$, for which $a[t] = a(-1)^t$.

Discrete harmonic signal (contd)

Because

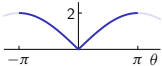
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The **fastest** harmonic signal is that for $\theta = \pm\pi$, for which $\alpha[t] = a(-1)^t$.

Discrete-time Fourier transform

The **discrete-time Fourier transform** (DTFT) of a discrete signal $x : \mathbb{Z} \rightarrow \mathbb{R}$ is the signal $X : [-\pi, \pi] \rightarrow \mathbb{C}$ such that

$$X(e^{j\theta}) = (\mathfrak{F}\{x\})(e^{j\theta}) := \sum_{t \in \mathbb{Z}} x[t] e^{-j\theta t}$$

where θ is called the angular frequency (in radians per step). The DTFT is well defined (uniform convergence, continuous X) if $x \in \ell_1$ and then

$$x[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) e^{j\theta t} d\theta =: (\mathfrak{F}^{-1}\{X\})(e^{j\theta})$$

If the domain of X is extended to $\mathbb{R}$, then X is a 2π -periodic function of θ .

If $x \in \ell_2$, then the DTFT sum might converge only in the (weaker) ℓ_2 -norm sense, so that X is defined *almost everywhere* and might not be continuous or even bounded. The use of distributions facilitates extending the DTFT to yet wider classes of signals.

Meaning

Similarly to the continuous-time case, the relation

$$x[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) e^{j\theta t} d\theta$$

means that

- x is a superposition of elementary harmonics $\frac{1}{2\pi} X(e^{j\theta}) \exp_{e^{j\theta}}$, with a continuum of frequencies θ , although now in the final range $[-\pi, \pi]$.

The signal X is then called the spectrum or frequency-domain representation of x , with the amplitude spectrum $|X|$ and the phase spectrum $\arg(X)$.

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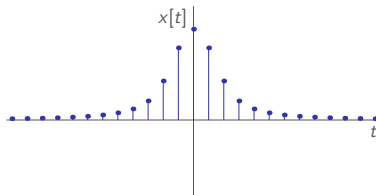
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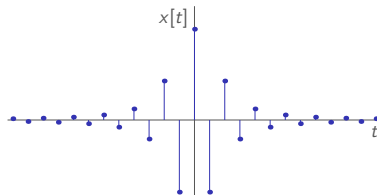
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Example

“slow” signal

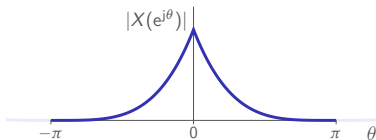
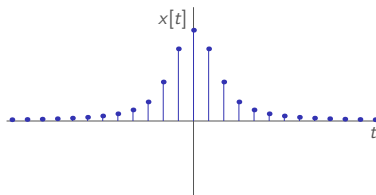


“fast” signal

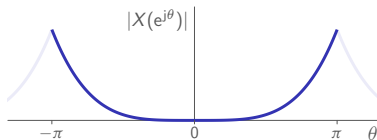
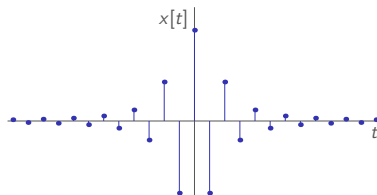


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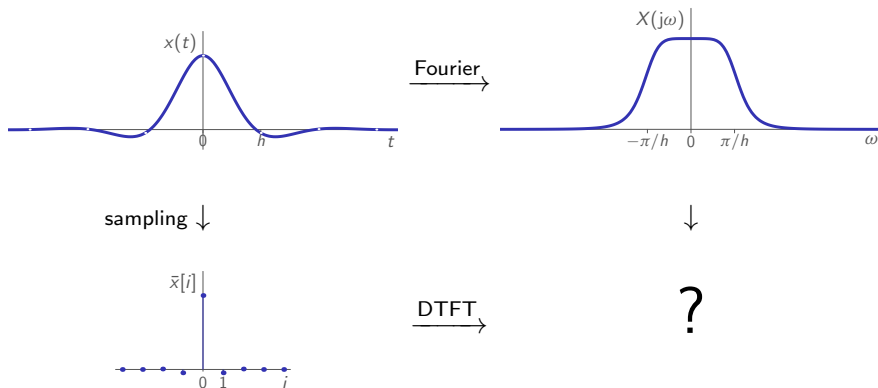


“fast” signal



Spectrum of sampled signal

Let x be an analog signal:



The question is

- can the DTFT of $\bar{x}$ be obtained directly from $X = \mathfrak{F}\{x\}$?

Preliminary: periodic summation

Consider a function $x : \mathbb{R} \rightarrow \mathbb{F}$, its periodic summation with period T is

$$x_T := \sum_{i \in \mathbb{Z}} \mathbb{S}_{iT} x \quad \implies \quad x_T(t) = \sum_{i \in \mathbb{Z}} x(t + iT)$$

The function $x_T : \mathbb{R} \rightarrow \mathbb{F}$ is T -periodic, because

$$x_T(t + T) = \sum_{i \in \mathbb{Z}} x(t + iT + T) \Big|_{j=i+1} = \sum_{j \in \mathbb{Z}} x(t + jT) = x_T(t)$$

Example: if $x = \text{tent} = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{else} \end{cases}$, then

$$x_T = \begin{cases} \text{tent} & \text{if } T = \frac{4}{3} \\ \text{tri} & \text{if } T = \frac{2}{3} \end{cases}$$

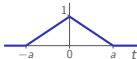
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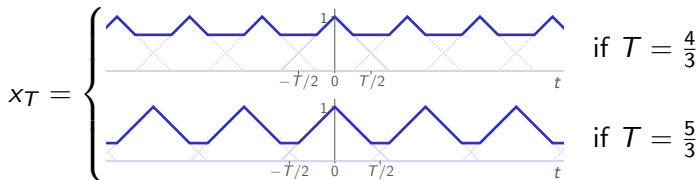
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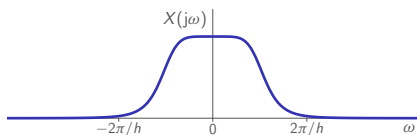
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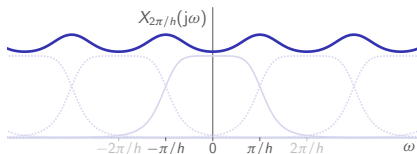
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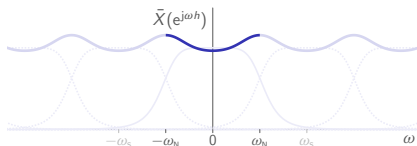
Spectrum of sampled signal (contd)



↓ periodic summation



↓ scaling by $\frac{1}{h}$



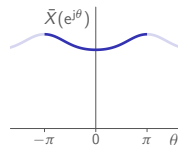
spectrum of $\bar{x}$:

$$\bar{X}(e^{j\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} X(j\frac{\theta + 2\pi i}{h})$$

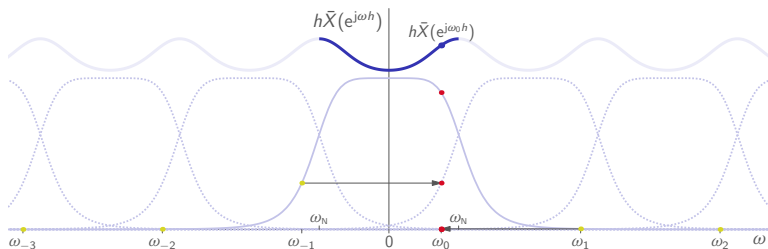
Nyquist frequency

$$\omega_N := \frac{\pi}{h} = \frac{\omega_s}{2}$$

$$\theta = \omega h \rightarrow$$



Spectrum of sampled signal: aliasing

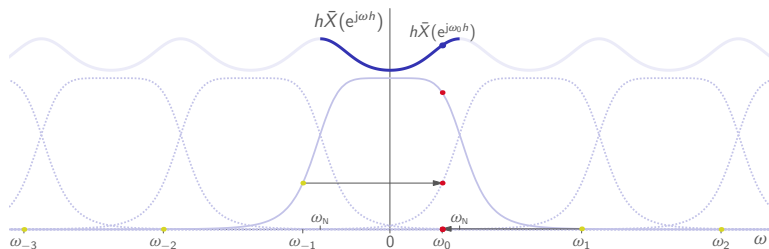


The spectrum of $\bar{x}$ at each discrete frequency $\theta_0 = \omega_0 h$ is a

- blend of analog spectra at all analog frequencies $\omega_i := \omega_0 + \omega_s i$, $i \in \mathbb{Z}$.

In this process every ω_i effectively tries to *alias* as θ_0 . This phenomenon is then dubbed **aliasing**, with respect to the **base frequency** ω_0 .

Spectrum of sampled signal: aliasing (contd)



Aliasing reveals the source of **information loss** under sampling,

- we can no longer tell $X(j\omega_i)$ from $X(j\omega_j)$ in their effect on $\bar{X}(e^{j\omega_0 h})$ unless we know how they depend on each other.

For instance, in this particular case $\bar{X}(e^{j\omega_0 h})$ has significant contributions of both $X(j\omega_0)$ and $X(j\omega_{-1})$ for most $\omega_0 > 0$, which might be problematic.

Spectrum of the output of ZOH

By the definition of the Fourier transform,

$$\begin{aligned}
 X(j\omega) &= \int_{\mathbb{R}} x(t) e^{-j\omega t} dt = \sum_{i \in \mathbb{Z}} \int_{ih}^{(i+1)h} x(t) e^{-j\omega t} dt \\
 &= \sum_{i \in \mathbb{Z}} \int_{ih}^{(i+1)h} \bar{x}[i] e^{-j\omega t} dt = \sum_{i \in \mathbb{Z}} \bar{x}[i] \int_{ih}^{(i+1)h} e^{-j\omega t} dt \\
 &= \sum_{i \in \mathbb{Z}} \bar{x}[i] \left(\frac{e^{-j\omega t}}{-j\omega} \right) \Big|_{ih}^{(i+1)h} = \sum_{i \in \mathbb{Z}} \bar{x}[i] \frac{e^{-j\omega h i} - e^{-j\omega h (i+1)}}{j\omega} \\
 &= \sum_{i \in \mathbb{Z}} \bar{x}[i] e^{-j\omega h i} \frac{1 - e^{-j\omega h}}{j\omega} \quad \text{cf. } \bar{X}(e^{j\theta}) = \sum_{i \in \mathbb{Z}} \bar{x}[i] e^{-j\theta i} \\
 &= h \frac{1 - e^{-j\omega h}}{j\omega h} \bar{X}(e^{j\omega h}) = h \operatorname{sinc}\left(\frac{\omega h}{2}\right) e^{-j\omega h/2} \bar{X}(e^{j\omega h})
 \end{aligned}$$

(note that $\bar{X}$ is ω_s -periodic as a function of ω).

Spectrum of the output of ZOH: example

With $\bar{x}$ as above,

