

Advanced Sampled-Data Control (038798) lecture no. 13

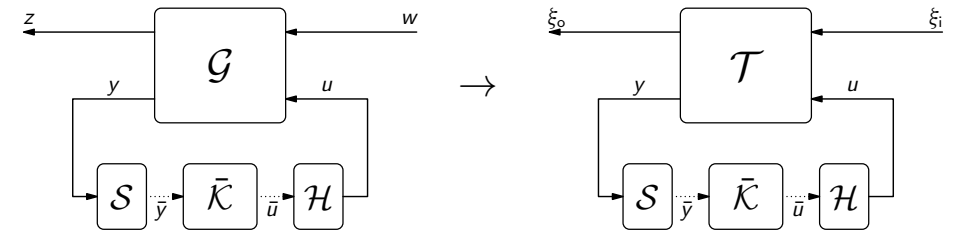
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1/48

Frequency-domain analysis

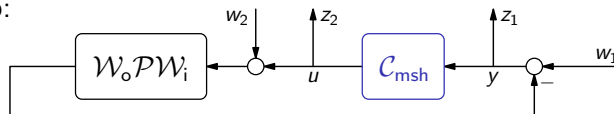


Once a sampled-data controller / estimator is designed for given generalized plant $\mathcal{G}$ (includes the model of the plant, disturbance/noise models, control goals and weights, et cetera), we might be interested to check properties of the resulted system. Toward this end, the way in which the spectrum of an exogenous signal of importance, say ξ_i , is transformed to the spectrum of a regulated signal of interest, say ξ_o , offers an invaluable insight.

2/48

Frequency-domain analysis: H_∞ loop shaping

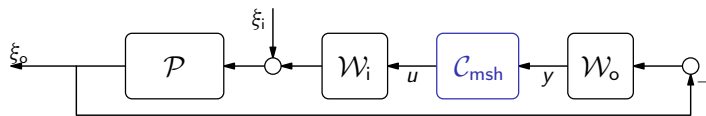
Design setup:



and then

$$G(s) = \begin{bmatrix} I & -P_{\text{msh}}(s) & -P_{\text{msh}}(s) \\ 0 & 0 & I \\ I & -P_{\text{msh}}(s) & -P_{\text{msh}}(s) \end{bmatrix},$$

where $P_{\text{msh}}(s) := W_o(s)P(s)W_i(s)$. *Disturbance attenuation analysis setup:*



and then

$$T(s) = \begin{bmatrix} P(s) & P(s)W_i(s) \\ -W_o(s)P(s) & -W_o(s)P(s)W_i(s) \end{bmatrix}.$$

3/48

Content

Frequency response of sampled-data systems: a “natural” approach

Frequency power response

Frequency power response in a special case

4/48

Frequency response interpretation

Remember from Lecture 1:

Let $u(t) = u_\omega e^{j\omega t}$, then (the *Frequency Response Theorem*)

$$\begin{aligned} y(t) &= \int_{\mathbb{R}} g(t-s) u_\omega e^{j\omega s} ds = \int_{\mathbb{R}} g(s) u_\omega e^{j\omega(t-s)} ds \\ &= \int_{\mathbb{R}} g(s) e^{-j\omega s} ds u_\omega e^{j\omega t} = [G(j\omega) u_\omega] e^{j\omega t}, \end{aligned}$$

so that the response to a harmonic input is also a harmonic signal with the same frequency, whose magnitude / direction are shaped by $G(j\omega)$. Usage:

- performance analysis of LTI systems
- system identification from harmonic inputs

Discrete version: $\bar{u}[i] = u_\theta e^{j\theta i} \implies \bar{y}[i] = [\bar{G}(e^{j\theta}) u_\theta] e^{j\theta i}$.

5/48

Frequency-response gain

In both analog and discrete cases, the frequency response “gains” are

$$\|G(j\omega)\| : \mathbb{R} \rightarrow \mathbb{R}^+ \quad \text{and} \quad \|\bar{G}(e^{j\theta})\| : [-\pi, \pi] \rightarrow \mathbb{R}^+$$

and they quantify the maximal amplification of a pure harmonic signal by a given LTI system. This

- is a very useful analysis tool (especially, as graphic visualization)
- adds insight to the H_∞ norm (as $\|\mathcal{G}\|_\infty = \sup_\omega \|G(j\omega)\|$)

We would like to have a

- similar tool for analyzing sampled-data systems.

Potential trouble toward this end is frequency *aliasing*, which mixes various input frequencies up in the response. But, at the same time, we know that *lifting* reclaims shift invariance...

6/48

Response to harmonic signals in the lifted domain

The signal

$$\check{u}[i](\tau) = \check{u}_\theta(\tau) e^{j\theta i}, \quad \text{for } \check{u}_\theta : [0, h) \rightarrow \mathbb{F}^n$$

appears to be a natural generalization of discrete **harmonic signals** in the lifted domain. Its application to a stable shift-invariant lifted system $\check{\mathcal{G}}$ with the impulse response $\check{g}[i]$ results in

$$\begin{aligned} \check{y}[i] &= \sum_{j \in \mathbb{Z}} \check{g}[i-j] \check{u}_\theta e^{j\theta j} = \sum_{j \in \mathbb{Z}} \check{g}[j] \check{u}_\theta e^{j\theta(i-j)} = \left[\sum_{j \in \mathbb{Z}} \check{g}[j] e^{-j\theta j} \right] \check{u}_\theta e^{j\theta i} \\ &= [\check{\mathcal{G}}(e^{j\theta}) \check{u}_\theta] e^{j\theta i}, \end{aligned}$$

which is a harmonic signal again (assuming convergence). In other words, the Frequency Response Theorem holds in the lifted domain as well.

7/48

Frequency-response gain of lifted systems

Similarly to the frequency-response gains of analog or discrete systems, we may define now the lifted frequency-response gain as

$$\|\check{\mathcal{G}}(e^{j\theta})\|_{\mathbb{L} \rightarrow \mathbb{L}} : [-\pi, \pi] \rightarrow \mathbb{R}^+$$

(we shall denote it $\|\check{\mathcal{G}}(e^{j\theta})\|$ for simplicity hereafter).

This appears natural and technically clean, but

- what is the meaning of all this?

How can we interpret the lifted Frequency Response Theorem? How can we use the lifted frequency-response gain (apart from expressing the H_∞ norm in terms of it)? Can it be efficiently calculated?

8/48

Examples of lifted harmonic signals

Let

1. $u(t) = u_{\omega} e^{j\omega_0 t}$, then

$$\check{u}[i](\tau) = u_{\omega_0} e^{j\omega_0(ih+\tau)} = u_{\omega_0} e^{j\omega_0 \tau} e^{j(\omega_0 h)i};$$

2. $u(t) = u_{\omega_l} e^{j\omega_l t}$, where $\omega_l := \omega_0 + \frac{2\pi}{h}l$ and $l \in \mathbb{Z}$ (with $e^{j\omega_l h} = e^{j\omega_0 h}$), then

$$\check{u}[i](\tau) = u_{\omega_l} e^{j\omega_l(ih+\tau)} = u_{\omega_l} e^{j\omega_l \tau} e^{j(\omega_0 h)i};$$

3. $u(t) = u_{\omega_0} e^{j\omega_0 t} + u_{\omega_l} e^{j\omega_l t}$, then

$$\check{u}[i](\tau) = u_{\omega_0} e^{j\omega_0(ih+\tau)} + u_{\omega_l} e^{j\omega_l(ih+\tau)} = (u_{\omega_0} e^{j\omega_0 \tau} + u_{\omega_l} e^{j\omega_l \tau}) e^{j(\omega_0 h)i}.$$

All these different analog signals result in lifted harmonic signals with

- the same frequency $\theta = \omega_0 h$ and different $\check{u}_{\theta}$.

9/48

(Rather) general lifted harmonic signal

Let $\check{u}[i] = \check{u}_{\theta} e^{j\theta i}$ for some $\check{u}_{\theta}$ such that its Fourier series

$$\check{u}_{\theta}(\tau) = \sum_{i \in \mathbb{Z}} c_i e^{j\omega_i \tau}, \quad \text{where } \omega_i := \frac{\theta}{h} + \frac{2\pi}{h}i$$

converges. This lifted signal $\check{u}$ corresponds then to

$$u(t) = \sum_{i \in \mathbb{Z}} c_i e^{j\omega_i t},$$

which is a

- linear combination of harmonic signals with all aliased frequencies w.r.t. the sampling period h . This, in turn, implies that the
- lifted frequency response reflects responses to *all possible combinations* of harmonic signals at all aliased frequencies.

This is natural / reasonable, but might not be quite handy (too much info).

10/48

Frequency-response gain of lifted systems (contd)

The frequency-response gain $\|\check{G}(e^{j\theta})\|$ may be viewed as

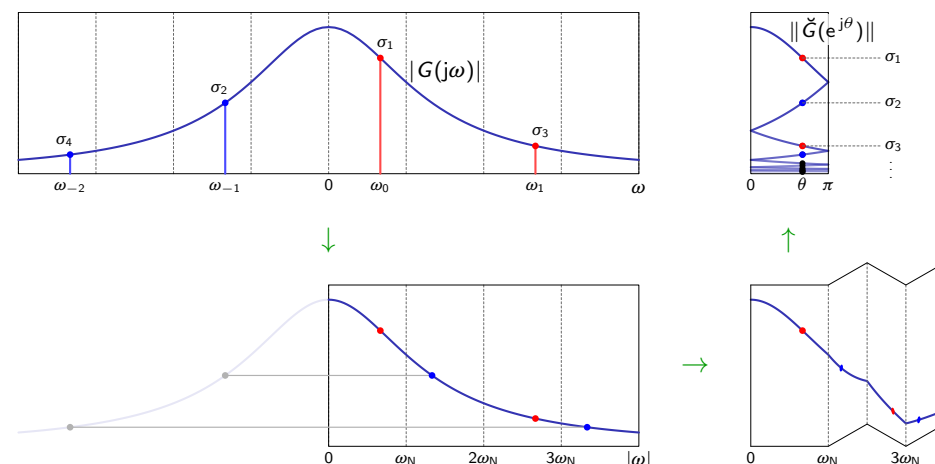
- a way to meaningfully quantify the lifted frequency response at each θ .

This, however, might not be the case...

11/48

Frequency-response gain of lifted LTI systems

Remember (Lecture 6, Slide 29), that if $\check{G}$ is the lifting of an LTI G , then



Thus,

$\check{G}(e^{j\theta})$ represents only a **segment** of what we can see in $\|G(j\omega)\|$.

12/48

Example 1

Consider the problem of designing a causal A/D converter $\mathcal{S}$ minimizing

$$\|\mathcal{S}_{\text{avg}}(\mathcal{D}_h - \mathcal{H}_{\text{ZOH}}\mathcal{S})\|_2,$$

where $\mathcal{D}_h$ is the h -delay operator and h is the sampling period. The solution is trivially

$$\mathcal{S} = \mathcal{S}_{\text{avg}}$$

because¹ $\mathcal{S}_{\text{avg}}(\mathcal{D}_h - \mathcal{H}_{\text{ZOH}}\mathcal{S}_{\text{avg}}) = 0$ (this property is called **consistency**). In this formulation

- $\mathcal{S}_{\text{avg}}$ serves as a weight
- $\mathcal{D}_h$ is added to render the result causal (smoothing formulation).

We may then be interested to investigate properties of the error system

$$\mathcal{G}_e = 1 - \mathcal{H}_{\text{ZOH}}\mathcal{S}_{\text{avg}}.$$

¹From Lecture 2, $\mathcal{S}_{\text{avg}} = \frac{1}{h}\bar{\mathcal{U}}\mathcal{H}_{\text{ZOH}}^*$, where $\bar{\mathcal{U}}$ is the backward shift, and $\mathcal{H}_{\text{ZOH}}^*\mathcal{H}_{\text{ZOH}} = h$ and then $\mathcal{S}_{\text{avg}}\mathcal{D}_h = \bar{\mathcal{U}}\mathcal{S}_{\text{avg}}$, by shift invariance.

Example 1: frequency-response gain

In the lifted domain,

$$\check{\mathcal{G}}_e(e^{j\theta}) = 1 - \check{\Phi}_{\text{ZOH}}(e^{j\theta})\check{\Psi}_{\text{avg}}(e^{j\theta}) = 1 - \frac{e^{-j\theta}}{h} \check{\Phi}_{\text{ZOH}}(e^{j\theta})\check{\Phi}_{\text{ZOH}}^*(e^{j\theta})$$

As $\|\check{\mathcal{O}}\| \leq \gamma \iff \check{\mathcal{O}}\check{\mathcal{O}}^* \leq \gamma^2 I$, we have that $\|\check{\mathcal{G}}_e(e^{j\theta})\| \leq \gamma_\theta$ iff

$$\begin{aligned} \left(1 - \frac{e^{-j\theta}}{h} \check{\Phi}_{\text{ZOH}}(e^{j\theta})\check{\Phi}_{\text{ZOH}}^*(e^{j\theta})\right) \left(1 - \frac{e^{j\theta}}{h} \check{\Phi}_{\text{ZOH}}(e^{j\theta})\check{\Phi}_{\text{ZOH}}^*(e^{j\theta})\right) \\ = 1 + \frac{1}{h}(1 - e^{-j\theta} - e^{j\theta})\check{\Phi}_{\text{ZOH}}(e^{j\theta})\check{\Phi}_{\text{ZOH}}^*(e^{j\theta}) \leq \gamma_\theta^2 I \end{aligned}$$

or, equivalently,

$$\frac{1 - 2\cos\theta}{h} \check{\Phi}_{\text{ZOH}}(e^{j\theta})\check{\Phi}_{\text{ZOH}}^*(e^{j\theta}) \leq (\gamma_\theta^2 - 1)I.$$

Example 1: frequency-response gain (contd)

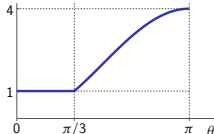
If $|\theta| \geq \frac{\pi}{3}$, then $1 - 2\cos\theta \geq 0$ and

$$\|\check{\mathcal{G}}_e(e^{j\theta})\| \leq \gamma_\theta \iff \|\check{\Phi}_{\text{ZOH}}(e^{j\theta})\|^2 = h \leq \frac{h(\gamma_\theta^2 - 1)}{1 - 2\cos\theta}$$

Thus, $\|\check{\mathcal{G}}_e(e^{j\theta})\| = \sqrt{2(1 - \cos\theta)} \in [1, 2]$. If $|\theta| < \frac{\pi}{3}$, then $1 - 2\cos\theta < 0$ and

$$\|\check{\mathcal{G}}_e(e^{j\theta})\| \leq \gamma_\theta \iff \check{\Phi}_{\text{ZOH}}(e^{j\theta})\check{\Phi}_{\text{ZOH}}^*(e^{j\theta}) \geq \frac{h(\gamma_\theta^2 - 1)}{1 - 2\cos\theta} I$$

but since $\check{\Phi}_{\text{ZOH}}(e^{j\theta})\check{\Phi}_{\text{ZOH}}^*(e^{j\theta})$ is a rank-1 operator, this is possible iff $\gamma_\theta \geq 1$. Thus, $\|\check{\mathcal{G}}_e(e^{j\theta})\|^2 = 1$ in this case. Summarizing:

$$\|\check{\mathcal{G}}_e(e^{j\theta})\|^2 = \begin{cases} 1 & \text{if } |\theta| \leq \frac{\pi}{3} \\ 2(1 - \cos\theta) & \text{if } |\theta| \geq \frac{\pi}{3} \end{cases} =$$


which makes little sense, as $\mathcal{S} = 0$ yields $\|\check{\mathcal{G}}_e(e^{j\theta})\|^2 = 1$ for all θ (better).

Example 2

Consider now the Sampling Theorem SHC $\mathcal{H}_{\text{sinc}}\mathcal{S}_{\text{ilp}}$ with

$$\Phi_{\text{sinc}}(j\omega) = h\mathbb{1}_{[-\omega_N, \omega_N]}(\omega) \quad \text{and} \quad \Psi_{\text{ilp}}(j\omega) = \mathbb{1}_{[-\omega_N, \omega_N]}(\omega).$$

We know (Lecture 6, Slide 13) that this SHC is LTI and in this case

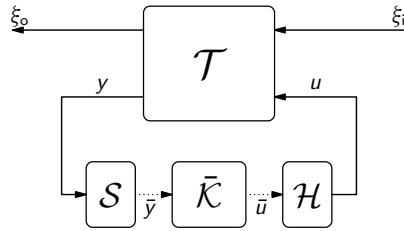
$$\mathcal{G}_e(j\omega) = 1 - \mathbb{1}_{[-\omega_N, \omega_N]}(\omega),$$

so that

$$\|\check{\mathcal{G}}_e(e^{j\theta})\| = 1.$$

This is no better than the reconstruction with the zero SHC, which is again an absurd...

Computational issues



Let

$$T(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] \quad \text{and} \quad \bar{K}(z) = z \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & 0 \end{array} \right]$$

and S and H be zero-order sampler and hold with

$$\psi(t) = \psi_0(t) \mathbb{1}_{[0,h]}(t) \quad \text{and} \quad \phi(t) = \phi_0(t) \mathbb{1}_{[0,h]}(t)$$

(general enough), such that $\psi_0(t)D_{21}$ contains no Dirac impulse terms.

17/48

Computational issues (contd)

Following the developments of Lecture 9, Slide 32, the lifted version has the plant

$$\left[\begin{array}{cc} \check{G}_{11}(z) & \check{G}_{12}(z) \\ \check{G}_{21}(z) & \check{G}_{22}(z) \end{array} \right] = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline \bar{C}_1 & \bar{D}_{11} & \bar{D}_{12} \\ \bar{C}_2 & \bar{D}_{21} & \bar{D}_{22} \end{array} \right],$$

where $\bar{A} = e^{Ah}$,

$$\dot{B}_j : \mathbb{L}^{m_j} \rightarrow \mathbb{R}^n \quad \check{\zeta} \mapsto \int_0^h e^{A(h-\sigma)} B_j \check{\zeta}(\sigma) d\sigma$$

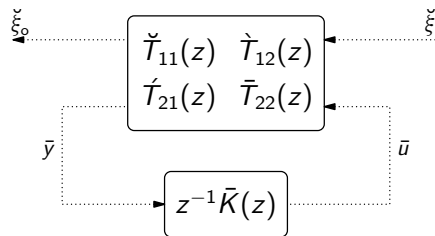
$$\dot{C}_i : \mathbb{R}^n \rightarrow \mathbb{L}^{p_i} \quad \bar{\chi} \mapsto C_i e^{A\tau} \bar{\chi}$$

$$\check{D}_{ij} : \mathbb{L}^{m_j} \rightarrow \mathbb{L}^{p_i} \quad \check{\zeta} \mapsto D_{ij} \check{\zeta}(\tau) + C_i \int_0^\tau e^{A(\tau-\sigma)} B_j \check{\zeta}(\sigma) d\sigma$$

$\begin{bmatrix} \bar{C}_2 & \bar{D}_{21} \end{bmatrix} = \dot{\Psi}_0 \begin{bmatrix} \dot{C}_1 & \dot{D}_{21} \end{bmatrix}$, $\begin{bmatrix} \bar{B}_2 \\ \bar{D}_{12} \end{bmatrix} = \begin{bmatrix} \dot{B}_2 \\ \dot{D}_{12} \end{bmatrix} \dot{\Phi}_0$, $\bar{D}_{22} = \dot{\Psi}_0 \check{D}_{22} \dot{\Phi}_0$, and $z^{-1}\dot{\Psi}_0$ and $\dot{\Phi}_0$ are the lifted transfer functions of S and H .

18/48

Computational issues (contd)

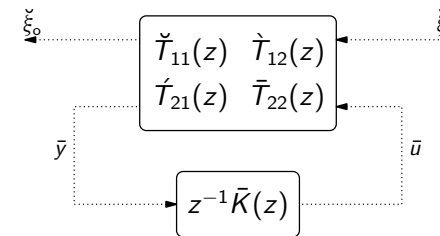


Then

$$\begin{aligned} \check{G}_e(z) &= \left[\begin{array}{cc|c} \bar{A} & \bar{B}_2 \bar{C}_K & \dot{B}_1 \\ \hline \bar{B}_K \bar{C}_2 & \bar{A}_K + \bar{B}_K \bar{D}_{22} \bar{C}_K & \bar{B}_K \dot{D}_{21} \\ \hline \bar{C}_1 & \bar{D}_{12} \bar{C}_K & \bar{D}_{11} \end{array} \right] \\ &= \check{D}_{11} + \begin{bmatrix} \dot{C}_1 & \dot{D}_{12} \end{bmatrix} \left[\begin{array}{cc|cc} \bar{A} & \bar{B}_2 \bar{C}_K & I & 0 \\ \hline \bar{B}_K \bar{C}_2 & \bar{A}_K + \bar{B}_K \bar{D}_{22} \bar{C}_K & 0 & \bar{B}_K \\ \hline I & 0 & 0 & 0 \\ 0 & \bar{C}_K & 0 & 0 \end{array} \right] \begin{bmatrix} \dot{B}_1 \\ \dot{D}_{21} \end{bmatrix}. \end{aligned}$$

19/48

Computational issues (contd)



In other words,

$$\check{G}_e(z) = \check{D}_{11} + \dot{C}_e \bar{G}_e(z) \dot{B}_e.$$

The compilation of $\|\check{G}_e(e^{j\theta})\|$ is then an

— iterative procedure at each θ ,

which includes calculations of matrix exponentials at each step and is thus computationally quite involved.

20/48

Summary

Thus, the frequency response gain of sampled-data system is

- ☹ not quite informative
- ☹ hard to compute.

This justifies a quest for alternatives...

21/48

Content

Frequency response of sampled-data systems: a “natural” approach

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Frequency power response in a special case

22/48

An alternative

Let $\mathcal{G}$ be a linear system with a scalar input. Its **frequency power response** (FPR) at a frequency $\omega \in \mathbb{R}$ is defined as

$$\text{FPR}_{\mathcal{G}}(\omega) := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \|(\mathcal{G}\alpha_{\omega})(t)\|^2 dt,$$

where $\alpha_{\omega}(t) := e^{j\omega t}$ is the elementary harmonic with the frequency ω . If $\mathcal{G}$ is time invariant,

$$\text{FPR}_{\mathcal{G}}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \|G(j\omega)\alpha_{\omega}\|^2 dt = \|G(j\omega)\|^2,$$

consistently with the standard frequency response gain. If $\mathcal{G}$ is h -periodic,

$$\|(\mathcal{G}e^{j\omega t})(t+h)\| = \|(\mathcal{G}e^{j\omega(t+h)})(t)\| = \|(\mathcal{G}e^{j\omega t})(t)e^{j\omega h}\| = \|(\mathcal{G}e^{j\omega t})(t)\|,$$

so that

$$\text{FPR}_{\mathcal{G}}(\omega) = \frac{1}{h} \int_0^h \|(\mathcal{G}\alpha_{\omega})(t)\|^2 dt.$$

23/48

Why?

The quantity

$$\text{FPR}_{\mathcal{G}}(\omega) := \frac{1}{h} \int_0^h \|(\mathcal{G}\alpha_{\omega})(t)\|^2 dt = \frac{1}{h} \int_0^h (\mathcal{G}\alpha_{\omega})^*(\mathcal{G}\alpha_{\omega})(t) dt$$

may be more informative, because it

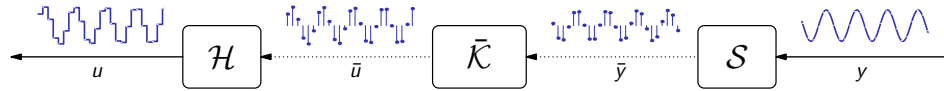
- does not mix input frequencies,
- does account for output aliasing.

In addition, it may be

- way easier to compute.

24/48

Propagating $y(t) = e^{j\omega t}$ in SHC



Let $\mathcal{S}$, $\bar{\mathcal{K}}$, and $\mathcal{H}$ be shift invariant and that the sampling is uniform. Then

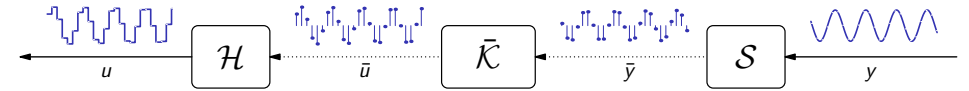
$$\bar{y}[j] = \int_{\mathbb{R}} \psi(jh - s) e^{j\omega s} ds = \int_{\mathbb{R}} \psi(t) e^{j\omega(jh - t)} dt = e^{j\omega hj} \Psi(j\omega),$$

from which $\bar{u}[j] = e^{j\omega hj} \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega)$, and then

$$\begin{aligned} u(ih + \tau) &= \sum_{j \in \mathbb{Z}} \phi((i - j)h + \tau) e^{j\omega hj} \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega) \\ &= \sum_{j \in \mathbb{Z}} \phi(jh + \tau) e^{j\omega h(i - j)} \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega) \\ &= e^{j\omega hi} \check{\phi}(e^{j\omega h}; \tau) \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega). \end{aligned}$$

25/48

Propagating $y(t) = e^{j\omega t}$ in SHC (contd)



Applying the Key Lifting Formula (with $\omega_i := \omega + \frac{2\pi}{h} i$),

$$u(ih + \tau) = \frac{1}{h} e^{j\omega hi} \sum_{j \in \mathbb{Z}} \Phi(j\omega_j) e^{j\omega_j \tau} \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega),$$

from which (as $e^{j\omega hi} = e^{j\omega_j hi}$ for all $i, j \in \mathbb{Z}$)

$$u(t) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Phi(j\omega_i) e^{j\omega_i t} \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega).$$

26/48

FPR in model matching

Consider the error system

$$\mathcal{G}_e = \mathcal{K} - \mathcal{H} \bar{\mathcal{K}} \mathcal{S}.$$

Its response to $e^{j\omega t}$ is $e(t) = e^{j\omega t} K(j\omega) - \frac{1}{h} \sum_{i \in \mathbb{Z}} \Phi(j\omega_i) e^{j\omega_i t} \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega)$, so that

$$\begin{aligned} \text{FPR}_e(\omega) &= \frac{1}{h} \int_0^h \left\| e^{j\omega t} K(j\omega) - \frac{1}{h} \sum_{i \in \mathbb{Z}} \Phi(j\omega_i) e^{j\omega_i t} \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega) \right\|^2 dt \\ &= \frac{1}{h} \int_0^h \left\| K(j\omega) - \left(\frac{1}{h} \sum_{i \in \mathbb{Z}} \Phi(j\omega_i) \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega) \right) e^{j\frac{2\pi}{h} it} \right\|^2 dt \\ &= \left\| K(j\omega) - \frac{1}{h} \Phi(j\omega) \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega) \right\|^2 \\ &\quad + \frac{1}{h^2} \sum_{i \in \mathbb{Z} \setminus \{0\}} \left\| \Phi(j\omega_i) \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega) \right\|^2 \end{aligned}$$

(because $\frac{1}{\sqrt{h}} e^{j\frac{2\pi}{h} it}$ are orthonormal).

27/48

FPR in model matching (contd)

If all components are SISO,

$$\begin{aligned} \text{FPR}_e(\omega) &= \left| K(j\omega) - \frac{1}{h} \Phi(j\omega) \bar{\mathcal{K}}(e^{j\omega h}) \Psi(j\omega) \right|^2 \\ &\quad + \frac{1}{h^2} \sum_{i \in \mathbb{Z} \setminus \{0\}} |\Phi(j\omega_i)|^2 |\bar{\mathcal{K}}(e^{j\omega h})|^2 |\Psi(j\omega)|^2 \end{aligned}$$

If ω is in the baseband, i.e. $\omega \in [-\omega_N, \omega_N]$, the second term above shows

- the contribution of the aliased frequencies to the error.

28/48

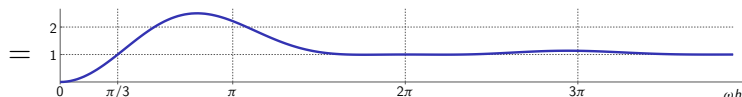
Example 1: error FPR

Consider again $\mathcal{G}_e = 1 - \mathcal{H}_{\text{ZOH}}\mathcal{S}_{\text{avg}}$, where

$$\Phi_{\text{ZOH}}(j\omega) = \frac{1 - e^{-j\omega h}}{j\omega} \quad \text{and} \quad \Psi_{\text{avg}}(j\omega) = \frac{1 - e^{-j\omega h}}{j\omega h}.$$

In this case

$$\text{FPR}_e(\omega) = \frac{4 + \omega^2 h^2 - 6 \cos(\omega h) + 2 \cos(2\omega h)}{\omega^2 h^2} \xrightarrow{\omega \rightarrow \infty} 1$$

$$=$$


and $\text{FPR}_e(\omega) < 1$ iff $|\omega h| < \frac{\pi}{3}$, which is logical.

29/48

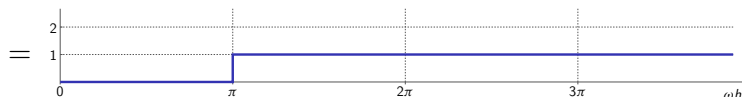
Example 2: error FPR

Consider $\mathcal{G}_e = 1 - \mathcal{H}_{\text{sinc}}\mathcal{S}_{\text{ilp}}$, where

$$\Phi_{\text{sinc}}(j\omega) = h \mathbb{1}_{[-\omega_N, \omega_N]}(\omega) \quad \text{and} \quad \Psi_{\text{ilp}}(j\omega) = \mathbb{1}_{[-\omega_N, \omega_N]}(\omega).$$

In this case

$$\text{FPR}_e(\omega) = 1 - \mathbb{1}_{[-\omega_N, \omega_N]}(\omega) \xrightarrow{\omega \rightarrow \infty} 1$$

$$=$$


and it is logical as well.

30/48

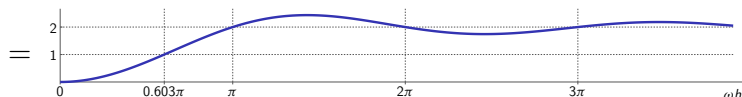
Example 3: error FPR

Consider now $\mathcal{G}_e = 1 - \mathcal{H}_{\text{ZOH}}\mathcal{S}_{\text{idl}}$, where

$$\Phi_{\text{ZOH}}(j\omega) = \frac{1 - e^{-j\omega h}}{j\omega} \quad \text{and} \quad \Psi_{\text{idl}}(j\omega) = 1.$$

In this case

$$\text{FPR}_e(\omega) = 2 - \frac{2 \sin(\omega h)}{\omega h} \xrightarrow{\omega \rightarrow \infty} 2$$

$$=$$


and $\text{FPR}_e(\omega) < 1$ iff $|\omega h| < 0.603355\pi$, which is better than with $\mathcal{S}_{\text{avg}}$.

31/48

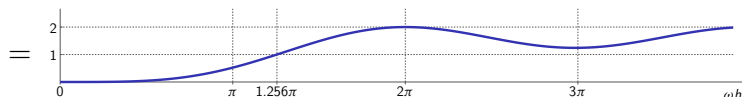
Example 4: error FPR

Finally, consider $\mathcal{G}_e = 1 - \mathcal{H}_{\text{FOH}}\mathcal{S}_{\text{idl}}$, where

$$\Phi_{\text{FOH}}(j\omega) = \frac{2(1 - \cos(\omega h))}{\omega^2 h} \quad \text{and} \quad \Psi_{\text{idl}}(j\omega) = 1.$$

In this case

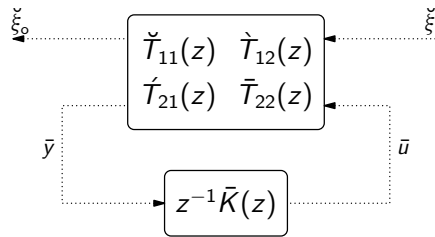
$$\text{FPR}_e(\omega) = \frac{5}{3} - \frac{4}{\omega^2 h^2} + \left(\frac{1}{3} + \frac{4}{\omega^2 h^2} \right) \cos(\omega h) \xrightarrow{\omega \rightarrow \infty} \frac{5 + \cos(\omega h)}{3}$$

$$=$$


and $\text{FPR}_e(\omega) < 1$ iff $|\omega h| < 1.25584\pi$, which is way better than with $\mathcal{H}_{\text{ZOH}}$. As a matter of fact, $\text{FPR}_e(\omega) < \frac{1}{2}$ iff $|\omega h| < 0.9858\pi$, so the error is below -3dB virtually up to the Nyquist frequency.

32/48

State-space computations



We know that in the lifted domain

$$\check{G}_e(z) = \check{D}_{11} + \begin{bmatrix} \check{C}_1 & \check{D}_{12} \end{bmatrix} \left[\begin{array}{cc|cc} \bar{A} & \bar{B}_2 \bar{C}_K & I & 0 \\ \bar{B}_K \bar{C}_2 & \bar{A}_K + \bar{B}_K \bar{D}_{22} \bar{C}_K & 0 & \bar{B}_K \\ \hline I & 0 & 0 & 0 \\ 0 & \bar{C}_K & 0 & 0 \end{array} \right] \begin{bmatrix} \check{B}_1 \\ \check{D}_{21} \end{bmatrix}.$$

We also know that $\check{T}_{11}(z) = \check{D}_{11} + \check{C}_1(zI - \bar{A})^{-1}\check{B}_1$.

33/48

State-space computations (contd)

Thus,

$$\check{G}_e(z) = \check{T}_{11}(z) + \underbrace{\begin{bmatrix} \check{C}_1 & \check{D}_{12} \end{bmatrix}}_{\phi_e(z)} \underbrace{\left[\begin{array}{cc|cc} \bar{A} & \bar{B}_2 \bar{C}_K & 0 & I & 0 \\ \bar{B}_K \bar{C}_2 & \bar{A}_K + \bar{B}_K \bar{D}_{22} \bar{C}_K & 0 & 0 & \bar{B}_K \\ \hline I & 0 & -I & 0 & 0 \\ 0 & \bar{C}_K & 0 & 0 & 0 \end{array} \right]}_{\bar{K}_a(z)} \underbrace{\begin{bmatrix} \check{B}_1 \\ \check{D}_{21} \end{bmatrix}}_{\psi_e(z)}.$$

Hence, in the time domain

$$\mathcal{G}_e = \mathcal{T}_{11} + \mathcal{H}_e \bar{K}_a \mathcal{S}_e$$

and we can apply the formula in Slide 27, with some more work, to compute $\text{FPR}_e \dots$

34/48

Content

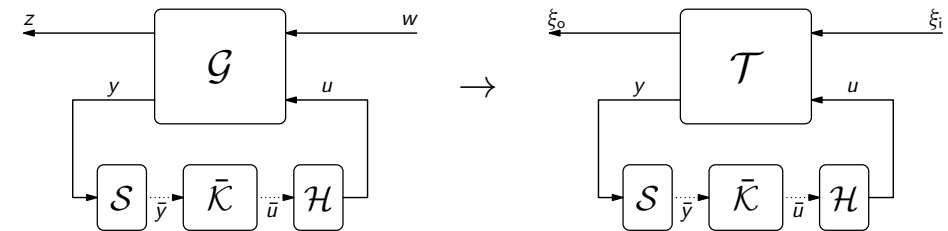
Frequency response of sampled-data systems: a “natural” approach

Frequency power response

Frequency power response in a special case

35/48

Special case



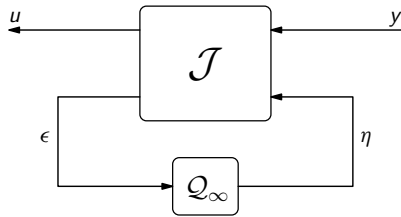
Let

$$G(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array} \right] \quad \text{and} \quad T(s) = \left[\begin{array}{c|cc} A & B_1 & B_u \\ \hline C_1 & D_{11} & D_{12} \\ C_y & D_{21} & 0 \end{array} \right]$$

(i.e. their (2,2) parts coincide) and assume that the sampled-data controller is designed by the Youla parameter in an *observer-based* parametrization to render the controller strictly proper in the lifted domain (as in Lecture 12).

36/48

Controller



where

$$Q_\infty : \begin{cases} \dot{x}_Q(t) = A x_Q(t) + L_y \epsilon(t), & x_Q(ih) = 0 \\ \eta(t) = F_u x_Q(t), \end{cases}$$

F_u and L_y are such that $A_F := A + B_u F_u$ and $A_L := A + L_y C_y$ are Hurwitz, and

$$J(s) = \left[\begin{array}{c|cc} A + B_u F_u + L_y C_y & -L_y & B_u \\ \hline F_u & 0 & I \\ -C_y & I & 0 \end{array} \right].$$

37/48

Error system

It can be shown (cf. Lecture 9, Slide 7) that $\mathcal{G}_e : \xi_i \mapsto \xi_o$ is

$$\mathcal{G}_e = \mathcal{N}_{11} - \mathcal{N}_{12} Q_\infty \mathcal{N}_{21},$$

where, introducing $B_L := B_1 + L_y D_{21}$ and $C_F := C_1 + D_{12} F_u$,

$$N(s) = \left[\begin{array}{c|c} N_{11}(s) & N_{12}(s) \\ \hline N_{21}(s) & 0 \end{array} \right] = \left[\begin{array}{cc|cc} A_F & -L_y C_y & -L_y D_{21} & B_u \\ 0 & A_L & B_L & 0 \\ \hline C_F & C_1 & D_{11} & D_{12} \\ 0 & C_y & D_{21} & 0 \end{array} \right].$$

The question now is

- how to compute its frequency power response, FPR_e ?

This is not a sample-and-hold circuit (Q_∞ is not a sampled-data element), and it has a plenty of structure to exploit...

38/48

FPR in the lifted domain

Let $\mathcal{G}_e : \xi_i \mapsto \xi_o$ be an h -periodic error system and $\check{\mathcal{G}}_e$ be its shift-invariant lifting. We know that

$$\xi_i(t) = e^{j\omega t} \implies \check{\xi}_i[i](\tau) = e^{j\omega\tau} e^{j\omega h i} =: \check{\alpha}_\omega(\tau) e^{j\omega h i}$$

and that

$$\check{\xi}_o[i] = [\check{\mathcal{G}}_e(e^{j\omega h}) \check{\alpha}_\omega] e^{j\omega h i}$$

and then

$$\begin{aligned} \text{FPR}_e(\omega) &= \frac{1}{h} \int_0^h \|\xi_o(t)\|^2 dt = \frac{1}{h} \int_0^h \|\check{\xi}_o[0](\tau)\|^2 d\tau = \frac{1}{h} \|\check{\xi}_o[0]\|_{\mathbb{L}}^2 \\ &= \frac{1}{h} \|\check{\mathcal{G}}_e(e^{j\omega h}) \check{\alpha}_\omega\|_{\mathbb{L}}^2, \end{aligned}$$

where $\mathbb{L} := L_2[0, h)$, as usual. Thus, we shall endeavor to

- determine the response of the lifted frequency response of $\check{\mathcal{G}}_e$ to $\check{\alpha}_\omega$.

39/48

hSTPBC representations of $\check{N}(z)$ and $\check{Q}_\infty(z)$

As follows from Lecture 11, Slide 46, a hSTPBC representation of $\check{N}(z)$ is

$$\begin{cases} \frac{d}{d\tau} \check{X}_N(z; \tau) = \begin{bmatrix} A_F & -L_y C_y \\ 0 & A_L \end{bmatrix} \check{X}_N(z; \tau) + \begin{bmatrix} -L_y D_{21} \\ B_L \end{bmatrix} \check{\Xi}_i(z; \tau) + \begin{bmatrix} B_u \\ 0 \end{bmatrix} \check{H}(z; \tau), \\ \quad z \check{X}_N(z; 0) - \check{X}_N(z; h) = 0 \\ \check{\Xi}_o(z; \tau) = \begin{bmatrix} C_F & C_1 \end{bmatrix} \check{X}_N(z; \tau) + D_{11} \check{\Xi}_i(z; \tau) + D_{12} \check{H}(z; \tau) \\ \check{E}(z; \tau) = \begin{bmatrix} 0 & C_y \end{bmatrix} \check{X}_N(z; \tau) + D_{21} \check{\Xi}_i(z; \tau) \end{cases}$$

and of $\check{Q}_\infty(z)$ is

$$\begin{cases} \frac{d}{d\tau} \check{X}_Q(z; \tau) = A \check{X}_Q(z; \tau) + L_y \check{E}(z; \tau), & z \check{X}_Q(z; 0) = 0 \\ \check{H}(z; \tau) = F_u \check{X}_Q(z; \tau). \end{cases}$$

40/48

hSTPBC representation of $\check{G}_e(z)$

Representing $\check{G}_e$ as an hSTPBC is then a matter of plugging in appropriate signals and aggregating states. In addition, a similarity transformation with

$$T = \begin{bmatrix} I & -I & -I \\ 0 & I & 0 \\ 0 & I & I \end{bmatrix} \quad \text{and} \quad S = T^{-1} = \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \\ 0 & -I & I \end{bmatrix}$$

may be applied to simplify it a bit, leading to $\check{G}_e(z)$ of the form

$$\begin{cases} \frac{d}{d\tau} \check{X}_e(z; \tau) = A_e \check{X}_e(z; \tau) + B_e \check{\Xi}_e(z; \tau), & z \check{X}_e(z; 0) + \Upsilon_e \check{X}_e(z; h) = 0 \\ \check{\Xi}_e(z; \tau) = C_e \check{X}_e(z; \tau) + D_{11} \check{\Xi}_e(z; \tau), \end{cases}$$

where

$$\begin{bmatrix} A_e & B_e \\ C_e & \end{bmatrix} := \begin{bmatrix} A_F & 0 & 0 & 0 \\ 0 & A_L & 0 & B_L \\ 0 & 0 & A & -B_1 \\ \hline C_F & 0 & -C_1 & \end{bmatrix} \quad \text{and} \quad \Upsilon_e := \begin{bmatrix} -I & I & I \\ 0 & -I & 0 \\ 0 & I & 0 \end{bmatrix}.$$

41/48

Response of $\check{G}_e(e^{j\omega h})$ to $\check{\alpha}_\omega$

Now, derive the response of $\check{G}_e(e^{j\omega h})$ to $\check{X}_e(e^{j\omega h}; \tau) = e^{j\omega\tau}$. By the formula from Lecture 10, Slide 15, we have (with $\Xi_e := e^{j\omega h}I + \Upsilon_e e^{A_e h}$):

$$\begin{aligned} \check{\Xi}_e(e^{j\omega h}; \tau) &= D_{11} e^{j\omega\tau} + C_e e^{A_e \tau} \Xi_e^{-1} \\ &\quad \times \left(e^{j\omega h} \int_0^\tau e^{(j\omega I - A_e)\sigma} d\sigma - \Upsilon_e e^{A_e h} \int_\tau^h e^{(j\omega I - A_e)\sigma} d\sigma \right) B_e \\ &= D_{11} e^{j\omega\tau} + C_e e^{A_e \tau} \Xi_e^{-1} \\ &\quad \times (\Xi_e e^{(j\omega I - A_e)\tau} - (I + \Upsilon_e) e^{j\omega h}) (j\omega I - A_e)^{-1} B_e \\ &= D_{11} e^{j\omega\tau} + C_e (e^{j\omega\tau} I - e^{A_e \tau} e^{j\omega h} \Xi_e^{-1} (I + \Upsilon_e)) (j\omega I - A_e)^{-1} B_e \\ &= \begin{bmatrix} e^{j\omega\tau} I & -e^{j\omega h} C_e e^{A_e \tau} \end{bmatrix} \begin{bmatrix} T_{11}(\omega) \\ \Xi_e^{-1} (I + \Upsilon_e) (j\omega I - A_e)^{-1} B_e \end{bmatrix} \end{aligned}$$

where the relations $\int_a^b e^{(j\omega I - A_e)\sigma} d\sigma = (e^{(j\omega I - A_e)b} - e^{(j\omega I - A_e)a}) (j\omega I - A_e)^{-1}$ and $D_{11} + C_e (sI - A_e)^{-1} B_e = D_{11} + C_1 (sI - A)^{-1} B_1 = T_{11}(s)$ were used.

42/48

Some algebra

Now,

$$\Xi_e = e^{j\omega h} I + \begin{bmatrix} -I & I & I \\ 0 & -I & 0 \\ 0 & I & 0 \end{bmatrix} e^{A_e h} = \begin{bmatrix} e^{j\omega h} I - e^{A_F h} & e^{A_L h} & e^{A_h} \\ 0 & e^{j\omega h} I - e^{A_L h} & 0 \\ 0 & e^{A_L h} & e^{j\omega h} I \end{bmatrix}$$

so that, denoting $\bar{\Theta}_F(z) := (zI - e^{A_F h})^{-1}$ and $\bar{\Theta}_L(z) := (zI - e^{A_L h})^{-1}$,

$$\Xi_e^{-1} = \begin{bmatrix} \bar{\Theta}_F(e^{j\omega h}) & \bar{\Theta}_F(e^{j\omega h})(e^{-j\omega h} e^{A_h} - I) e^{A_L h} \bar{\Theta}_L(e^{j\omega h}) & -\bar{\Theta}_F(e^{j\omega h}) e^{-j\omega h} e^{A_h} \\ 0 & \bar{\Theta}_L(e^{j\omega h}) & 0 \\ 0 & -e^{-j\omega h} e^{A_L h} \bar{\Theta}_L(e^{j\omega h}) & e^{-j\omega h} I \end{bmatrix}$$

and then

$$\Xi_e^{-1} (I + \Upsilon_e) = (e^{j\omega h} I - e^{A_e h})^{-1} \begin{bmatrix} I \\ 0 \\ I \end{bmatrix} (I - e^{-j\omega h} e^{A_h}) \begin{bmatrix} 0 & I & I \end{bmatrix}.$$

43/48

Response of $\check{G}_e(e^{j\omega h})$ to $\check{\alpha}_\omega$ (contd)

With some more (straightforward) algebra we end up with

$$\check{\Xi}_e(e^{j\omega h}; \tau) = \begin{bmatrix} e^{j\omega\tau} I & C_a e^{A_a \tau} (I - e^{-(j\omega I - A_a)h})^{-1} B_a \end{bmatrix} \begin{bmatrix} T_{11}(j\omega) \\ G_s(j\omega) \end{bmatrix},$$

where

$$\begin{bmatrix} A_a & B_a \\ C_a & \end{bmatrix} := \begin{bmatrix} A_F & 0 & I \\ 0 & A & I \\ \hline -C_F & C_1 & \end{bmatrix}$$

and

$$G_s(s) := \left[\begin{array}{c|c|c} A_L & 0 & B_L \\ \hline 0 & A & B_1 \\ \hline I & -I & 0 \end{array} \right] - \left[\begin{array}{c|c|c} A_L & 0 & B_L \\ \hline 0 & A & B_1 \\ \hline e^{A_h} & -e^{A_h} & 0 \end{array} \right] e^{-sh}.$$

44/48

Finally, $\text{FPR}_e(\omega) \dots$

It is readily verified that

$$\int_0^h \begin{bmatrix} e^{-j\omega h} I \\ e^{A_a' \tau} C_a' \end{bmatrix} \begin{bmatrix} e^{j\omega h} I & C_a e^{A_a \tau} \end{bmatrix} d\tau = \begin{bmatrix} hI & C_a(j\omega I - A_a)^{-1}(I - e^{-j\omega h} e^{A_a h}) \\ (\cdot)^* & \int_0^h e^{A_a' \tau} C_a' C_a e^{A_a \tau} d\tau \end{bmatrix}$$

Thus,

$$\text{FPR}_e(\omega) = \frac{1}{h} \begin{bmatrix} T_{11}(\tilde{j}\omega) & G_s(\tilde{j}\omega) \end{bmatrix} \begin{bmatrix} hI & G_a(j\omega) \\ G_s(j\omega) & \bar{Q}_a(e^{j\omega h}) \end{bmatrix} \begin{bmatrix} T_{11}(j\omega) \\ G_s(j\omega) \end{bmatrix},$$

where $G_a(s) := C_a(sI - A_a)^{-1}B_a$ and

$$\bar{Q}_a(z) := z \begin{bmatrix} B_a' & 0 \end{bmatrix} \left(zI - \exp \left(\begin{bmatrix} -A_a' & -C_a' C_a \\ 0 & A_a \end{bmatrix} h \right) \right)^{-1} \begin{bmatrix} 0 \\ B_a \end{bmatrix}.$$

45/48

Example: “ H_2 loop shaping”

Consider design problem with the generalized plant as in Slide 3 with

$$P(s) = \frac{1}{s}, \quad W_i(s) = k, \quad \text{and} \quad W_o(s) = 1,$$

but let us use the H_2 optimization instead of H_∞ (and ignore the fact that the cost is infinite then, because $G_{zw}(s)$ is not strictly proper). It results in the analog case in

$$K(s) = \frac{k^2}{s + 2k}.$$

Consider then the frequency response analysis of the sensitivity function, i.e.

$$T(s) = \begin{bmatrix} 1 & -P(s)W_i(s) \\ 1 & -P(s)W_i(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sqrt{k} \\ \sqrt{k} & 1 & 0 \\ \sqrt{k} & 1 & 0 \end{bmatrix}$$

with which $G_e(s) = S(s) = \frac{1}{1+P(s)K(s)} = \frac{s(s+2k)}{(s+k)^2}$ in the analog case. In this setting $|S(j\omega)| \leq 1/\sqrt{2}$ whenever $\omega \leq \sqrt{\sqrt{10} - 3}k \approx 0.4028k$ (bandwidth).

46/48

Example: FPR of the sampled-data redesign

Redesigning this controller along the line of Lectures 10 and 11, with

$$A = 0, \quad B_u = -\sqrt{k}, \quad F_u = \sqrt{k}, \quad C_y = \sqrt{k}, \quad \text{and} \quad L_y = -\sqrt{k},$$

yields

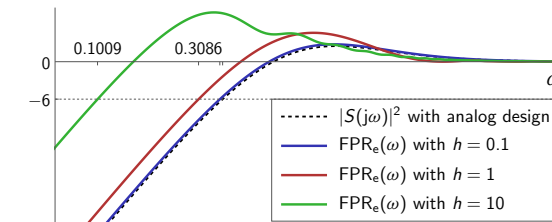
$$\begin{aligned} \text{FPR}_e(\tilde{\omega}) = & \frac{\tilde{\omega}^4 + 3\kappa^2 \tilde{\omega}^2 + 2\kappa^3(\kappa + 2)}{(\tilde{\omega}^2 + \kappa^2)^2} \\ & - \frac{4\kappa^3 \cos(\tilde{\omega}) - 2\kappa^2(\tilde{\omega}^2 - \kappa^2) \sin(\tilde{\omega})/\tilde{\omega}}{(\tilde{\omega}^2 + \kappa^2)^2} \\ & - \frac{\kappa(1 - \cos(\tilde{\omega})) \sinh(\kappa)}{(\tilde{\omega}^2 + \kappa^2)(\cosh(\kappa) - \cos(\tilde{\omega}))}. \end{aligned}$$

where $\kappa := kh$ and $\tilde{\omega} := \omega h$. Expectably, $\lim_{\tilde{\omega} \rightarrow 0} \text{FPR}_e(\tilde{\omega}) = 0$, which is a consequence of the fact that $T_{22}(s)$ contains an integral action.

47/48

Example: FPR of the sampled-data redesign (contd)

For $k = 1$:



and, expectably,

- closed-loop performance deteriorates as h grows...

48/48