

Control Theory (00350188)

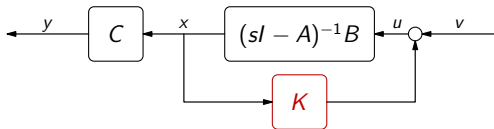
lecture no. 9

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT



State feedback



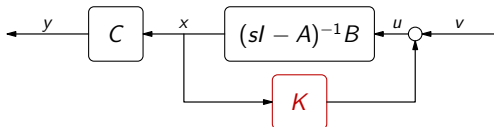
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- stabilizing
- shaping closed-loop modes
- optimizing quadratic cost function
- ...

The elephant in the room:

- what if the state vector cannot be measured directly?

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Outline

State observer

Observability

Example: 2-mass system (observability)

Minimality

State observer: pole placement

Observer-based feedback

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State reconstruction

Consider state equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0, \\ y(t) = Cx(t). \end{cases}$$

If the state vector cannot be measured (this is what typically happens), then it could be reconstructed from the measured y . Such reconstructor is called **state observer** or simply **observer**.

Naïve observer

A possible approach is to construct a **virtual plant**, like

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \quad \hat{x}(0) = \hat{x}_0,$$

for some initial guess $\hat{x}_0$. The observation error $\epsilon := x - \hat{x}$ satisfies

$$\dot{\epsilon}(t) = A\epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0$$

which are autonomous dynamics, driven only by the mismatch between $\hat{x}(0)$ and $x(0)$.

Good news:

- if A is Hurwitz, then $\lim_{t \rightarrow \infty} \epsilon(t) = 0$, i.e. $\hat{x}(t) \rightarrow x(t)$ asymptotically no matter what u is, provided we know it, of course

Bad news:

- we cannot affect error dynamics,
- if A is unstable, $\hat{x}$ doesn't converge to x .

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A possible approach is to construct a **virtual plant**, like

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Luenberger observer

Naïve observer ignores the

- information about x , available in the measurement y .

Consider adding a function of measured mismatch $y - C\hat{x}$ (aka innovations signal) in the form

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)), & \hat{x}(0) &= \hat{x}_0 \\ &= (A + LC)\hat{x}(t) + Bu(t) - Ly(t), & \hat{x}(0) &= \hat{x}_0\end{aligned}$$

for a gain $L \in \mathbb{R}^{n \times 1}$. In this case,

$$\dot{\epsilon}(t) = (A + LC)\epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0.$$

Now we potentially

- can affect the error dynamics.

Q: what freedom we have in assigning $\text{spec}(A + LC)$ by the choice of L ?

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Special case: observer form

Assume that

$$A = \begin{bmatrix} -a_{n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & \cdots & 1 \\ -a_0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad C = [1 \ 0 \ \cdots \ 0].$$

Choosing

$$L = \begin{bmatrix} l_{n-1} \\ \vdots \\ l_1 \\ l_0 \end{bmatrix} \quad \Rightarrow \quad A_L = A + LC = \begin{bmatrix} -(a_{n-1} - l_{n-1}) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(a_1 - l_1) & 0 & \cdots & 1 \\ -(a_0 - l_0) & 0 & \cdots & 0 \end{bmatrix}$$

is still an observer form (companion matrix) and its characteristic polynomial

$$\chi_{A_L}(\lambda) = \lambda^n + (a_{n-1} - l_{n-1})\lambda^{n-1} + \cdots + (a_1 - l_1)\lambda + (a_0 - l_0).$$

Special case: observer form (contd)

Therefore, any desired observer characteristic polynomial, say

$$\hat{\chi}(s) = s^n + \hat{\chi}_{n-1}s^{n-1} + \cdots + \hat{\chi}_1s + \hat{\chi}_0$$

for some coefficients $\hat{\chi}_i > 0$, can be assigned by

$$L = \begin{bmatrix} a_{n-1} - \hat{\chi}_{n-1} \\ \vdots \\ a_1 - \hat{\chi}_1 \\ a_0 - \hat{\chi}_0 \end{bmatrix} \implies A_L = \begin{bmatrix} -\hat{\chi}_{n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\hat{\chi}_1 & 0 & \cdots & 1 \\ -\hat{\chi}_0 & 0 & \cdots & 0 \end{bmatrix}$$

Q: under what condition this can be said about an arbitrary realization?

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Outline

State observer

Observability

Example: 2-mass system (observability)

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Observability: definition

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0, \\ y(t) = Cx(t) + Du(t). \end{cases}$$

This system (or the pair (C, A)) is said to be

- **observable** if any initial state x_0 can be reconstructed from time history of $u(t)$ and $y(t)$ in interval $[0, t_1]$ for every $t_1 > 0$ and $u(t)$.

Simplifying observation:

- Without loss of generality we can assume that $u(t) \equiv 0$. Indeed, as

$$y(t) = Ce^{At}x_0 + Du(t) + C \int_0^t e^{A(t-s)}Bu(s)ds,$$

x_0 reconstructable from time history of $y(t), u(t)$ iff it reconstructable from time history of $\tilde{y}(t) := y(t) - Du(t) - C \int_0^t e^{A(t-s)}Bu(s)ds$.

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Observability and observability matrix

Matrix

$$M_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

called the **observability matrix**.

Theorem

Pair (C, A) is observable if and only if $\det M_o \neq 0$.

Proof

If $u = 0$, then $y(t) = Ce^{At}x_0$ and

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = M_o x_0.$$

We have:

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We have:

1. If $\det M_o \neq 0$, x_0 can be obtained from $n - 1$ derivatives of y at $t = 0$.
2. If $\det M_o = 0$, then $\exists v \neq 0$ such that $M_o v = 0$, i.e. that $CA^i v = 0$ for all $i = 0, \dots, n - 1$. Then, by Cayley-Hamilton,

$$CA^i v = 0, \quad \forall i \in \mathbb{Z}^+ \implies Ce^{At} v \equiv 0.$$

Therefore, if $x_0 = v$, then $y(t) = Ce^{At}x_0 \equiv 0$ and this initial condition is indistinguishable from $x(0) = 0$. $\square$

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Observability and similarity

If $\tilde{A} = TAT^{-1}$ and $\tilde{C} = CT^{-1}$ for some nonsingular T , then

$$\begin{aligned}\tilde{M}_o &:= \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T^{-1} \\ &= M_o T^{-1}\end{aligned}$$

i.e.

- observability is not affected by similarity transformations.

Ovservability: some other tests

Theorem

The following statements are equivalent:

1. (C, A) is observable;
2. $\det M_o \neq 0$;
3. $\det W_o(t) \neq 0$ for all $t > 0$, where $W_o(t) := \int_0^t e^{A's} C' C e^{As} ds \in \mathbb{R}^{n \times n}$;
4. $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \in \mathbb{C}^{(n+1) \times n}$ has full column rank $\forall \lambda \in \mathbb{C}$ (**PBH test**);
5. eigenvalues of $A + LC$ can be freely assigned by $L \in \mathbb{R}^n$;
6. (A', C') is controllable.

The last statement shows

- duality between observability and controllability properties.

Observability: some other tests

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The last statement shows

- **duality** between observability and controllability properties.

Observability: some useful facts

The following observations/definitions are important:

- $W_o(t)$ -test leads to a derivative-free reconstruction algorithm. Let

$$\tilde{x}(t) := [W_o(t_1)]^{-1} \int_0^t e^{A's} C' y(s) ds.$$

In this case

$$\tilde{x}(t_1) = [W_o(t_1)]^{-1} \int_0^{t_1} e^{A's} C' C e^{As} x_0 ds = x_0.$$

- If (C, A) is not observable, the PBH test fails for some $\lambda_i \in \mathbb{C}$. These λ_i are eigenvalues of A and called **unobservable modes** of (C, A) .
- If λ is an unobservable mode of (C, A) , then it is eigenvalue of $A + LC$ for any L .

Observability: some useful facts

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Detectability

Pair (C, A) is said to be

- **detectable** if all its unobservable modes are stable (in open LHP).

Detectability means that there exists $L \in \mathbb{R}^n$ such that

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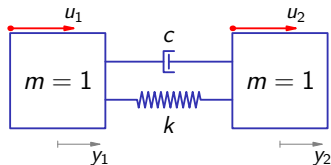
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Setup

Consider again



with

$$\begin{cases} \underbrace{\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_C = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_C \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} \end{cases}$$

Observability under $y = \gamma_1 y_1 + \gamma_2 y_2$

In this case

$$y(t) = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \gamma_1 & \gamma_2 & 0 & 0 \end{bmatrix} x(t)$$

Observability matrix (denoting $\delta_\gamma := \gamma_1 - \gamma_2$):

$$M_o = \begin{bmatrix} \gamma_1 & \gamma_2 & 0 & 0 \\ 0 & 0 & \gamma_1 & \gamma_2 \\ -k\delta_\gamma & k\delta_\gamma & -c\delta_\gamma & c\delta_\gamma \\ 2ck\delta_\gamma & -2ck\delta_\gamma & -(k-2c^2)\delta_\gamma & (k-2c^2)\delta_\gamma \end{bmatrix}$$

with $\det M_o = -k^2(\gamma_1^2 - \gamma_2^2)^2$. Thus, the system is

- unobservable for $\gamma_1 = \pm\gamma_2$.

What could it mean?

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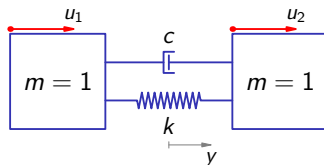
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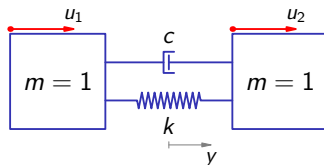


PBH test:

$$\text{rank} \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ -k & k & -c - \lambda & c \\ k & -k & c & -c - \lambda \\ 1 & 1 & 0 & 0 \end{bmatrix} \bigg|_{\lambda = -c \pm \sqrt{c^2 - 2k}} = 3,$$

(rank lost at **unobservable modes** of A). This agrees with our intuition that oscillations cannot be seen via the center of mass.

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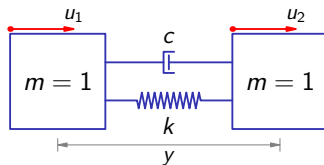
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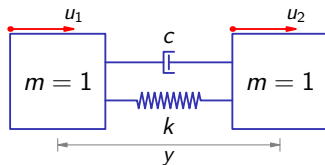


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(rank lost at **unobservable mode** of A). This agrees with our intuition that rigid body motion cannot be seen via relative position of the masses.

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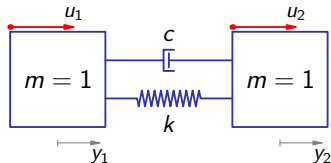
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Transfer functions for $y = \gamma_1 y_1 + \gamma_2 y_2$



Transfer function from u_1 to y :

$$P_1(s) = \frac{\gamma_1 s^2 + c(\gamma_1 + \gamma_2)s + k(\gamma_1 + \gamma_2)}{s^2(s^2 + 2cs + 2k)}$$

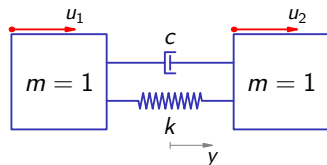
and transfer function from u_2 to y :

$$P_2(s) = \frac{\gamma_2 s^2 + c(\gamma_1 + \gamma_2)s + k(\gamma_1 + \gamma_2)}{s^2(s^2 + 2cs + 2k)}$$

(both obtained via $C(sI - A)^{-1}B$).

Transfer functions for $y = \gamma_1 y_1 + \gamma_2 y_2$ (contd)

$\gamma_1 = \gamma_2$:



then

$$P_1(s) = P_2(s) = \frac{\gamma_1}{s^2}.$$

$\gamma_1 = -\gamma_2$:

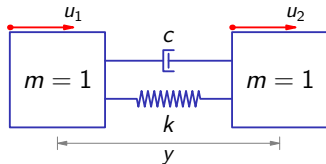
then

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Transfer functions for $y = \gamma_1 y_1 + \gamma_2 y_2$ (contd)

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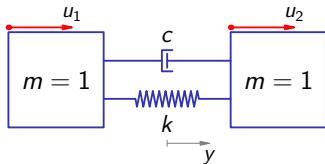
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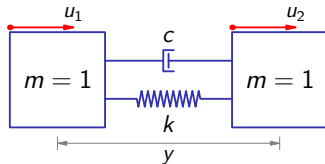
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In both cases we have **pole/zero cancellations** (of different modes though).

Outline

State observer

Observability

Example: 2-mass system (observability)

Minimality

State observer: pole placement

Observer-based feedback

Minimal state-space realization

Example

Let $G(s) = \frac{1}{s+1}$. The following are its state-space realizations:

$$\begin{cases} \dot{x} = -x + u, & x(0) = 0 \\ y = x \end{cases} \quad \text{and} \quad \begin{cases} \dot{\tilde{x}} = -\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, & \tilde{x}(0) = 0, \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}. \end{cases}$$

The first of them has state dimension $n = 1$, while the second one— $n = 2$. This indicates that there is *redundancy* in $\tilde{x}$ (it accumulates somebody else history as well).

We may be interested to avoid redundancy. To this end, the notion of minimal state-space realization, i.e. a realization with minimal possible dimension, plays a key role.

Minimal state-space realization

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Minimality criterion

Theorem

Realization

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = 0, \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is minimal iff it is both controllable and observable.

Explanations

- uncontrollable part of x cannot be affected by input u ,
- unobservable part of x is invisible from output y .

Important fact

- every two minimal realizations of the same system are similar (i.e. there is a similarity transformation between them).

Minimality criterion

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Minimality and poles

Theorem

If

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = 0, \\ y(t) = Cx(t) + Du(t) \end{cases}$$

is minimal, then $\lambda \in \mathbb{C}$ is a pole of $G(s) = D + C(sI - A)^{-1}B$ iff it is an eigenvalue of A .

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Luenberger observer: choice of L

Let (C, A) be observable, then for an arbitrary polynomial

$$\hat{\chi}_{cl}(s) = s^n + \hat{\chi}_{n-1}s^{n-1} + \cdots + \hat{\chi}_1s + \hat{\chi}_0$$

there exists observer gain L such that $\hat{\chi}_{cl}(s)$ is characteristic polynomial of observer error, i.e. $\hat{\chi}_{cl}(s) = \det(sI - A_L)$.

The gain L leading to a required $\hat{\chi}_{cl}(s)$ can be chosen by the counterpart of Ackermann's formula:

$$L = -\hat{\chi}_{cl}(A)M_o^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

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$$L = -\hat{\chi}_{\text{cl}}(A)M_o^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

¹Apply Ackermann's formula to $(A + LC)' = A' + C'L'$ and then transpose the result.

Example: two-tank system (contd)

Suppose that fluid height only in the first tank can be measured, i.e.

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), & \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

(here $y = h_1 - h_{1,\text{eq}}$). To reconstruct $x_2(t)$ we build state observer (**virtual sensor**) in the form

$$\begin{bmatrix} \dot{\hat{x}}_1(t) \\ \dot{\hat{x}}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (y(t) - \hat{x}_1(t))$$

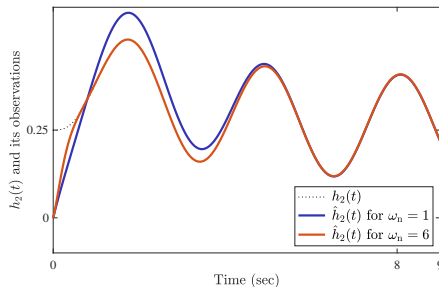
where

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = -\hat{\chi}_{\text{cl}}(A) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} -3 + 2\hat{\xi}\hat{\omega}_n \\ 5 - 4\hat{\xi}\hat{\omega}_n + \hat{\omega}_n^2 \end{bmatrix}$$

for a desired $\hat{\chi}_{\text{cl}}(s) = s^2 + 2\hat{\xi}\hat{\omega}_n s + \hat{\omega}_n^2$.

Example: simulations

With $u(t) = q(t) - q_{\text{eq}} = 0.5 \sin(2t)$, $\hat{\xi} = 0.8$, and $\hat{\omega}_n = \{1, 5\}$,



under $L = \begin{bmatrix} 1.4 \\ -2.8 \end{bmatrix}$ and $L = - \begin{bmatrix} 6.6 \\ 21.8 \end{bmatrix}$.

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Output feedback: naïve approach

Consider

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) \end{cases}$$

in which the state vector x is not measured. Hence, state feedback cannot be used. Under this circumstance, we may try to

- combine state feedback and state observer

instead, i.e. to use observed state in control law as if it were the true state.

This results to the following control law:

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L(y(t) - C\hat{x}(t)), & \hat{x}(0) = \hat{x}_0 \\ u(t) = K\hat{x}(t) + v(t) \end{cases}$$

which is called **observer-based controller**.

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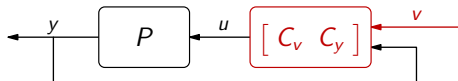
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Observer-based controller

Observer-based control law can be rewritten as

$$\begin{cases} \dot{\hat{x}}(t) = (A + BK + LC)\hat{x}(t) - Ly(t) + Bv(t), & \hat{x}(0) = \hat{x}_0 \\ u(t) = K\hat{x}(t) + v(t) \end{cases}$$

which is a system having v and y as its inputs and u as its output:



with

$$\begin{bmatrix} C_v(s) & C_y(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} + K(sI - (A + BK + LC))^{-1} \begin{bmatrix} B & -L \end{bmatrix},$$

where $C_v : v \mapsto u$ and $C_y : y \mapsto u$. Note that

- controller $-C_y$ is the state-space counterpart of the feedback controller in the standard unity-feedback case with negative feedback.

Closed-loop system

State equation of the closed-loop system $v \mapsto y = Cx$ is:

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v(t) \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \end{cases}$$

with initial conditions $\begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix}$. What can we say about its modes / stability?

A key observation is that

→ dynamics of the observer error $e = x - \hat{x}$ do not depend on u .

So change the state vector to

$$\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix},$$

i.e. use the similarity transformation with $T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = T^{-1}$.

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A key observation is that

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So change the state vector to

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i.e. use the similarity transformation with $T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = T^{-1}$.

Closed-loop system: similarity transformation

We have:

$$\begin{aligned}\tilde{A}_{cl} &= T A_{cl} T^{-1} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & BK \\ -LC & A + BK + LC \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \\ \tilde{B}_{cl} &= T B_{cl} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix} \\ \tilde{C}_{cl} &= C_{cl} T^{-1} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} C & 0 \end{bmatrix}\end{aligned}$$

Note that

– the pair $(\tilde{A}_{cl}, \tilde{B}_{cl})$ has all modes of A_L uncontrollable in it.

Indeed,

$$\begin{bmatrix} 0 & \tilde{\eta}_2^T \end{bmatrix} \begin{bmatrix} A_K - \lambda I & -BK & B \\ 0 & A_L - \lambda I & 0 \end{bmatrix} = \begin{bmatrix} 0 & \tilde{\eta}_2^T (A_L - \lambda I) & 0 \end{bmatrix} = 0$$

for every right (nonzero) eigenvector $\tilde{\eta}_2$ of A_L , so PBH yields the conclusion.

Closed-loop system: similarity transformation

We have:

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$$\tilde{B}_{cl} = T B_{cl} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

$$\tilde{C}_{cl} = C_{cl} T^{-1} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = \begin{bmatrix} C & 0 \end{bmatrix}$$

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for every right (nonzero) eigenvector $\tilde{\eta}_2$ of A_L , so PBH yields the conclusion.

The separation

Thus, we end up with the closed-loop system

$$\begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\epsilon}(t) \end{bmatrix} = \begin{bmatrix} A_K & -BK \\ 0 & A_L \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t), & \begin{bmatrix} x(0) \\ \epsilon(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_0 - \hat{x}_0 \end{bmatrix} \\ y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \epsilon(t) \end{bmatrix} \end{cases}$$

The roots of the closed-loop characteristic polynomial

$$\chi_{cl}(s) = \det(sI - A_K) \det(sI - A_L)$$

are the union of the state-feedback and observer modes. Thus, all we need to do to stabilize the system is to

- design stabilizing state feedback
 - design stable observer
- separately. This is known as the *separation principle*.

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- design stabilizing state feedback i.e. its gain K
- design stable observer i.e. its gain L

separately. This is known as the **separation principle**.

Closed-loop system $v \mapsto y$

The “ ϵ ” part of the closed-loop behavior

$$\dot{\epsilon}(t) = A_L \epsilon(t), \quad \epsilon(0) = x_0 - \hat{x}_0 \quad \implies \quad \epsilon(t) = e^{A_L t} (x_0 - \hat{x}_0)$$

The “ x ” part is then

$$\dot{x}(t) = A_K x(t) + Bv(t) - BK\epsilon(t) = A_K x(t) + B(v(t) - Ke^{A_L t}(x_0 - \hat{x}_0))$$

i.e. including an observer is

- equivalent to adding an **exponentially decaying signal** to v .

Moreover, if $x_0 = \hat{x}_0$, then $\epsilon = 0$ and

$$\dot{x}(t) = A_K x(t) + Bv(t), \quad x(0) = x_0$$

exactly like in the case of measured state.