

Control Theory (00350188)

lecture no. 7

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Outline

State feedback

Controllability

Example: 2-mass system

State feedback: pole placement

State feedback and reference response

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State feedback: pole placement

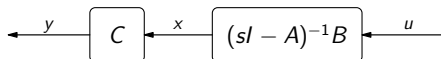
State feedback and reference response

Setup

Let

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) \end{cases}$$

Equivalently,



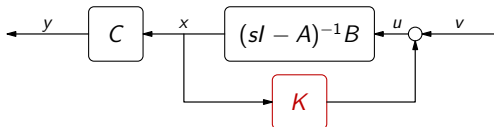
taking into account that if $x_0 = 0$, then

$$Y(s) = CX(s) \quad \text{and} \quad X(s) = (sI - A)^{-1}BU(s).$$

In what follows, we assume that

- the whole **state x is measurable**.

State feedback



The control law

$$u = Kx + v$$

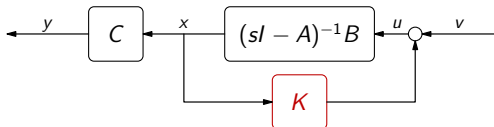
for a gain¹ $K \in \mathbb{R}^{1 \times n}$ and exogenous signal v is dubbed the **state feedback**.

Frequently, we study the case of

$v = 0$ and nonzero initial conditions $x(0) = x_0$, (regular problem) which plays a role of the **test input**, much like the step reference does in the classical unity-feedback setup.

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State feedback



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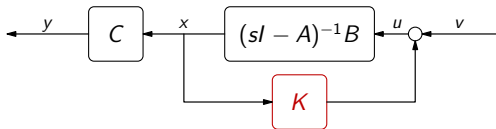
Frequently, we study the case of

- $v = 0$ and nonzero initial conditions $x(0) = x_0$, (regulator problem)

which plays a role of the **test input**, much like the step reference does in the classical unity-feedback setup.

¹This K is for Kalman.

State feedback: closed-loop dynamics



If $u = Kx + v$, then

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x}(t) = \overbrace{(A + BK)}^{:= A_K} x(t) + Bv(t) \\ y(t) = Cx(t) \end{cases}$$

so that

$$\underbrace{T_{xv}(s) = (sI - A_K)^{-1}B}$$

$$\Downarrow$$

$$\overbrace{T_{yv}(s) = C(sI - A_K)^{-1}B \quad \text{and} \quad T_{uv}(s) = 1 + K(sI - A_K)^{-1}B}$$

(because $y = Cx$ and $u = v + Kx$).

State feedback and system zeros

Let

$$P(s) = C(sI - A)^{-1}B = \frac{N_P(s)}{\det(sI - A)}.$$

Since (mind that $A_K = A + BK$)

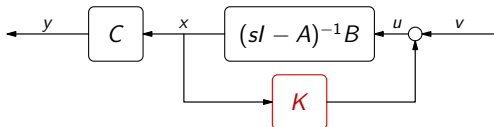
$$T_{uv}(s) = 1 + K(sI - A_K)^{-1}B \iff \frac{1}{T_{uv}(s)} = 1 - K(sI - A)^{-1}B,$$

we have that

$$\begin{aligned} T_{uv}(s) &= 1 + K(sI - A_K)^{-1}B = \frac{\text{some } n\text{-order monic polynomial}}{\det(sI - A_K)} \\ &= \frac{1}{1 - K(sI - A)^{-1}B} = \frac{\det(sI - A)}{\text{some } n\text{-order monic polynomial}} \\ &= \frac{\det(sI - A)}{\det(sI - A_K)} \end{aligned}$$

(mind that $T_{uv}(\infty) = 1$ and $\det(sI - M)$ is always monic).

State feedback and system zeros (contd)



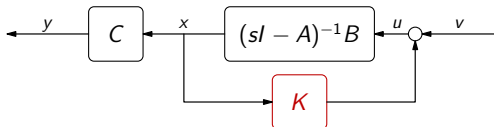
Thus,

$$T_{yv}(s) = P(s)T_{uv}(s) = \frac{N_P(s)}{\det(sI - A)} \frac{\det(sI - A)}{\det(sI - A_K)} = \frac{N_P(s)}{\det(sI - A_K)}.$$

Hence,

– state feedback does not move zeros
(unless canceled by roots of $\det(sI - A_K)$, of course).

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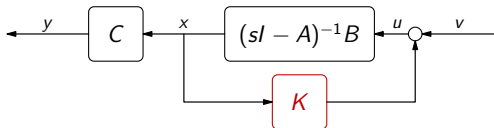
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State feedback and closed-loop poles



With

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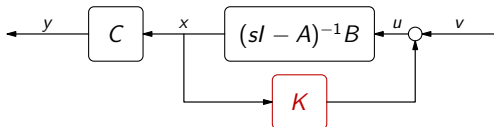
closed-loop poles are

- among the roots of $\det(sI - A_K)$,

which are the eigenvalues of $A_K = A + BK$.

Q: what freedom we have in assigning $\text{spec}(A + BK)$ by the choice of K ?

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Special case: companion form

Assume that

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

If $K = [k_0 \ k_1 \ \cdots \ k_{n-1}]$, then

$$A_K = A + BK = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -(a_0 - k_0) & -(a_1 - k_1) & \cdots & -(a_{n-1} - k_{n-1}) \end{bmatrix}$$

is still a companion matrix and its characteristic polynomial

$$\chi_{A_K}(\lambda) = \lambda^n + (a_{n-1} - k_{n-1})\lambda^{n-1} + \cdots + (a_1 - k_1)\lambda + (a_0 - k_0).$$

Special case: companion form (contd)

Therefore, any desired closed-loop characteristic polynomial, say

$$\chi_{\text{cl}}(s) = s^n + \chi_{n-1}s^{n-1} + \cdots + \chi_1s + \chi_0$$

for some coefficients $\chi_i > 0$, can be assigned by

$$K = \begin{bmatrix} a_0 - \chi_0 & a_1 - \chi_1 & \cdots & a_{n-1} - \chi_{n-1} \end{bmatrix},$$

for which

$$A_K = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\chi_0 & -\chi_1 & \cdots & -\chi_{n-1} \end{bmatrix}$$

Q: under what condition this can be said about an arbitrary realization?

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Controllability: definition and criterion

Consider

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

This system (or the pair (A, B)) is said to be

- **controllable** if for every x_0 and x_1 , $\exists t_1$ and $u(t) : [0, t_1] \rightarrow \mathbb{R}$ such that $x(t_1) = x_1$.

Matrix

$$M_c := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times n}$$

called the **controllability matrix**.

Theorem

(A, B) is controllable if and only if $\det M_c \neq 0$.

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Preliminary simplifying observation

Without loss of generality we can take $x_0 = 0$. Indeed,

$$x(t_1) = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds$$

implies

$$\tilde{x}(t_1) := x(t_1) - e^{At_1}x_0 = \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds.$$

Thus, the very same u transferring x from $x(0) = x_0$ to $x(t_1) = x_1$ transfers it from $x(0) = 0$ to $x(t_1) = \tilde{x}(t_1)$. Hence, if every $\tilde{x}(t_1)$ is reachable from $x(0) = 0$, then every $x_1 = \tilde{x}(t_1) + e^{At_1}x_0$ is reachable from $x(0) = x_0$.

Remark Alternatively, we may take $x_1 = 0$, based on the relation

$$0 = e^{At_1}(x_0 - e^{-At_1}x(t_1)) + \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds,$$

which is possible because e^{At_1} is always invertible.

Outline of the proof

Remember, $e^{At} = \sum_{i=0}^{n-1} g_i(t) A^i$. Taking $x(0) = 0$,

$$\begin{aligned} x(t_1) &= \int_0^{t_1} e^{A(t_1-s)} B u(s) ds = \int_0^{t_1} \left(\sum_{i=0}^{n-1} A^i g_i(t_1-s) \right) B u(s) ds \\ &= \sum_{i=0}^{n-1} A^i B \int_0^{t_1} g_i(t_1-s) u(s) ds = \sum_{i=0}^{n-1} A^i B \gamma_i \\ &= M_c \gamma, \end{aligned}$$

where $\gamma_i := \int_0^{t_1} g_i(t_1-s) u(s) ds$. We have:

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where $\gamma_i := \int_0^{t_1} g_i(t_1-s) u(s) ds$. We have:

1. If $\det M_c = 0$, then $\exists x_1$ such that $x_1 = M_c \gamma$ is not solvable in γ , hence this x_1 is not reachable by u .

2. If $\det M_c \neq 0$, then every x_1 is reachable with $\gamma = M_c^{-1} x_1$. Moreover, it can be shown that n equations $\gamma_i = \int_0^{t_1} g_i(t_1-s) u(s) ds$ are always solvable in u . □

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Controllability and similarity transformations

If $\tilde{A} = TAT^{-1}$ and $\tilde{B} = TB$, then (remember, $(TAT^{-1})^k = TA^kT^{-1}$)

$$\begin{aligned}\tilde{M}_c &= [\tilde{B} \quad \tilde{A}\tilde{B} \quad \dots \quad \tilde{A}^{n-1}\tilde{B}] \\ &= [TB \quad TAT^{-1}TB \quad \dots \quad TA^{n-1}T^{-1}TB] \\ &= T[B \quad AB \quad \dots \quad A^{n-1}B] \\ &= TM_c,\end{aligned}$$

i.e.

- controllability is not affected by similarity transformations.

Given two similar controllable realizations, the transformation matrix

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Controllability: some other tests

Theorem

The following statements are equivalent:

1. (A, B) is controllable;
2. $\det M_c \neq 0$;
3. $\det W_c(t) \neq 0$ for all $t > 0$, where $W_c(t) := \int_0^t e^{As} B B' e^{A's} ds \in \mathbb{R}^{n \times n}$;
4. $\begin{bmatrix} A - \lambda I & B \end{bmatrix} \in \mathbb{C}^{n \times n+1}$ has full row rank $\forall \lambda \in \mathbb{C}$ (**PBH test**).

Matrix $W_c(t)$ and control law for $0 = x_0 \rightarrow x_1$

If

$$u(t) = u_{\min}(t) := B'e^{A'(t_1-t)}[W_c(t_1)]^{-1}x_1,$$

then

$$\begin{aligned}x(t_1) &= \int_0^{t_1} e^{A(t_1-s)} B u(s) ds = \int_0^{t_1} e^{A(t_1-s)} B B' e^{A'(t_1-s)} [W_c(t_1)]^{-1} x_1 ds \\&= \int_0^{t_1} e^{As} B B' e^{A's} ds [W_c(t_1)]^{-1} x_1 = x_1.\end{aligned}$$

In fact, $u_{\min}$ has

minimal energy, $E_u := \int_0^{t_1} u^2(t) dt$,

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Minimum energy proof

If $u = u_{\min} + u_{\delta}$, then, by linearity,

$$x(t_1) = x_1 + \int_0^{t_1} e^{A(t_1-s)} B u_{\delta}(s) ds.$$

Hence, $x(t_1) = x_1$ iff u_{δ} satisfies

$$\int_0^{t_1} e^{A(t_1-s)} B u_{\delta}(s) ds = 0.$$

Now,

$$\begin{aligned} E_u &= \int_0^{t_1} (u_{\min}(t) + u_{\delta}(t))^2 dt \\ &= \int_0^{t_1} u_{\min}^2(t) dt + \int_0^{t_1} u_{\delta}^2(t) dt + 2 \int_0^{t_1} u_{\min}(t) u_{\delta}(t) dt \\ &= E_{u_{\min}} + E_{u_{\delta}} + 2x_1' [W_c(t_1)]^{-1} \int_0^{t_1} e^{A(t_1-t)} B u_{\delta}(t) dt = E_{u_{\min}} + E_{u_{\delta}} \end{aligned}$$

Because $E_{u_{\delta}} \geq 0$, the minimum is attained by $u_{\delta} = 0$. □

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$$= E_{u_{\min}} + E_{u_{\delta}} + 2x_1^T [W_c(t_1)]^{-1} \int_0^{t_1} e^{A(t_1-s)} B u_{\delta}(s) ds = E_{u_{\min}} + E_{u_{\delta}}$$

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Because $E_u \geq 0$, the minimum is attained by $u_{\delta} = 0$.



Minimum energy: example

If $P(s) = 1/(s^2 + s)$, then its physical realization (having $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$) is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

and has

$$M_c = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad W_c(t) = \frac{1}{2} \begin{bmatrix} 4e^{-t} - e^{-2t} + 2t - 3 & (1 - e^{-t})^2 \\ (1 - e^{-t})^2 & 1 - e^{-2t} \end{bmatrix}$$

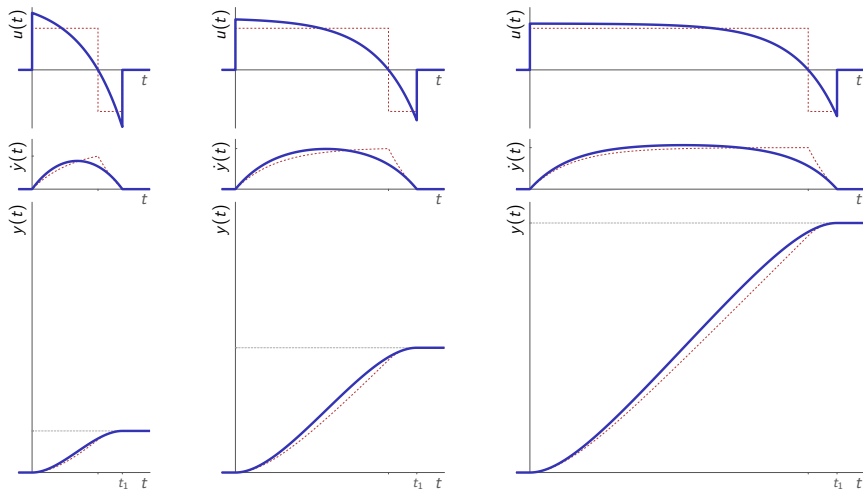
with

$$\det M_c = -1 \quad \text{and} \quad \det W_c(t) = (1 - e^{-t})(e^{-t}(t + 2) + t - 2)/2 > 0$$

(controllable). If $x_1 = \begin{bmatrix} y_f \\ 0 \end{bmatrix}$, then

$$u_{\min}(t) = \frac{1 + e^{-t_1} - 2e^{t-t_1}}{e^{-t_1}(t_1 + 2) + t_1 - 2} (\mathbb{1}(t) - \mathbb{1}(t - t_1)) y_f.$$

Minimum energy: example (contd)



$t_1 = t_f$ of minimum-time bang-bang solutions from Lecture 2 (dashed red lines).

Uncontrollable modes and stabilizability

If (A, B) not controllable, PBH test fails for some $\lambda_i \in \mathbb{C}$ such that

$$\text{rank} \begin{bmatrix} A - \lambda_i I & B \end{bmatrix} < n.$$

These λ_i are eigenvalues of A . Indeed, if PBH fails, $\exists \tilde{\eta}_i \neq 0$ such that

$$\tilde{\eta}_i' \begin{bmatrix} A - \lambda_i I & B \end{bmatrix} = \begin{bmatrix} \tilde{\eta}_i'(A - \lambda_i I) & \tilde{\eta}_i' B \end{bmatrix} = 0 \implies \begin{cases} \tilde{\eta}_i' A = \lambda_i \tilde{\eta}_i' \\ \tilde{\eta}_i' B = 0 \end{cases}$$

i.e. this $\tilde{\eta}_i$ is a left eigenvector of A . λ_i 's at which PBH fails called
— uncontrollable modes of (A, B) .

Pair (A, B) is said to be

— stabilizable if all its uncontrollable modes are stable
(i.e. in the open LHP).

Uncontrollable modes and stabilizability

If (A, B) not controllable, PBH test fails for some $\lambda_i \in \mathbb{C}$ such that

$$\text{rank} \begin{bmatrix} A - \lambda_i I & B \end{bmatrix} < n.$$

These λ_i are eigenvalues of A . Indeed, if PBH fails, $\exists \tilde{\eta}_i \neq 0$ such that

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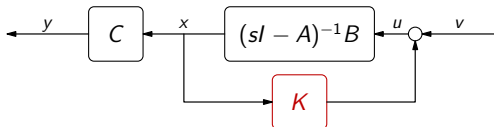
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Uncontrollable modes and state feedback



with

$$T_{yv}(s) = \frac{N_P(s)}{\det(sI - A_K)} \quad \text{and} \quad T_{uv}(s) = \frac{\det(sI - A)}{\det(sI - A_K)}.$$

If $\lambda \in \text{spec}(A)$ is an uncontrollable mode of (A, B) , then

$$\tilde{\eta}'_i A_K = \tilde{\eta}'_i (A + BK) = \tilde{\eta}'_i A = \lambda \tilde{\eta}'_i \implies \tilde{\eta}'_i (A + BK - \lambda I) = 0.$$

In other words,

- uncontrollable modes are eigenvalues of A_K for every K .

Hence,

- state feedback cannot affect uncontrollable modes.

Outline

State feedback

Controllability

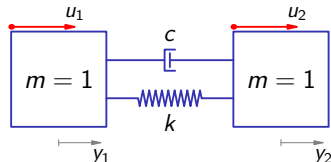
Example: 2-mass system

State feedback: pole placement

State feedback and reference response

Setup

Consider the following 2-mass system:



with external forces u_1 and u_2 . It is described by the following equations:

$$\begin{bmatrix} \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

State-space model

Possible realization (non-unique):

$$\begin{cases} \underbrace{\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix}}_A \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}}_y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_C \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} \end{cases}$$

System has 4 modes (eigenvalues of A):

- $\lambda_1 = \lambda_2 = 0$ rigid body motion
- $\lambda_{3,4} = -c \pm \sqrt{c^2 - 2k}$ spring-damper dynamics

Controllability

If $u_1 = \alpha_1 u$ and $u_2 = \alpha_2 u$ for a force u and constants α_1 and α_2 , then

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & -c & c \\ k & -k & c & -c \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} u(t)$$

Controllability matrix:

$$M_c = \begin{bmatrix} 0 & \alpha_1 & -c(\alpha_1 - \alpha_2) & -(2c^2 - k)(\alpha_1 - \alpha_2) \\ 0 & \alpha_2 & c(\alpha_1 - \alpha_2) & -(2c^2 - k)(\alpha_1 - \alpha_2) \\ \alpha_1 & -c(\alpha_1 - \alpha_2) & (2c^2 - k)(\alpha_1 - \alpha_2) & -4c(c^2 - k)(\alpha_1 - \alpha_2) \\ \alpha_2 & c(\alpha_1 - \alpha_2) & -(2c^2 - k)(\alpha_1 - \alpha_2) & 4c(c^2 - k)(\alpha_1 - \alpha_2) \end{bmatrix}$$

with $\det M_c = -k^2(\alpha_1^2 - \alpha_2^2)^2$. Thus, the system is

- uncontrollable for $\alpha_2 = \pm\alpha_1$.

What could it mean?

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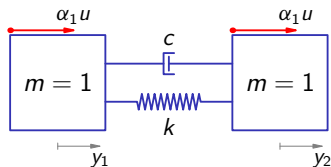
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Example 1: controllability with $\alpha_2 = \alpha_1$ ($u_1 = u_2$)



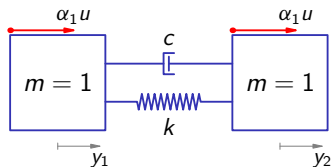
PBH test:

$$\text{rank} \begin{bmatrix} -\lambda & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 1 & 0 \\ -k & k & -c - \lambda & c & \alpha_1 \\ k & -k & c & -c - \lambda & \alpha_1 \end{bmatrix} \bigg|_{\lambda = -c \pm \sqrt{c^2 - 2k}} = 3,$$

(rank lost at **uncontrollable modes** of A). This agrees with our intuition that

— if equal forces applied to each mass, oscillations not excited.

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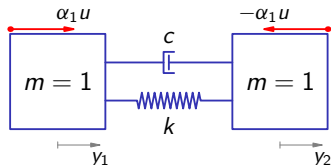
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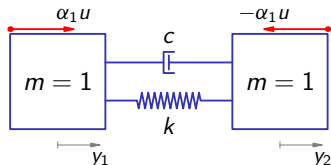
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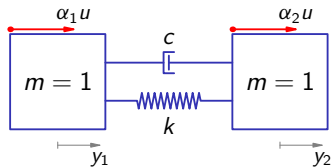
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Transfer functions for $u_1 = \alpha_1 u$ and $u_2 = \alpha_2 u$



Transfer function from u to y_1 :

$$P_1(s) = \frac{\alpha_1 s^2 + c(\alpha_1 + \alpha_2)s + k(\alpha_1 + \alpha_2)}{s^2(s^2 + 2cs + 2k)}$$

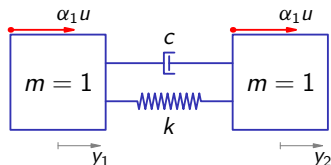
and transfer function from u to y_2 :

$$P_2(s) = \frac{\alpha_2 s^2 + c(\alpha_1 + \alpha_2)s + k(\alpha_1 + \alpha_2)}{s^2(s^2 + 2cs + 2k)}$$

(both obtained via $C(sI - A)^{-1}B$).

Transfer functions for $u_1 = \alpha_1 u$ and $u_2 = \alpha_2 u$ (contd)

$$\alpha_2 = \alpha_1:$$



then

$$P_1(s) = P_2(s) = \frac{\alpha_1}{s^2}.$$

$$\alpha_2 = -\alpha_1:$$

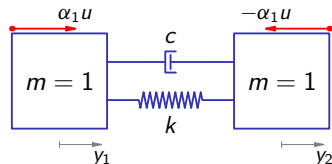
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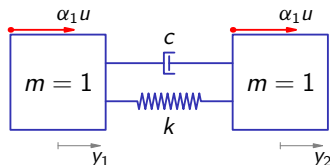
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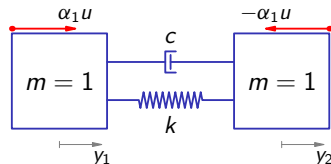
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In both cases we have **pole/zero cancellations** (of different modes though).

Outline

State feedback

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State feedback and reference response

Preliminary observations

We know by now that

- if (A, B) uncontrollable, then $\text{spec}(A_K)$ cannot be assigned arbitrarily;
- if (A, B) is in the companion form, i.e.

$$(A, B) = (A_{\text{cf}}, B_{\text{cf}}) := \left(\begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

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then $\text{spec}(A_K)$ can be assigned arbitrarily, with $K_i = a_i - \chi_i$.

Remark: It can be shown that

$$M_{\text{c,cf}} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}^{-1} \implies \det M_{\text{c,cf}} = (-1)^{\lfloor n/2 \rfloor}.$$

Pole placement for arbitrary realizations: logic

Given controllable (A, B) , all we need to do is to

- transform $(A, B) \rightarrow (A_{\text{cf}}, B_{\text{cf}}) = (TAT^{-1}, TB)$
- design K_{cf} such that $\text{spec}(A_{\text{cf}} + B_{\text{cf}}K_{\text{cf}})$ matches the roots of a required $\chi_{\text{cl}}(s)$
- transform $K_{\text{cf}} \rightarrow K = K_{\text{cf}}T$ for the original realization for this choice

$$A + BK = T^{-1}A_{\text{cf}}T + T^{-1}B_{\text{cf}}K_{\text{cf}}T = T^{-1}(A_{\text{cf}} + B_{\text{cf}}K_{\text{cf}})T$$

has the same spectrum as $A_{\text{cf}} + B_{\text{cf}}K_{\text{cf}}$

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Elegant algorithm to construct required $K_{\text{cf}}T$ w/o explicit calculation of T is offered by **Ackermann's formula**.

Ackermann's formula: preliminaries

1. by the Cayley–Hamilton theorem, $\chi_{A_{\text{cf}}}(A_{\text{cf}}) = 0$, so that

$$\begin{aligned}
 \chi_{\text{cl}}(A_{\text{cf}}) &= A_{\text{cf}}^n + \chi_{n-1}A_{\text{cf}}^{n-1} + \cdots + \chi_1A_{\text{cf}} + \chi_0I \\
 &= -a_{n-1}A_{\text{cf}}^{n-1} - \cdots - a_1A_{\text{cf}} - a_0I \\
 &\quad + \chi_{n-1}A_{\text{cf}}^{n-1} + \cdots + \chi_1A_{\text{cf}} + \chi_0I \\
 &= (\chi_{n-1} - a_{n-1})A_{\text{cf}}^{n-1} + \cdots + (\chi_1 - a_1)A_{\text{cf}} + (\chi_0 - a_0)I.
 \end{aligned}$$

2. if e_j is the j th standard basis in $\mathbb{R}^n$, then $\forall j = 1, \dots, n-1$,

$$e_j' A_{\text{cf}} = e_{j+1}' \quad \text{or, equivalently,} \quad e_1' A_{\text{cf}} = e_2'.$$

3. if $A_{\text{cf}} = TAT^{-1}$ and $B_{\text{cf}} = TB$, then

$$M_{\text{cdf}} = TM_c \iff T = M_{\text{cdf}}M_c^{-1}.$$

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2. if e_i is the i th standard basis in $\mathbb{R}^n$, then $\forall i = 1, \dots, n-1$,

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$$M_{\text{c,cf}} = TM_{\text{c}} \iff T = M_{\text{c,cf}}M_{\text{c}}^{-1}.$$

Ackermann's formula: preliminaries (contd)

4. Combining 1 and 2:

$$\begin{aligned}
 K_{\text{cf}} &= \begin{bmatrix} a_0 - \chi_0 & a_1 - \chi_1 & \cdots & a_{n-1} - \chi_{n-1} \end{bmatrix} \\
 &= (a_0 - \chi_0)e'_1 + (a_1 - \chi_1)e'_2 + \cdots + (a_{n-1} - \chi_{n-1})e'_n \\
 &= (a_0 - \chi_0)e'_1 + (a_1 - \chi_1)e'_1 A_{\text{cl}} + \cdots + (a_{n-1} - \chi_{n-1})e'_1 A_{\text{cl}}^{n-1} \\
 &= e'_1 ((a_0 - \chi_0)I + (a_1 - \chi_1)A_{\text{cl}} + \cdots + (a_{n-1} - \chi_{n-1})A_{\text{cl}}^{n-1}) \\
 &= -e'_1 \chi_{\text{cl}}(A_{\text{cf}})
 \end{aligned}$$

5. By 2 and the fact that $B_{\text{cf}} = e_n$:

$$\begin{aligned}
 e'_1 M_{\text{cf}} &= e'_1 \begin{bmatrix} B_{\text{cf}} & A_{\text{cf}} B_{\text{cf}} & \cdots & A_{\text{cf}}^{n-1} B_{\text{cf}} \end{bmatrix} \\
 &= \begin{bmatrix} e'_1 B_{\text{cf}} & e'_2 B_{\text{cf}} & \cdots & e'_n B_{\text{cf}} \end{bmatrix} \\
 &= \begin{bmatrix} e'_1 e_n & e'_2 e_n & \cdots & e'_n e_n \end{bmatrix} \\
 &= e'_n
 \end{aligned}$$

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Ackermann's formula: derivation

We have:

$$K_{\text{cf}} = -e'_1 \chi_{\text{cl}}(A_{\text{cf}}) \quad \text{by item 4}$$

$$= -e'_1 \chi_{\text{cl}}(TAT^{-1}) = -e'_1 T \chi_{\text{cl}}(A) T^{-1}$$

$$= -e'_1 M_{\text{c,cf}} M_{\text{c}}^{-1} \chi_{\text{cl}}(A) T^{-1} \quad \text{by item 3}$$

$$= -e'_n M_{\text{c}}^{-1} \chi_{\text{cl}}(A) T^{-1} \quad \text{by item 5}$$

Now it is time to return to the original coordinates:

$$\begin{aligned} K &= K_{\text{cf}} T = -e'_n M_{\text{c}}^{-1} \chi_{\text{cl}}(A) T^{-1} T \\ &= -e'_n M_{\text{c}}^{-1} \chi_{\text{cl}}(A), \end{aligned}$$

voilà!

Ackermann's formula

The feedback gain assigning the closed-loop poles to the roots of $\chi_{\text{cl}}(s)$ is

$$K = - \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_c^{-1} \chi_{\text{cl}}(A),$$

where

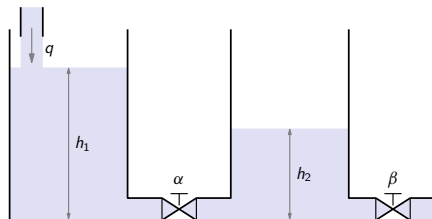
$$M_c = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

and

$$\chi_{\text{cl}}(A) = A^n + \chi_{n-1}A^{n-1} + \cdots + \chi_1A + \chi_0I.$$

This gain K is called **Ackermann's formula** and indeed depends only on the original (controllable) realization.

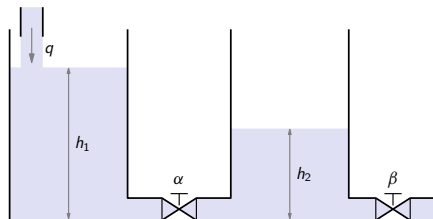
Example: two-tank system



Here:

- q is the control flow,
- h_1 and h_2 are fluid heights,
- α is the resistance to the valve between the tanks,
- β is the resistances of the output valve,
- crossing areas of each tank is σ .

Example: two-tank system (contd)



Its nonlinear dynamics (provided $h_1 > h_2$)

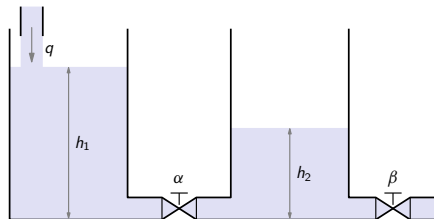
$$\sigma \begin{bmatrix} \dot{h}_1(t) \\ \dot{h}_2(t) \end{bmatrix} = \begin{bmatrix} -\alpha \sqrt{h_1(t) - h_2(t)} + q \\ \alpha \sqrt{h_1(t) - h_2(t)} - \beta \sqrt{h_2(t)} \end{bmatrix}$$

The equilibria are

$$h_{2,\text{eq}} = \frac{\alpha^2}{\alpha^2 + \beta^2} h_{1,\text{eq}} < h_{1,\text{eq}} \quad \text{and} \quad q_{\text{eq}} = \sqrt{\frac{\alpha^2 \beta^2}{\alpha^2 + \beta^2} h_{1,\text{eq}}}$$

for any $h_{1,\text{eq}} > 0$.

Example: two-tank system (contd)



With $x_i = h_i - h_{i,\text{eq}}$ and $u = q - q_{\text{eq}}$, the linearized dynamics are

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \frac{\alpha \sqrt{1 + (\alpha/\beta)^2}}{2\sigma \sqrt{h_{1,\text{eq}}}} \begin{bmatrix} -1 & 1 \\ 1 & -1 - (\beta/\alpha)^2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \frac{1}{\sigma} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

with negative real poles at

$$s_{1,2} = -\frac{\alpha \sqrt{1 + (\alpha/\beta)^2}}{2\sigma \sqrt{h_{1,\text{eq}}}} \frac{2\alpha^2 + \beta^2 \pm \sqrt{4\alpha^4 + \beta^4}}{2\alpha^2}.$$

Example: two-tank system (contd)

Let $\sigma = 1$, $\alpha = \beta = 1$ and choose

$$h_{1,\text{eq}} = 1/2 \quad \implies \quad h_{2,\text{eq}} = 1/4 \quad \text{and} \quad q_{\text{eq}} = 1/2.$$

If $h_1(0) = h_2(0) = 0$, then $x_1(0) = -1/2$ and $x_2(0) = -1/4$ and we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = - \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}.$$

with modes at $s_1 = -2.618$ and $s_2 = -0.382$ and

$$P_1(s) = \frac{s+2}{(s+2.618)(s+0.382)} \quad \text{and} \quad P_2(s) = \frac{1}{(s+2.618)(s+0.382)}$$

as the transfer functions of the systems $u \mapsto x_1$ and $u \mapsto x_2$, respectively.

Our goal is to

regulate x from $x(0)$ to $\lim_{t \rightarrow \infty} x(t) = 0$ in a desired manner

(regulator problem), which is effectively the set-point tracking of h_{eq} .

Example: two-tank system (contd)

Let $\sigma = 1$, $\alpha = \beta = 1$ and choose

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Our **goal** is to

- regulate x from $x(0)$ to $\lim_{t \rightarrow \infty} x(t) = 0$ in a desired matter (regulator problem), which is effectively the set-point tracking of h_{eq} .

Example: state feedback

Controllability matrix

$$M_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \implies M_c^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let's choose

$$\chi_{cl}(s) = s^2 + 2\zeta\omega_n s + \omega_n^2.$$

Ackermann's formula $K = - \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} M_c^{-1} \chi_{cl}(A)$ reads then

$$\begin{aligned} K &= - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}^2 + 2\zeta\omega_n \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} + \omega_n^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 - 2\zeta\omega_n + \omega_n^2 & -3 + 2\zeta\omega_n \\ -3 + 2\zeta\omega_n & 5 - 4\zeta\omega_n + \omega_n^2 \end{bmatrix} \\ &= - \begin{bmatrix} -3 + 2\zeta\omega_n & 5 - 4\zeta\omega_n + \omega_n^2 \end{bmatrix} \end{aligned}$$

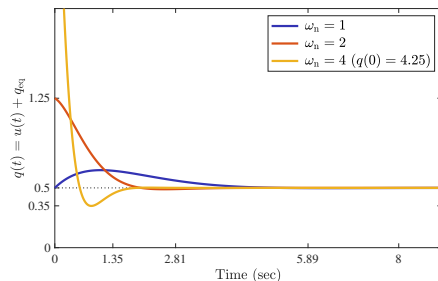
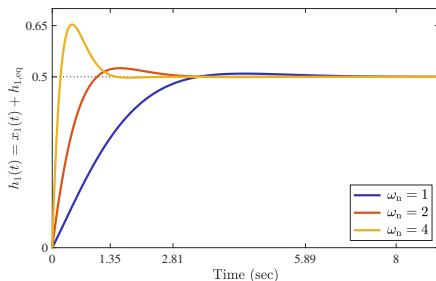
(the absolute values of both components of K eventually grow with ω_n).

Example: simulations

The control law

$$u(t) = Kx(t) \implies q(t) = Kh(t) + q_{eq} - Kh_{eq}.$$

With $\sigma = 1$, $\alpha = \beta = 1$, $\zeta = 0.8$, and $\omega_n = \{1, 2, 5\}$,

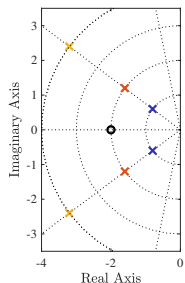
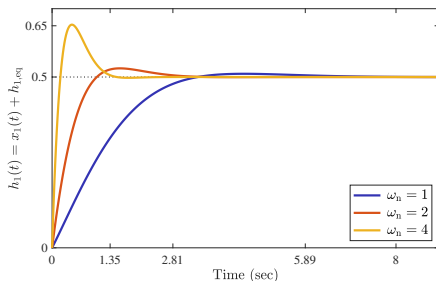


under $K = - \begin{bmatrix} -1.4 & 2.8 \end{bmatrix}$, $K = - \begin{bmatrix} 0.2 & 2.6 \end{bmatrix}$, $K = - \begin{bmatrix} 3.4 & 8.2 \end{bmatrix}$. Thus,

- faster poles $\implies$ faster response & larger control effort
- faster poles $\implies$ large overshoot (why?)

Example: simulations (contd)

The reason can be seen in the pole-zero map:



Because system zeros are not moved,

- when poles move left, zero at -2 becomes dominant.

Outline

State feedback

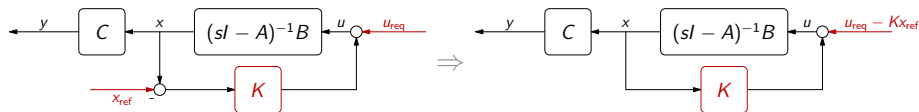
Controllability

Example: 2-mass system

State feedback: pole placement

State feedback and reference response

Architecture motivated by 2DOF



As usual, two requirements

1. x_{ref} and u_{req} are bounded
2. $x_{\text{ref}} = (sI - A)^{-1}Bu_{\text{req}}$

stability

consistency

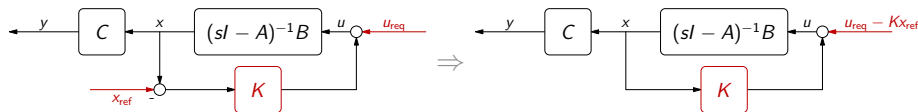
In this case

$$u = u_{\text{req}} + K((sI - A)^{-1}Bu - x_{\text{ref}}) \implies u = \frac{u_{\text{req}} - Kx_{\text{ref}}}{1 - K(sI - A)^{-1}B} = u_{\text{req}}$$

and $x = (sI - A)^{-1}Bu = x_{\text{ref}}$, regardless K (provided it is stabilizing).

$$\text{CONSIDER } x_{\text{ref}}: y_{\text{ref}} \rightarrow u_{\text{req}} = \frac{y_{\text{ref}}}{C(sI - A)^{-1}B} \rightarrow x_{\text{ref}} = (sI - A)^{-1}Bu_{\text{req}}$$

Architecture motivated by 2DOF



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1. x_{ref} and u_{req} are bounded
2. $x_{\text{ref}} = (sI - A)^{-1}B u_{\text{req}}$

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In this case

$$u = u_{\text{req}} + K((sI - A)^{-1}Bu - x_{\text{ref}}) \implies u = \frac{u_{\text{req}} - Kx_{\text{ref}}}{1 - K(sI - A)^{-1}B} = u_{\text{req}}$$

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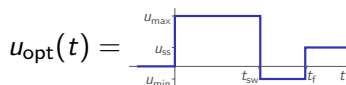
Generating x_{ref} : $y_{\text{ref}} \rightarrow u_{\text{req}} = \frac{y_{\text{ref}}}{C(sI - A)^{-1}B} \rightarrow x_{\text{ref}} = (sI - A)^{-1}B u_{\text{req}}$

Example: time-optimal response

We may choose y_{ref} as the

- fastest filling of tank 1 to $x_1 = x_{1,f}$ under $u(t) \in [u_{\min}, u_{\max}]$

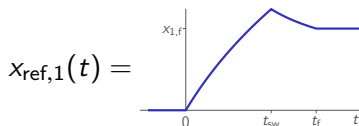
for some $-q_{\text{eq}} \leq u_{\min} < u_{\max}$. The optimal bang-bang



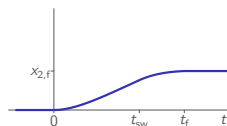
has

$$U_{\text{opt}}(s) = \frac{u_{\max} - (u_{\max} - u_{\min})e^{-st_{\text{sw}}} + (x_{1,f}/P_1(0) - u_{\min})e^{-st_f}}{s},$$

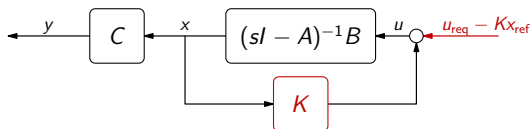
where t_{sw} and t_f are chosen to render $P_1(s)U_{\text{opt}}(s)$ FIR. In this case



and $x_{\text{ref},2}(t) =$



Example: simulations (contd)



With $x_{1,f} = 0.5$, $u_{\min} = -0.5 = -q_{\text{eq}}$, $u_{\max} = 1.5$, and $u_{\text{req}} = u_{\text{opt}}$,

