

Control Theory (00350188)

lecture no. 6

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT



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Outline

Preliminaries: linear algebra

State-space models

System interconnections in terms of state-space realizations

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Linear algebra, notation and terminology

A matrix $A \in \mathbb{R}^{n \times m}$ is an $n \times m$ table of real numbers,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} := [a_{ij}],$$

accompanied by their manipulation rules (addition, multiplication, etc). The number of linearly independent rows (or columns) in A is called its rank and denoted $\text{rank } A$. A matrix $A \in \mathbb{R}^{n \times m}$ is said to

- have full row (column) rank if $\text{rank } A = n$ ($\text{rank } A = m$)
- be square if $n = m$, tall if $n > m$, and fat if $n < m$
- be diagonal ($A = \text{diag}\{a_i\}$) if it is square and $a_{ij} = 0$ whenever $i \neq j$
- be lower (upper) triangular if $a_{ij} = 0$ whenever $i > j$ ($i < j$)
- be symmetric if $A = A'$, where the transpose $A' := [a_{ji}]$

The identity matrix $I_n := \text{diag}\{1\} \in \mathbb{R}^{n \times n}$ (if the dimension is clear, just I). It is a convention that $A^0 = I$ for every nonzero square A .

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Block matrices

Let $A \in \mathbb{R}^{n \times m}$ and let $\sum_{i=1}^v n_i = n$ and $\sum_{i=1}^\mu m_i = m$ for some $n_i, m_i \in \mathbb{N}$. Then A can be formally split as

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1\mu} \\ A_{21} & A_{22} & \cdots & A_{2\mu} \\ \vdots & \vdots & \ddots & \vdots \\ A_{v1} & A_{v2} & \cdots & A_{v\mu} \end{bmatrix} = [A_{ij}]$$

for $A_{ij} = [a_{ij,kl}] \in \mathbb{R}^{n_i \times m_j}$ with entries $a_{ij,kl} = a_{pq}$, where $p = \sum_{r=1}^{i-1} n_r + k$ and $q = \sum_{r=1}^{j-1} m_r + l$. For example, we may present

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

for $A_{11} = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, $A_{13} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 9 & 10 \end{bmatrix}$, $A_{22} = 11$, $A_{23} = 12$. This split is not unique and is normally done merely for convenience.

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Block matrices (contd)

We say that a block matrix $A = [A_{ij}]$ is

- block-diagonal ($A = \text{diag}\{A_i\}$) if $v = \mu$ and $A_{ij} = 0$ whenever $i \neq j$
- block lower (upper) triangular if $A_{ij} = 0$ whenever $i > j$ ($i < j$)

If A is square, i.e. $n = m$, a natural split is with $n_i = m_i$ for all i . Then all A_{ij} are square too.

Operations on block matrices are performed in terms of their blocks, like

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix},$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

Sum, product, inverse of block-diagonal (block-triangular) matrices remain block-diagonal (block-triangular).

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Eigenvalues & eigenvectors

Given a square matrix $A \in \mathbb{R}^{n \times n}$, its **eigenvalues** are the solutions $\lambda \in \mathbb{C}$ to

$$\chi_A(\lambda) := \det(\lambda I - A) = \lambda^n + \chi_{n-1}\lambda^{n-1} + \cdots + \chi_1\lambda + \chi_0 = 0$$

(**characteristic equation**). There are n (not necessarily different) eigenvalues of A and $\text{spec}(A)$ denotes their set. If $\lambda_i \in \text{spec}(A)$ is such that $\text{Im } \lambda_i \neq 0$, then $\bar{\lambda}_i \in \text{spec}(A)$ too. Also, $\lambda_i \in \text{spec}(A) \implies \lambda_i t \in \text{spec}(At)$, $\forall t \in \mathbb{R}$.

Right and left **eigenvectors** associated with $\lambda_i \in \text{spec}(A)$ are nonzero vectors η_i and $\tilde{\eta}_i$ such that

$$(\lambda_i I - A)\eta_i = 0 \quad \text{and} \quad \tilde{\eta}_i'(\lambda_i I - A) = 0,$$

respectively. Note that

$$A^k \eta_i = \eta_i \lambda_i^k \quad \text{and} \quad \tilde{\eta}_i' A^k = \lambda_i^k \tilde{\eta}_i', \quad \forall k \in \mathbb{N}$$

so $\lambda_i \in \text{spec}(A) \implies \lambda_i^k \in \text{spec}(A^k)$, keeping left and right eigenvectors.

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Eigenvalues & eigenvectors of block-triangular matrices

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Because

$$\begin{bmatrix} \lambda I - A_{11} & -A_{12} \\ 0 & \lambda I - A_{22} \end{bmatrix} \begin{bmatrix} \eta \\ 0 \end{bmatrix} = \begin{bmatrix} (\lambda I - A_{11})\eta \\ 0 \end{bmatrix},$$

every eigenvalue of A_{11} is that of A (its right eigenvector is $\begin{bmatrix} \eta_i \\ 0 \end{bmatrix}$). Because

$$\begin{bmatrix} 0 & \tilde{\eta}' \end{bmatrix} \begin{bmatrix} \lambda I - A_{11} & -A_{12} \\ 0 & \lambda I - A_{22} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{\eta}'(\lambda I - A_{22}) \end{bmatrix},$$

every eigenvalue of A_{22} is that of A (its left eigenvector is $\begin{bmatrix} 0 & \tilde{\eta}_i' \end{bmatrix}$). Thus

- λ_i is an eigenvalue of A iff it is an eigenvalue of A_{11} or A_{22} .

The same arguments

- work for block-lower-triangular and block-diagonal matrices too.

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Cayley–Hamilton

In essence: each square matrix satisfies its own characteristic equation:

$$\chi_A(A) := A^n + \chi_{n-1}A^{n-1} + \cdots + \chi_1 A + \chi_0 I_n = 0.$$

Important consequences:

- A^k for all $k \geq n$ is a linear combination of A^i , $i = 0, \dots, n-1$, like

$$\begin{aligned} A^n &= -\chi_{n-1}A^{n-1} - \cdots - \chi_1 A - \chi_0 I_n \\ A^{n+1} &= -\chi_{n-1}A^n - \cdots - \chi_1 A^2 - \chi_0 A \\ &= \chi_{n-1}(\chi_{n-1}A^{n-1} + \cdots + \chi_1 A + \chi_0 I_n) - \cdots - \chi_1 A^2 - \chi_0 A \\ &= (\chi_{n-1}^2 - \chi_{n-2})A^{n-1} + \cdots + (\chi_{n-1}\chi_1 - \chi_0)A + \chi_{n-1}\chi_0 I_n \\ &\vdots \end{aligned}$$

- A^{-1} , if exists, is also a linear combination of A^i , $i = 0, \dots, n-1$:

$$A^{-1} = -\frac{1}{\chi_0}(\chi_1 I + \cdots + \chi_{n-1}A^{n-2} + A^{n-1}).$$

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Matrix functions

Let $f(x) = \sum_{i=0}^{\infty} f_i x^i$ be analytic. Its matrix version $f(A)$ is defined as

$$f(A) := \sum_{i=0}^{\infty} f_i A^i.$$

It is readily seen that $f(A)\eta_i = \eta_i f(\lambda_i)$ for all eigenvalue-eigenvector pairs. By Cayley–Hamilton, we know that $\exists g_i, i = 0, \dots, n-1$ such that

$$f(A) = \sum_{i=0}^{n-1} g_i A^i.$$

This does **not** imply that $f(x) = g(x) := \sum_{i=0}^{n-1} g_i x^i$ for all x , but still

$$f(\lambda_i) = g(\lambda_i) \quad \text{and} \quad \left. \frac{d^j f(x)}{dx^j} \right|_{x=\lambda_i} = \left. \frac{d^j g(x)}{dx^j} \right|_{x=\lambda_i}, \quad \forall j = 1, \dots, \mu_i - 1$$

for each eigenvalue λ_i of A having multiplicity μ_i . These equations can be used to calculate all n coefficients g_i and, hence, $f(A)$.

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Matrix functions (contd)

If all n eigenvalues of A are different,

$$f(\lambda_i) = \sum_{j=0}^{n-1} g_j \lambda_i^j = \begin{bmatrix} g_0 & g_1 & \cdots & g_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_i \\ \vdots \\ \lambda_i^{n-1} \end{bmatrix}.$$

Thus, $\begin{bmatrix} f(\lambda_1) & f(\lambda_2) & \cdots & f(\lambda_n) \end{bmatrix} = \begin{bmatrix} g_0 & g_1 & \cdots & g_{n-1} \end{bmatrix} V$, where

$$V := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

is the Vandermonde matrix (with $\det V = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \neq 0$). Hence,

$$\begin{bmatrix} g_0 & g_1 & \cdots & g_{n-1} \end{bmatrix} = \begin{bmatrix} f(\lambda_1) & f(\lambda_2) & \cdots & f(\lambda_n) \end{bmatrix} V^{-1}.$$

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Matrix exponential

The matrix exponential is defined (here $t \in \mathbb{R}$, we shall need it later on) as

$$\exp(At) = e^{At} := I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$

Example

Let $At = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} t$ with eigenvalues $\lambda_1 = -t$ and $\lambda_2 = 2t$. Then

$$\begin{bmatrix} g_0(t) & g_1(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -t & 2t \end{bmatrix}^{-1} \iff \begin{cases} g_0(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} \\ g_1(t) = -\frac{1}{3t}e^{-t} + \frac{1}{3t}e^{2t} \end{cases}$$

and hence

$$\exp\left(\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} t\right) = g_0(t) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + g_1(t) \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} t = \begin{bmatrix} e^{-t} & \frac{1}{3}(e^{2t} - e^{-t}) \\ 0 & e^{2t} \end{bmatrix}$$

Note, the exponential of this triangular matrix is triangular too (general).

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Similar matrices

Let $A \in \mathbb{R}^{n \times n}$ and $T \in \mathbb{R}^{n \times n}$ with $\det T \neq 0$. The matrices

$$A \quad \text{and} \quad TAT^{-1}$$

are said to be **similar** and $A \mapsto TAT^{-1}$ is called a similarity transformation. Similarity transformations do not affect characteristic equations:

$$\begin{aligned} \chi_{TAT^{-1}}(s) &= \det(\lambda I - TAT^{-1}) = \det(T(\lambda I - A)T^{-1}) \\ &= \det(T) \det(\lambda I - A) \det(T^{-1}) = \det(\lambda I - A) \\ &= \chi_A(s). \end{aligned}$$

Some definitions / facts:

- a matrix $A \in \mathbb{R}^{n \times n}$ is said to be diagonalizable if there is a $T \in \mathbb{R}^{n \times n}$ such that $TAT^{-1} = \text{diag}\{\lambda_i\}$;
- symmetric matrices are always diagonalizable;
- $f(TAT^{-1}) = Tf(A)T^{-1}$ for any analytic function $f(x)$ follows by the very fact that $(TAT^{-1})^k = TA^kT^{-1}$ for all $k \in \mathbb{N}$

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Sign-definite matrices

A symmetric matrix $A = A' \in \mathbb{R}^{n \times n}$ is said to be

- positive definite ($A > 0$) if $x'Ax > 0$ for all $x \neq 0$,
- positive semidefinite ($A \geq 0$) if $x'Ax \geq 0$ for all x ,
- negative definite ($A < 0$) if $x'Ax < 0$ for all $x \neq 0$ (or if $-A > 0$),
- negative semidefinite ($A \leq 0$) if $x'Ax \leq 0$ for all x (or $-A \geq 0$).

Clearly, $A > 0 \implies \det A \neq 0$. In fact,

- $A > 0$ (≥ 0) iff all its eigenvalues are positive (nonnegative)

Given two matrices $A = A' \in \mathbb{R}^{n \times n}$ and $B = B' \in \mathbb{R}^{n \times n}$, we say

- $A > B$ if $A - B > 0$,
- $A \geq B$ if $A - B \geq 0$,
- $A < B$ if $B - A > 0$ (equivalently, $A - B < 0$),
- $A \leq B$ if $B - A \geq 0$ (equivalently, $A - B \leq 0$).

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Sign-definite matrices (contd)

Example

If $A = \begin{bmatrix} \alpha & -1 \\ -1 & 1 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$, then

$$\begin{aligned} x'Ax &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha x_1^2 - 2x_1x_2 + x_2^2 \\ &= (\alpha - 1)x_1^2 + (x_1 - x_2)^2 \end{aligned}$$

and if

- $\alpha > 1$ then $x'Ax > 0$ unless $x_1 = x_2 = 0$, for which $x'Ax = 0 \implies A > 0$
- $\alpha = 1$ then $x'Ax \geq 0$ unless $x_1 = x_2$, for which $x'Ax = 0 \implies A \geq 0$
- $\alpha < 1$ then $x'Ax$ might be both positive (e.g. if $x_1 = 0 \neq x_2$) and negative (e.g. if $x_1 = x_2 \neq 0$) $\implies A \not\geq 0$

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Sign-definite block matrices

Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{for square } A_{11} \text{ and } A_{22}$$

be symmetric, i.e. A_{11} and A_{22} are symmetric and $A_{12} = A_{21}'$. In this case

$$A > 0 \implies \begin{cases} A_{11} > 0 & \text{as } \begin{bmatrix} x_1' & 0 \end{bmatrix} A \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1' A_{11} x_1 > 0 \text{ for all } x_1 \neq 0 \\ A_{22} > 0 & \text{as } \begin{bmatrix} 0 & x_2' \end{bmatrix} A \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2' A_{22} x_2 > 0 \text{ for all } x_2 \neq 0 \end{cases}$$

Then (this technique is known as **completing the square**)

$$\begin{aligned} x'Ax &= x_1' A_{11} x_1 + x_1' A_{12} x_2 + x_2' A_{21} x_1 + x_2' A_{22} x_2 \\ &= x_1' (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + x_1' A_{12} A_{22}^{-1} A_{21} x_1 + 2x_2' A_{21} x_1 + x_2' A_{22} x_2 \\ &= x_1' (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (x_1' A_{12} A_{22}^{-1} + x_2') A_{22} (A_{22}^{-1} A_{21} x_1 + x_2) \end{aligned}$$

and $A > 0 \iff (A_{22} > 0) \wedge (A_{11} - A_{12} A_{22}^{-1} A_{21} > 0)$. By dual arguments,
 $A > 0 \iff (A_{11} > 0) \wedge (A_{22} - A_{21} A_{11}^{-1} A_{12} > 0)$.

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Outline

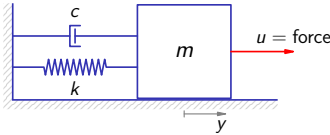
Preliminaries: linear algebra

State-space models

System interconnections in terms of state-space realizations

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Spring-mass-damper system 1



Dynamics,

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t),$$

can be rewritten as

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

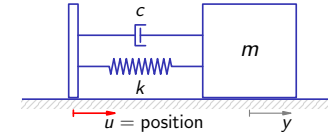
where

$$x(t) := \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

is the **state vector**, having clear physical meaning.

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Spring-mass-damper system 2



Dynamics,

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = c\dot{u}(t) + ku(t),$$

can be rewritten as

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} k/m & c/m \end{bmatrix} x(t) \end{cases}$$

where

$$x(t) := \begin{bmatrix} v(t) \\ \dot{v}(t) \end{bmatrix} \quad \text{for } \ddot{v}(t) = u(t) - y(t)$$

is the **state vector**, having no clear physical meaning.

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State-space realizations

A state-space description (realization) of a linear SISO system $G : u \mapsto y$ is

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) + Du(t), \end{cases}$$

where

- $u(t) \in \mathbb{R}$ is an **input** signal,
- $y(t) \in \mathbb{R}$ is an **output** signal,
- $x(t) \in \mathbb{R}^n$ is a **state vector** (internal variable) with initial condition x_0 .

State and realization are not unique. For example, let $\tilde{x} := Tx$, then

$$G : \begin{cases} \dot{\tilde{x}}(t) = TAT^{-1}\tilde{x}(t) + TBu(t), & \tilde{x}(0) = Tx_0, \\ y(t) = CT^{-1}\tilde{x}(t) + Du(t) \end{cases}$$

describes the very same mapping $u \mapsto y$ (similarity transformation).

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Solution of the state equation

The evolution of the state vector $x(t)$ at $t > 0$ is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds.$$

Substituting this expression into the output equation, we end up with

$$y(t) = Ce^{At}x_0 + Du(t) + C \int_0^t e^{A(t-s)}Bu(s)ds.$$

This solution prompts the following interpretation of the state vector:

- the state vector $x(t)$ is a **history accumulator**: there is no need to know input *history* to calculate future outputs, the knowledge of the current state and *future* inputs is sufficient.

Indeed, given any t_0 , then for all $t > t_0$

$$y(t) = \underbrace{Ce^{A(t-t_0)}x(t_0)}_{\text{history (up to } t_0)} + \underbrace{Du(t) + C \int_{t_0}^t e^{A(t-s)}Bu(s)ds}_{\text{future (from } t_0 \text{ and up to } t)}.$$

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Impulse response via state-space realizations

The response to $u(t) = \delta(t)$ and zero initial conditions $x_0 = 0$ is

$$y(t) = D\delta(t) + C \int_0^t e^{A(t-s)} B \delta(s) ds = D\delta(t) + Ce^{At} B \mathbb{1}(t).$$

If $D = 0$, this response is reminiscent of that to initial condition. Namely,

$$y(t) = Ce^{At} B = Ce^{At} x_0 \quad \text{whenever } x_0 = B.$$

for all $t \geq 0$. This implies that

- the effect of initial conditions can also be expressed as the effect of external signal

(Dirac delta in this case).

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From state space to transfer functions

The transfer function is the Laplace transform of the impulse response (with zero initial conditions). Hence,

$$G(s) = \mathcal{L}\{D\delta(t) + Ce^{At} B\} = D + C(sI - A)^{-1} B.$$

This can also be seen via rewriting the state model in the s -domain:

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases} \implies \begin{cases} X(s) = (sI - A)^{-1} BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

Because

$$D + C(sI - A)^{-1} B = D + \frac{C \operatorname{adj}(sI - A) B}{\det(sI - A)},$$

where $\operatorname{adj}(sI - A)$ is the adjugate matrix having polynomial entries,

- **poles** of $D + C(sI - A)^{-1} B$ are the eigenvalues of A , i.e. in $\operatorname{spec}(A)$ (it's a bit more complicated, we'll study that later on in the course).

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Properness

Because

$$\lim_{|s| \rightarrow \infty} C(sI - A)^{-1} B = \lim_{|s| \rightarrow \infty} \frac{1}{s} C(I - \frac{1}{s} A)^{-1} B = \lim_{|z| \rightarrow 0} z C(I - zA)^{-1} B = 0$$

we have that

$$\lim_{|s| \rightarrow \infty} D + C(sI - A)^{-1} B = D.$$

Compare

$$\lim_{|s| \rightarrow \infty} \frac{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = b_n.$$

Thus, the transfer function

- $D + C(sI - A)^{-1} B$ is strictly proper iff $D = 0$ (or bi-proper iff $D \neq 0$).

The parameter **D** is called the **feed-through** term of the realization.

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Pole excess and high-frequency gain

Let

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

be strictly proper (i.e. $n > m$). Its pole excess, $n - m$, can be interpreted as

- the minimum $k \in \mathbb{N}$ such that $s^k G(s)$ is bi-proper.

Now, because

$$s(sI - A)^{-1} = I + A(sI - A)^{-1},$$

we have that

$$\begin{aligned} sC(sI - A)^{-1} B &= CB + CA(sI - A)^{-1} B \\ s^2 C(sI - A)^{-1} B &= sCB + CAB + CA^2(sI - A)^{-1} B \\ &\vdots \\ s^k C(sI - A)^{-1} B &= \sum_{i=1}^{k-1} s^i CA^{k-1-i} B + CA^{k-1} B + CA^k(sI - A)^{-1} B \end{aligned}$$

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Pole excess and high-frequency gain (contd)

Thus, given $G(s) = D + C(sI - A)^{-1}B$ for¹ $D = 0$, its

- pole excess is the minimum $k \in \mathbb{N}$ such that $CA^{k-1}B \neq 0$.

The quantities $CA^{k-1}B$, $k \in \mathbb{N}$, are known as **Markov parameters** of $G(s)$.

The high-frequency gain of $G(s)$,

$$b_m = \lim_{|s| \rightarrow \infty} s^{n-m} G(s).$$

In state space,

$$s^{n-m} C(sI - A)^{-1} B = CA^{n-m-1} B + CA^{n-m}(sI - A)^{-1} B.$$

Hence, the high-frequency gain of $G(s) = C(sI - A)^{-1}B$ is

$$b_m = CA^{n-m-1} B,$$

where n is the dimension of A and $n - m$ is the pole excess of $G(s)$.

¹If $D \neq 0$, then the pole excess is always zero.

From transfer functions to state space

Let $G(s) = \frac{b_{n-1}s^{n-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$. Its state-space realization in

- **companion form**

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -a_0 & -a_1 & \dots & -a_{n-1} & 1 \\ \hline b_0 & b_1 & \dots & b_{n-1} & 0 \end{array} \right],$$

- **observer form**

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cccc|c} -a_{n-1} & 1 & \dots & 0 & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_1 & 0 & \dots & 1 & b_1 \\ -a_0 & 0 & \dots & 0 & b_0 \\ \hline 1 & 0 & \dots & 0 & 0 \end{array} \right].$$

If $b_n \neq 0$, $\frac{b_n s^n + b_{n-1}s^{n-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0} = b_n + \frac{(b_{n-1} - b_n a_{n-1})s^{n-1} + \dots + b_0 - b_n a_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$ & $D = b_n$.

Outline

Preliminaries: linear algebra

State-space models

System interconnections in terms of state-space realizations

Problem

We know how to carry out parallel, cascade, and feedback interconnections in terms of transfer functions $G_1(s)$ and $G_2(s)$. Indeed:

- parallel: $G_1(s) + G_2(s)$
- series: $G_2(s)G_1(s)$
- feedback: $\frac{G_1(s)}{1 \mp G_1(s)G_2(s)}$

The question is: how this can be done in state space?

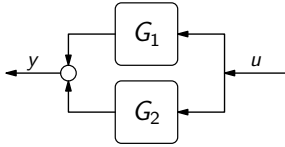
We assume that

$$G_1 : \begin{cases} \dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) \\ y_1(t) = C_1 x_1(t) + D_1 u_1(t) \end{cases} \quad \text{and} \quad G_2 : \begin{cases} \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t) \\ y_2(t) = C_2 x_2(t) + D_2 u_2(t) \end{cases}$$

A possible way of interconnecting systems is to

- determine what inputs / outputs to equate
- unite state vectors

Parallel interconnection



This corresponds to

- $u_1 = u_2 = u$
- $y = y_1 + y_2$

Then

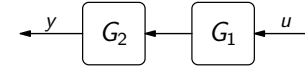
$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + (D_1 + D_2)u(t) \end{cases}$$

We already know (spectrum of block-diagonal matrices) that

- poles of $G(s)$ are $\text{spec}(A_1) \cup \text{spec}(A_2)$

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Series / cascade interconnection



This corresponds to

- $u_1 = u$
- $u_2 = y_1$
- $y = y_2$

Then

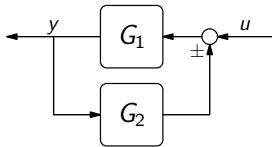
$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + D_2 D_1 u(t) \end{cases}$$

We already know (spectrum of block-triangular matrices) that

- poles of $G(s)$ are $\text{spec}(A_1) \cup \text{spec}(A_2)$

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Feedback interconnection



This corresponds to

- $u_1 = u \pm y_2$
- $u_2 = y_1$
- $y = y_1$

Then, assuming that $D_1 = 0$ for simplicity (enough for our purposes),

$$G : \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 \pm B_1 D_2 C_1 & \pm B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

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Inversion

Let

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

be bi-proper (i.e. $D \neq 0$). Its inverse is the system mapping $y \mapsto u$. Then

$$u(t) = -D^{-1}Cx(t) + D^{-1}y(t)$$

and

$$\dot{x}(t) = Ax(t) + B(-D^{-1}Cx(t) + D^{-1}y(t))$$

Therefore,

$$G^{-1} : \begin{cases} \dot{x}(t) = (A - BD^{-1}C)x(t) + BD^{-1}y(t) \\ u(t) = -D^{-1}Cx(t) + D^{-1}y(t) \end{cases}$$

and **zeros of $G(s)$** (= poles of $G^{-1}(s)$) are in **$\text{spec}(A - BD^{-1}C)$** .

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