

Control Theory (00350188)

lecture no. 8

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Optimization-based design

The underlying idea:

- express requirements in terms of a size of a **cost function** to be minimized by the control law.

Important to remember that

- no performance index can reflect all our requirements.

We therefore shall use

- optimization as design tool,

rather than the control goal per se. Every controller is optimal with respect to some cost function. But not every optimal controller makes sense. Thus, optimization methods should be judged by

- simplicity of their solutions
- simplicity of tuning their properties via weighting functions
- byproducts (what do we get for granted)

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Outline

Linear-Quadratic Regulator (LQR) problem: formulation

Mathematical preliminaries

Linear-Quadratic Regulator (LQR) problem: solution

LQR: solution properties

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LQR: solution properties

The LQR problem

Given

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

find u that

- stabilizes the systems ($\lim_{t \rightarrow \infty} x(t) = 0$),
- minimizes

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

Assumptions:

A₁: (A, B) is stabilizable

necessary for a stabilizing controller to exist

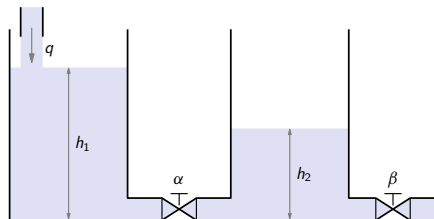
A₂: $R > 0$ and $Q - SR^{-1}S' \geq 0$

guarantees that the optimal u is bounded and that $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0 \implies \mathcal{J} \geq 0$

A₃: $\begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank for all ω

guarantees that the optimal u exists and is unique

Example: two-tank system from Lecture 7



With $\alpha = \beta = \sigma = 1$, $h_{\text{eq}} = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$, and $q_{\text{eq}} = 0.5$, the linearized dynamics

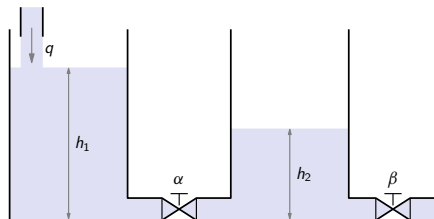
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

where $x_i = h_i - h_{i,\text{eq}}$ and $u = q - q_{\text{eq}}$.

Our goal is to

— regulate x from $x(0)$ to $\lim_{t \rightarrow \infty} x(t) = 0$ in a desired manner (regulator problem), which is effectively the set-point tracking of h_{eq} .

Example: two-tank system from Lecture 7



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Example: LQR formulations

We may consider several quantities to penalize:

1. level deviation from the steady state, $\gamma_i := \int_0^\infty x_i^2(t) dt$, for $i = 1, 2$
2. control effort, $\gamma_u := \int_0^\infty u^2(t) dt$
3. rate of level asynchronization, $\gamma_{12} := \int_0^\infty (\dot{x}_1(t) - 2\dot{x}_2(t))^2 dt$

These goals are conflicting, e.g. faster response needs higher control effort, so the design is

- the art of tradeoffs,

whose essence is to seek for a right blend of a number of them via weights.

Example: building Q , R , and S

First, express each penalty via LQR quadratic forms. To this end, note that

$$x_i = \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad u = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \text{and} \quad \dot{x}_1 - 2\dot{x}_2 = C_{12} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

where $C_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}$, $C_2 := \begin{bmatrix} 0 & 1 \end{bmatrix}$, $C_{12} := C_1 - 2C_2 = \begin{bmatrix} 1 & -2 \end{bmatrix}$. Hence,

$$\gamma_i = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} C_i' C_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

$$\gamma_u = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

and

$$\gamma_{12} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} A' C_{12}' C_{12} A & A' C_{12}' C_{12} B \\ B' C_{12}' C_{12} A & B' C_{12}' C_{12} B \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

Example: building Q , R , and S (contd)

Now we could easily mix them. For example, for every $r > 0$

$$\gamma_i + r\gamma_u = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \underbrace{\begin{bmatrix} C_i' C_i & 0 \\ 0 & r \end{bmatrix}}_{\geq 0} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt.$$

which is a standard LQR cost.

Likewise, for every $\lambda \in (0, 1)$

$$\lambda\gamma_i + (1 - \lambda)\gamma_{12} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

where

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} \lambda C_i' C_i + (1 - \lambda) A' C_{12}' C_{12} A & (1 - \lambda) A' C_{12}' C_{12} B \\ (1 - \lambda) B' C_{12}' C_{12} A & (1 - \lambda) B' C_{12}' C_{12} B \end{bmatrix} \geq 0$$

because $Q - SR^{-1}S = \lambda C_i' C_i \geq 0$ (note that $C_{12}B = 1 \neq 0$).

Outline

Linear-Quadratic Regulator (LQR) problem: formulation

Mathematical preliminaries

Linear-Quadratic Regulator (LQR) problem: solution

LQR: solution properties

CARE

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $Q = Q' \in \mathbb{R}^{n \times n}$, $R = R' \in \mathbb{R}^{m \times m}$, and $S \in \mathbb{R}^{n \times m}$ be given. The matrix equation

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

is called the **continuous-time algebraic Riccati equation** (CARE). Its solution $\bar{X} \in \mathbb{R}^{n \times n}$ is said to be stabilizing if the matrix

$$A_K := A - BR^{-1}(S' + B'\bar{X})$$

is Hurwitz¹. The stabilizing $\bar{X}$, if exists, is unique and satisfies $\bar{X} = \bar{X}'$. We are interested in CAREs for

$$R > 0 \quad \text{and} \quad \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$$

(the latter is equivalent to $Q - SR^{-1}S' \geq 0$).

¹MATLAB: `icare(A,B,Q,R,S)` or `icare(A,B,Q,R)` if $S = 0$.

CARE: example

Let $A = a$, $B = b$, $Q = q \geq 0$, $R = r > 0$, and $S = 0$. The CARE reads

$$2a\bar{X} + q - \frac{b^2}{r}\bar{X}^2 = 0.$$

1. If $b = 0$, then the CARE becomes $2a\bar{X} + q = 0$ and $A_K = a$.
 - if $a \geq 0$, then no stabilizing solution exists
 - if $a < 0$, then $\bar{X} = -q/(2a) \geq 0$ is the stabilizing solution

2. If $b \neq 0$, then the CARE above is quadratic, solvable by

$$\bar{X} = \frac{ar \pm \sqrt{(a^2r + b^2q)r}}{b^2} \Rightarrow A_K = a - \frac{b^2}{r}\bar{X} = \mp \frac{\sqrt{a^2r + b^2q}}{\sqrt{r}}$$

and it is Hurwitz for “+”, unless $a = q = 0$. Thus, the stabilizing

$$\bar{X} = \frac{ar + \sqrt{(a^2r + b^2q)r}}{b^2} > 0.$$

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and it is Hurwitz for “+”, unless $a = q = 0$. Thus, the stabilizing

$$\bar{X} = \frac{ar + \sqrt{(a^2r + b^2q)r}}{b^2} > 0.$$

CARE: existence

Let $R > 0$ and $Q - SR^{-1}S' \geq 0$. If

- (A, B) is stabilizable,

- $\begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank $\forall \omega \in \mathbb{R}$,

then the stabilizing solution $\bar{X}$ to

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

exists and is such that $\bar{X} \geq 0$.

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Solution

Theorem

If $\mathcal{A}_{1-3}$ hold, then the unique stabilizing controller that minimizes

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

is

$$u(t) = -R^{-1}(S' + B'\bar{X})x(t),$$

where $\bar{X} = \bar{X}' \geq 0$ is the stabilizing solution to the CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0.$$

The optimal cost is then

$$\mathcal{J}_{opt} = x_0' \bar{X} x_0.$$

Remark. The optimal control law is a state feedback, with the gain $K = -R^{-1}(S' + B'\bar{X})$. The closed-loop “A” matrix $A + BK$ is **Hurwitz** because $\bar{X}$ is the stabilizing solution.

Proof

Denote $K := -R^{-1}(S' + B'\bar{X})$ and define

$$\begin{aligned} M_X &:= \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} + \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix} \\ &= \begin{bmatrix} Q + A'\bar{X} + \bar{X}A & S + \bar{X}B \\ S' + B'\bar{X} & R \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} \end{aligned}$$

because

$$Q + A'\bar{X} + \bar{X}A = (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = K'RK \quad \& \quad S' + B'\bar{X} = -RK$$

for every Riccati solution $\bar{X}$. Thus,

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} - \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix}$$

which is a key relation.

Proof (contd)

Now, the quadratic form in the cost $\mathcal{J}$ is

$$\begin{aligned}
 & \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
 &= \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
 &\quad - \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \left(\begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix} \right) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
 &= (u(t) - Kx(t))' R (u(t) - Kx(t)) - x'(t) \bar{X} \dot{x}(t) - \dot{x}'(t) \bar{X} x(t) \\
 &= (u(t) - Kx(t))' R (u(t) - Kx(t)) - \frac{d}{dt} (x'(t) \bar{X} x(t))
 \end{aligned}$$

Proof (contd)

Thus,

$$\begin{aligned}\mathcal{J} &= \int_0^\infty \left((u(t) - Kx(t))' R (u(t) - Kx(t)) - \frac{d}{dt} (x'(t) \bar{X} x(t)) \right) dt \\ &= \int_0^\infty (u(t) - Kx(t))' R (u(t) - Kx(t)) dt + x_0' \bar{X} x_0 - \lim_{t \rightarrow \infty} x'(t) \bar{X} x(t).\end{aligned}$$

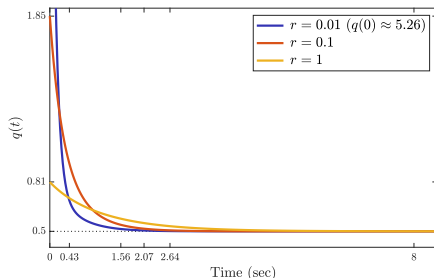
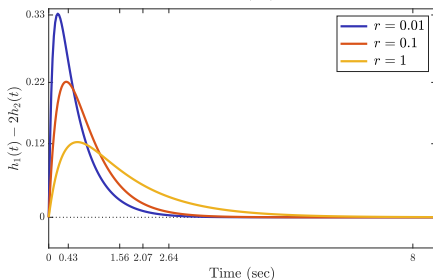
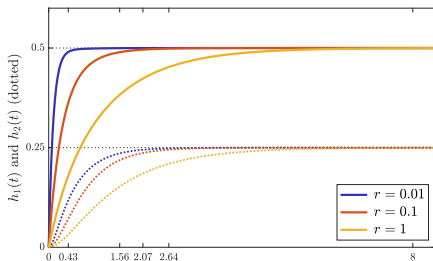
If u is stabilizing, then $\lim_{t \rightarrow \infty} x'(t) \bar{X} x(t) = 0$. Because $R > 0$,

$$\mathcal{J} = \int_0^\infty (u(t) - Kx(t))' R (u(t) - Kx(t)) dt + x_0' \bar{X} x_0 \geq x_0' \bar{X} x_0$$

whenever the system is asymptotically stable and the equality is attained by $u = Kx$, which *is* stabilizing ($\bar{X}$ is the stabilizing solution). □

Example: design 1

Consider $\mathcal{J} = \gamma_1 + r\gamma_u$ for various control energy weights $r > 0$. Results:



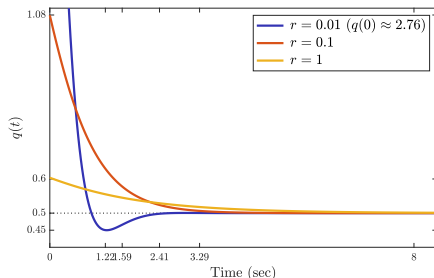
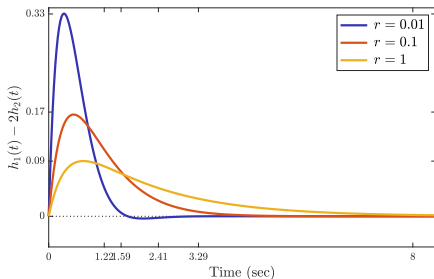
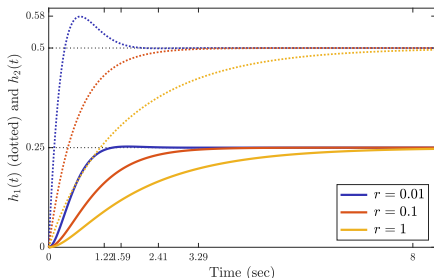
The gains and closed-loop eigenvalues are

$$K = - \begin{bmatrix} 9.1264 & 0.7722 \\ 2.4595 & 0.4841 \\ 0.5316 & 0.1729 \end{bmatrix}$$

and $\{-10.15, -1.97\}$, $\{-3.75, -1.71\}$, and $\{-2.7, -0.83\}$.

Example: design 2

Consider $\mathcal{J} = \gamma_2 + r\gamma_u$ for various control energy weights $r > 0$. Results:



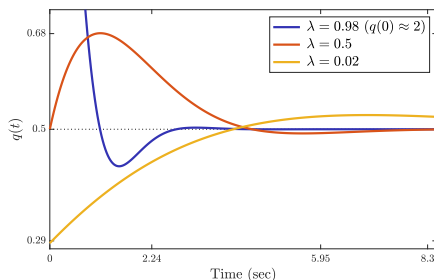
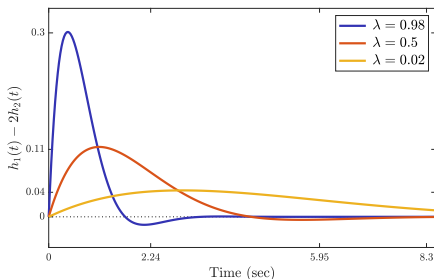
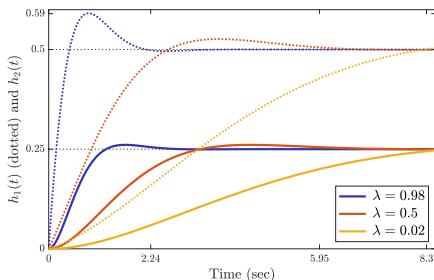
The gains and closed-loop eigenvalues are

$$K = - \begin{bmatrix} 2.2057 & 4.6384 \\ 0.6923 & 0.9320 \\ 0.1350 & 0.1441 \end{bmatrix}$$

and $\{-2.6 \pm j1.81\}$, $\{-2.15, -1.54\}$, and $\{-2.59, -0.55\}$.

Example: design 3

Consider $\mathcal{J} = \lambda\gamma_2 + (1 - \lambda)\gamma_{12}$ for various weights $\lambda \in (0, 1)$. Results:



The gains and closed-loop eigenvalues are

$$K = - \begin{bmatrix} 0.7417 & 4.5167 \\ -1.5858 & 3.1716 \\ -2.4655 & 4.0738 \end{bmatrix}$$

and $\{-1.87 \pm j1.87\}$, $\{-0.71 \pm j0.71\}$, and $\{-0.27 \pm j0.27\}$.

LQR with guaranteed exponential decay

Consider now the cost

$$\mathcal{J}_\alpha = \int_0^\infty e^{2\alpha t} \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

for some $\alpha \geq 0$. To render $\mathcal{J}_\alpha$ finite,

- x and u must decay faster than $e^{-\alpha t}$.

Variables x_α and u_α such that $x_\alpha(t) = e^{\alpha t} x(t)$ and $u_\alpha(t) = e^{\alpha t} u(t)$ satisfy

$$\dot{x}_\alpha(t) = \alpha e^{\alpha t} x(t) + e^{\alpha t} \dot{x}(t) = (\alpha I + A)x_\alpha(t) + Bu_\alpha(t)$$

and the cost in terms of them reads

$$\mathcal{J}_\alpha = \int_0^\infty \begin{bmatrix} x'_\alpha(t) & u'_\alpha(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x_\alpha(t) \\ u_\alpha(t) \end{bmatrix} dt.$$

This is the original LQR, modulo the substitution $A \rightarrow \alpha I + A$ in the CARE.

LQR with guaranteed exponential decay

Consider now the cost

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This is the original LQR, modulo the substitution $A \rightarrow \alpha I + A$ in the CARE.

LQR with guaranteed exponential decay (contd)

Thus, the control law minimizing $\mathcal{J}_\alpha$ is

$$u(t) = K_\alpha x(t),$$

where $K_\alpha := -R^{-1}(S' + B'\bar{X}_\alpha)$ and $\bar{X}_\alpha = \bar{X}'_\alpha \geq 0$ is the stabilizing solution (i.e. such that $\alpha I + A + BK_\alpha$ is Hurwitz) to

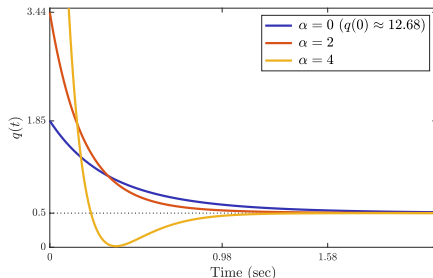
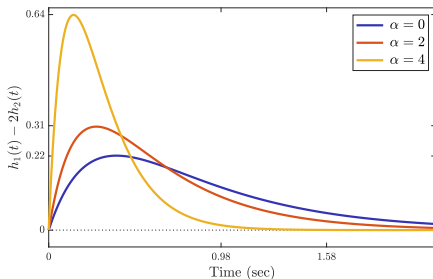
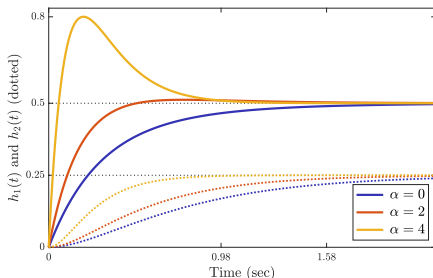
$$(\alpha I + A)'\bar{X}_\alpha + \bar{X}_\alpha(\alpha I + A) + Q - (S + \bar{X}_\alpha B)R^{-1}(S' + B'\bar{X}_\alpha) = 0.$$

Moreover, because $\lambda \in \text{spec}(\alpha I + A) \iff \lambda - \alpha \in \text{spec}(A)$,

- $\alpha I + A + BK_\alpha$ is Hurwitz iff $\text{spec}(A + BK_\alpha) \in \{s \in \mathbb{C} \mid \text{Re } s < -\alpha\}$ and we end up with the property that
- minimizing $\mathcal{J}_\alpha$ ensures that **closed-loop eigenvalues have $\text{Re } \lambda_i < -\alpha$** .

Example: design 1 with various α 's

Consider $\mathcal{J} = \gamma_1 + r\gamma_u$ for various decay guarantees $\alpha \geq 0$. Results:



The gains and closed-loop eigenvalues are

$$K = - \begin{bmatrix} 2.4595 & 0.4841 \\ 4.8730 & 2 \\ 11.4128 & 25.8879 \end{bmatrix}$$

and $\{-3.75, -1.71\}$, $\{-5.59, -2.28\}$, and $\{-8.6, -5.72\}$, to the left of $-\alpha$, indeed.

So beware, “more stable” $\not\Rightarrow$ “better”.

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Scaling the cost $\mathcal{J}$

Optimal solutions for

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \quad \text{and} \quad \gamma \begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$$

coincide. Indeed, the CARE for the latter reads

$$A'\bar{X}_\gamma + \bar{X}_\gamma A + \gamma Q - (\gamma S + \bar{X}_\gamma B)(\gamma R)^{-1}(\gamma S' + B'\bar{X}_\gamma) = 0$$

or, equivalently,

$$A'(\gamma^{-1}\bar{X}_\gamma) + (\gamma^{-1}\bar{X}_\gamma)A + Q - (S + (\gamma^{-1}\bar{X}_\gamma)B)R^{-1}(S' + B'(\gamma^{-1}\bar{X}_\gamma)) = 0$$

Hence, $\bar{X}_\gamma = \gamma\bar{X}$ and

$$K_\gamma = -(\gamma R)^{-1}(\gamma S' + B'\bar{X}_\gamma) = -R^{-1}(S' + B'\bar{X})$$

is independent of γ .

but the optimal cost is proportional to γ

CARE manipulations

Remember,

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} - \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix}.$$

Hence,

$$\begin{aligned} & \begin{bmatrix} B'(-sI - A')^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} \\ &= (I - B'(-sI - A')^{-1}K')R(I - K(sI - A)^{-1}B) \\ &\quad + B'(-sI - A')^{-1}\bar{X}(A(sI - A)^{-1} + I)B \\ &\quad + B'((-sI - A')^{-1}A' + I)\bar{X}(sI - A)^{-1}B \\ &= (I - B'(-sI - A')^{-1}K')R(I - K(sI - A)^{-1}B) \\ &\quad + sB'(-sI - A')^{-1}\bar{X}(sI - A)^{-1}B - sB'(-sI - A')^{-1}\bar{X}(sI - A)^{-1}B, \end{aligned}$$

as $A(sI - A)^{-1} + I = s(sI - A)^{-1}$ & $(-sI - A')^{-1}A' + I = -s(-sI - A')^{-1}$,
cf. Lect. 6, slide 24.

CARE manipulations

Remember,

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} - \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix}.$$

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as $A(sI - A)^{-1} + I = s(sI - A)^{-1}$ & $(-sI - A')^{-1}A' + I = -s(-sI - A')^{-1}$,
cf. Lect. 6, Slide 24.

CARE manipulations (contd)

Thus, we end up with the relation

$$\begin{aligned} \begin{bmatrix} B'(-sI - A')^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} \\ = (I - B'(-sI - A')^{-1}K')R(I - K(sI - A)^{-1}B) \end{aligned}$$

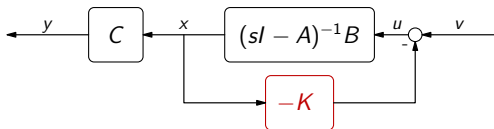
or

$$\begin{bmatrix} B'(-sI - A')^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} = (I + L'_{\text{sf}}(-s))R(I + L_{\text{sf}}(s))$$

where

$$L_{\text{sf}}(s) := -K(sI - A)^{-1}B$$

is the (strictly proper) loop transfer function in



Return difference equality for $m = 1$ and $S = 0$

Denote $r = R$ (to emphasize that it is scalar) and rewrite the relation as

$$(1 + L_{\text{sf}}(-s))(1 + L_{\text{sf}}(s)) = 1 + \frac{1}{r} B'(-sI - A')^{-1} Q(sI - A)^{-1} B,$$

If $s = j\omega$, then $L_{\text{sf}}(-j\omega) = \overline{L_{\text{sf}}(j\omega)}$ and the above reads

$$|1 + L_{\text{sf}}(j\omega)|^2 = 1 + \frac{1}{r} B'(-j\omega I - A')^{-1} Q(j\omega I - A)^{-1} B$$

known as² the **return difference equality**. Note that $Q \geq 0$ implies that

$$B'(-j\omega I - A')^{-1} Q(j\omega I - A)^{-1} B = [(j\omega I - A)^{-1} B]' Q [(j\omega I - A)^{-1} B] \geq 0$$

Hence, the return difference equality ensures that

$$|1 + L_{\text{sf}}(j\omega)|^2 \geq 1, \quad \forall \omega \in \mathbb{R}.$$

² $1 + L_{\text{sf}}(s)$ is the return-difference t.f., remember it from the Nyquist criterion proof.

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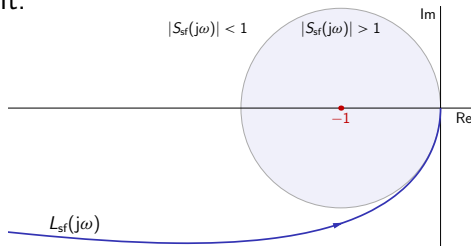
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Return difference equality: implication

The inequality

$$|1 + L_{\text{sf}}(j\omega)|^2 \geq 1 \iff |S_{\text{sf}}(j\omega)| \leq 1, \quad \forall \omega \in \mathbb{R}$$

implies that the polar plot of $L_{\text{sf}}(j\omega)$ is outside the open unit disk centered at the critical point:

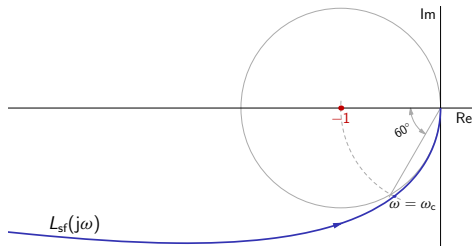


Remark There is no conflict with Bode's sensitivity integral

$$\int_0^\infty \ln |S_{\text{sf}}(j\omega)| d\omega = \pi \sum_{i=1}^m \operatorname{Re} p_i,$$

It just means that the pole excess of $L_{\text{sf}}(s)$ equals 1 (remember, $L_{\text{sf}}(s)$ is strictly proper).

Stability margins of LQR ($m = 1$ and $S = 0$)



An immediate consequence is that **LQR** optimal loop **guarantees**

- gain margin $\mu_g = \infty$ $L_{sf}(j\omega)$ does not cross the real axis in $[-1, 0)$
- phase margin $\mu_{ph} \geq 60^\circ$ $L_{sf}(j\omega)$ is further from -1 than $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

whenever the LQR parameter $S = 0$.

Closed-loop poles ($m = 1$ and $S = 0$)

Because $1 + L_{\text{sf}}(s) = \chi_{\text{cl}}(s)/\chi_{\text{ol}}(s)$,

$$(1 + L_{\text{sf}}(-s))(1 + L_{\text{sf}}(s)) = 1 + \frac{1}{r} B'(-sI - A')^{-1} Q(sI - A)^{-1} B$$

reads

$$\frac{\chi_{\text{cl}}(-s)\chi_{\text{cl}}(s)}{\chi_{\text{ol}}(-s)\chi_{\text{ol}}(s)} = 1 + \frac{1}{r} G_*(s),$$

where

$$G_*(s) := B'(-sI - A')^{-1} Q(sI - A)^{-1} B$$

is such that $G_*(-s) = G_*(s)$, so that its poles / zeros are symmetric about the imaginary axis. If $x'Qx = y^2$, then $Q = C'C$ and $G_*(s) = P(-s)P(s)$.

The solutions of

$$1 + \frac{1}{r} G_*(s) = 0 \iff -r = G_*(s),$$

which is a root-locus form under $k = 1/r$, determine then both the roots of $\chi_{\text{cl}}(s)$ and those of $\chi_{\text{cl}}(-s)$.

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which is a **root-locus form** under $k = 1/r$, determine then both the roots of $\chi_{\text{cl}}(s)$ (closed-loop LQR poles) and those of $\chi_{\text{cl}}(-s)$ (their reflection about the $j\omega$ -axis).

Root locus with respect to $k = 1/r$

- has $2n$ loci symmetric about both real and imaginary axes
- starts for $r = \infty$ at roots of $\chi_{ol}(-s)\chi_{ol}(s)$
- ends as $r \downarrow 0$ at zeros of $G_*(s)$ or ∞ (depending on the cost)
- has an even number of asymptotes, whose centroid is at $\sigma_c = 0$
- if $G_*(s)$ is strictly proper (even pole excess), then asymptotes angles

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8
2	0	π						
4	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$				
6	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$		
8	$\frac{\pi}{8}$	$\frac{3\pi}{8}$	$\frac{5\pi}{8}$	$\frac{7\pi}{8}$	$\frac{9\pi}{8}$	$\frac{11\pi}{8}$	$\frac{13\pi}{8}$	$\frac{15\pi}{8}$

half of them are in the LHP (and none on the imaginary axis).

- there are exactly n roots in the open LHP at each $r \in (0, \infty)$ (because $G_*(j\omega) \geq 0$, $-x = G_*(j\omega)$ cannot be solved in $r > 0$)

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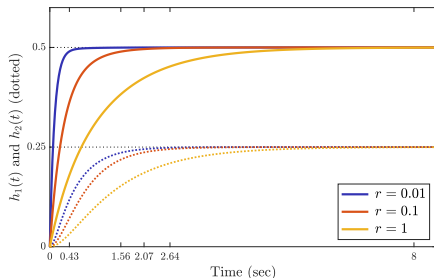
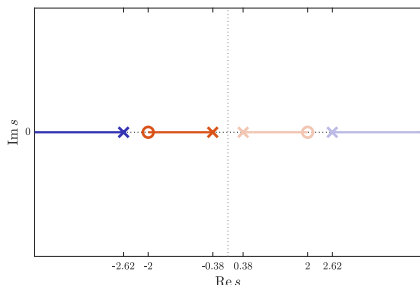
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Example: design 1

LQR for $\mathcal{J} = \gamma_1 + r\gamma_u$ has $S = 0$ and

$$G_*(s) = -\frac{(s+2)(s-2)}{(s+2.618)(s+0.382)(s-0.382)(s-2.618)} = P_1(-s)P_1(s)$$

The root locus with respect to $1/r$:



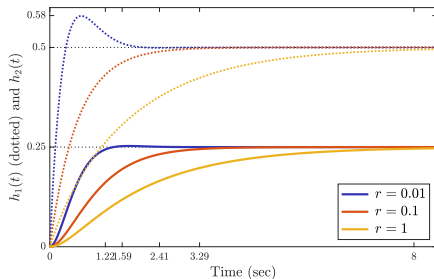
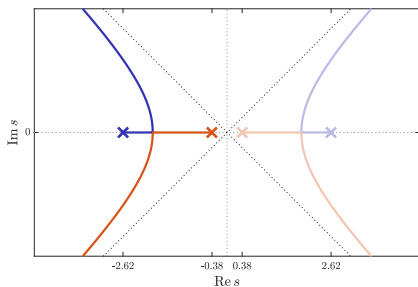
Closed-loop poles were at $\{-10.15, -1.97\}$, $\{-3.75, -1.71\}$, and $\{-2.7, -0.83\}$.

Example: design 2

LQR for $\mathcal{J} = \gamma_2 + r\gamma_u$ has $S = 0$ and

$$G_*(s) = \frac{1}{(s + 2.618)(s + 0.382)(s - 0.382)(s - 2.618)} = P_2(-s)P_2(s)$$

The root locus with respect to $1/r$:



Closed-loop poles were at $\{-2.6 \pm j1.81\}$, $\{-2.15, -1.54\}$, and $\{-2.59, -0.55\}$.

Limiting behavior: cheap control, $r \rightarrow 0$

The LQR problem with the cost

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \quad r \rightarrow 0$$

is known as the **cheap control** LQR problem. By root-locus arguments, the roots of $\chi_{cl}(s)$ for $1/r \rightarrow \infty$

- either approach the stable zeros of $G_*(s)$
- or go to $\operatorname{Re} s = -\infty$ along the LHP asymptotes
(if the pole excess of $G_*(s)$ is larger than 2, then the damping ratio of the “worst” asymptote is at most $1/\sqrt{2}$)

Limiting behavior: expensive control, $r \rightarrow \infty$

We already know that the solution to the LQR for

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

and

$$\mathcal{J}_r = \int_0^\infty \begin{bmatrix} x'(t) & u(t) \end{bmatrix} \begin{bmatrix} Q/r & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt = \frac{1}{r} \mathcal{J}$$

coincide. Thus, the LQR problem for the limiting case of $r \rightarrow \infty$ effectively corresponds to

$$\mathcal{J}_\infty = \int_0^\infty u^2(t) dt \quad \text{or} \quad \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is the problem of

- stabilizing with the lowest control energy.

Limiting behavior: expensive control, $r \rightarrow \infty$ (contd)

Assumptions:

$\mathcal{A}_1$: (A, B) is stabilizable

$\mathcal{A}_2$: $R > 0$ and $Q - SR^{-1}S' \geq 0$

for $Q = 0$, $S = 0$, and $R = 1$ obviously holds

$\mathcal{A}_3$: $\begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank for all ω

for $Q = 0$, $S = 0$, and $R = 1$ is equivalent to $\text{spec}(A) \cap j\mathbb{R} = \emptyset$

By root-locus arguments, the roots of $\chi_d(s)$ for $1/r = 0$ are the stable roots of $\chi_{ol}(-s)\chi_{ol}(s)$. In other words,

if λ is an OLHP root of $\chi_{ol}(s)$, then it is also a root of $\chi_d(s)$,

if λ is an ORHP root of $\chi_{ol}(s)$, then $-\lambda$ is a root of $\chi_d(s)$.

Thus, the optimal controller for this problem

- reflects unstable poles about the $j\omega$ -axis and
- keeps stable poles untouched.

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By root-locus arguments, the roots of $\chi_{\text{cl}}(s)$ for $1/r = 0$ are the stable roots of $\chi_{\text{ol}}(-s)\chi_{\text{ol}}(s)$. In other words,

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Limiting behavior: expensive control, $r \rightarrow \infty$ (contd)

If $\mathcal{A}_3$ does not hold, i.e. if for some ω

$$\text{rank} \begin{bmatrix} A - j\omega I & B \\ Q & S \\ S' & R \end{bmatrix} = \text{rank} \begin{bmatrix} A - j\omega I & B \\ 0 & 0 \\ 0 & r \end{bmatrix} = \text{rank}(A - j\omega I) + 1 < n + 1,$$

then

- no optimal stabilizing solution exists.

For example, let $\dot{x}(t) = u(t)$ and $u(t) = kx(t)$ for $k < 0$. In this case

$$\dot{x}(t) = kx(t) \implies x(t) = e^{kt}x_0 \implies u(t) = ke^{kt}x_0$$

and

$$\mathcal{J}_\infty = \int_0^\infty u^2(t) dt = \int_0^\infty k^2 e^{2kt} x_0^2 dt = \frac{|k|}{2} x_0^2.$$

It can be made arbitrarily small, but the optimal $k = 0$ is not stabilizing.

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When is state feedback optimal?

Theorem

If $m = 1$, (A, B) is controllable, and $K \in \mathbb{R}^{1 \times n}$ satisfies

1. $A + BK$ is stable,
2. $|1 - K(j\omega I - A)^{-1}B| \geq 1$ for all $\omega \in \mathbb{R}$,

then there is $0 \leq Q = Q' \in \mathbb{R}^{n \times n}$ such that this K is LQR optimal for

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

If (A, B) is not controllable, then there is Q for which $K(sI - A)^{-1}B$ is the optimal loop.

This result shows that LQR optimization is

→ sufficiently rich, so that we won't miss any "good" controller by using Q and r as design parameter under $S = 0$.

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This result shows that LQR optimization is

- sufficiently rich, so that we won't miss any “good” controller by using Q and r as design parameter under $S = 0$.

When is state feedback optimal? (contd)

The condition $S = 0$ is essential. If S can also be chosen freely, then

- all stabilizing state-feedback gains are LQR-optimal, regardless the second condition. For example, the cost

$$\mathcal{J} = \int_0^\infty (u(t) - K_\star x(t))^2 dt$$

which corresponds to

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K'_\star \\ I \end{bmatrix} \begin{bmatrix} -K_\star & I \end{bmatrix} = \begin{bmatrix} K'_\star K_\star & -K'_\star \\ -K_\star & I \end{bmatrix},$$

produces the optimal $u(t) = K_\star x(t)$.

in this case $\bar{X} = 0$ and $\mathcal{J}_{\text{opt}} = 0$